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AL-RISĀLA AL-MUHĪTĪYYA: A SUMMARY

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Abstract. In this paper we present an English summary of al-Risāla al-muhītīyya (“The Treatise on the Circumference”). This treatise is one of the most significant mathematical achievements of Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī which he completed in July 1244 (Sha'bān 827 A.H.L.). Also, we present a complete English translation of the titles of all ten sections as well as the title of the introduction and the title of the conclusion. Additionally, we provide comments throughout the paper as necessary.

1. Preliminaries

Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī (also known as al-Kāshānī, Kāshī, or Kāshānī) was one of the most renowned mathematicians in the Islamic world. Al-Kāshī’s exact date of birth is not known, but it was sometime during the second half of the fourteenth century in the city of Kāshān in central Iran. Al-Kāshī died on the morning of Wednesday, June 22, 1429 (Ramadān 19, 832 A.H.L.) outside of Samarqand at the observatory he had helped to build in Uzbekistān. For a more detailed discussion of Al-Kāshī’s life and his mathematical achievements we refer the reader to the author’s article in the Journal of Recreational Mathematics [5].

One of the most significant mathematical achievements of al-Kāshī is al-Risāla al-muhītīyya (“The Treatise on the Circumference”), which is also known as Risāla-i muhītīyya, or simply Muhītīyya. This 58-page long treatise, written longhand in Arabic by al-Kāshī himself, was completed in Samarqand in July 1244 (Sha'bān 827 A.H.L.) [1]. We argued in [2] that [1] is the original manuscript of al-Risāla al-muhītīyya which was donated by Nāder Shāh (King Nāderi) in 1145 A.H.S. to the library of the Āstāne Qudse Razawī, Mashhad, Iran. It is Number 162 of the mathematics collection (Number 5389 of the general collection) of the central library of the Āstāne Qudse Razawī. We include photocopies of the first page (Plate A) and the last page (Plate B) of this original manuscript of al-Risāla al-muhītīyya with this paper. Another handwritten manuscript of al-Risāla al-muhītīyya exists in the Kitābkhān-i Shōrā-ye Molī (National Library of the Parliament), Tehran, Iran. Other manuscripts of this treatise exist.
in Germany, India, Russia, Turkey, and various libraries throughout the Middle East and elsewhere.

Paul L. Luckey was the first to write an impressive German translation of most of al-Risāla al-muhītīyya in 1949. Luckey’s German translation with a detailed commentary, along with copies of most pages from the typed Arabic manuscript number 576 of the Istanbul Military Museum, was published in Berlin, posthumously in 1953 [10]. The typed pages of the Arabic manuscript of Al-Risāla al-muhītīyya that were used by Luckey and were published along with his translation contain some typographical errors [3]. A. Qurbānī in Kāshānī Nāmeh [12], a monograph on al-Kāshī and his achievements in mathematics and astronomy, has written a Persian summary of Al-Risāla al-muhītīyya, based on Āstāne Qudse Razawi’s manuscript, Luckey’s German translation, Rosenfeld and Youschkevitch’s Russian translation, as well as other sources. Qurbānī’s book will be our main source for the writing of this article. In addition, Boris A. Rosenfeld and Adolf P. Youschkevitch wrote a Russian account of Al-Risāla al-muhītīyya with commentaries in 1954 [18]. According to A. Qurbānī and Professor B. A. Rosenfeld himself, the handwritten manuscript that was available to Rosenfeld and Youschkevitch contained some errors, which carried over into the Russian translation. However, the corrected version of [17] along with extended commentaries was published in 1956 in a book by B. A. Rosenfeld [16]. According to Professor Rosenfeld, [16] contains an exposition by the Turkish astronomer Mirim Chelebi (d. 1525). Chelebi was the grandson of both the Turkish astronomer and mathematician Salāh al-Dīn Mūsā Qādī zade al-Rūmī (1360–1437), and Alā al-Dīn ʿAlī al-Qūshji (d. 1474), another Turkish astronomer who was also a mathematician, philosopher, and linguist.

In this paper we present an English summary of Al-Risāla al-muhītīyya. Also, we present a complete English translation of the titles of all ten sections as well as the title of the introduction and the title of the conclusion. Additionally, we provide comments throughout the paper as necessary. In this summary we present those topics and ideas of Al-Risāla al-muhītīyya which we believe are mathematically significant, and easy to communicate in modern notations without getting into unnecessary, insignificant, lengthy discussions, and repeated calculations.

As we will see in the introduction of Al-Risāla al-muhītīyya, the calculations of π by Archimedes (c. 287–212 BC), Abu’l-Wafā’ al-Buzjānī (940–998), and Abū Rayḥān al-Bīrūnī (973–after 1050) motivated al-Kāshī to write this treatise on improving the estimation of π by these three renowned mathematicians. Another motivation for al-Kāshī to write this treatise was the fact that there was a real demand for more precise trigonometric tables in connection with advanced research in astronomy. Al-Kāshī used an
inscribed regular polygon and a circumscribed regular polygon each with $3 \times 2^{28}$ sides to calculate the ratio of the circumference to its diameter of the corresponding circle to obtain an approximation for $\pi$ that far surpassed all approximation of $\pi$ by all previous mathematicians and astronomers throughout the globe at that time. In the review of [10], E. S. Kennedy stated that al-Risāla al-muhitīyya was a landmark of Islamic computational technique [MR0055264 (14, 1051a)]. Al-Kāshī’s method of obtaining this approximation of $\pi$ was very similar to that of Archimedes. Paul Luckey in the introduction of [10] states that al-Risāla al-muhitīyya is a masterpiece of sexagesimal calculations. However, to make his work accessible to non-astronomers al-Kāshī also performed his calculations in the decimal system and hence made use of decimal fractions. Contrary to popular belief by some scientists and mathematicians, especially in some Persian circles, he may not be the first who discovered decimal fractions. However, he did discover decimal fractions on his own and for his own use without knowledge or use of others’ works (the footnotes are listed in Section 14).

Comment 1.1. The author in [5, p. 36] stated that Luckey translated the entire al-Risāla al-muhitīyya into German and in [3, p. 507] he again wrote that Luckey’s translation was a comprehensive one. After studying the German manuscript more closely, the author regrets to admit that Luckey did not translate the entire al-Risāla al-muhitīyya. For example, al-Kāshī started al-Risāla al-muhitīyya with the usual phrase, “In the Name of God, Most Gracious, Most Merciful,” but the Arabic copy of al-Risāla al-muhitīyya that Luckey used did not have this phrase and hence, it was not in the German translation. An example of a major omission in the German translation was in Section 4, where out of 28 tables (pages) Luckey has translated only four tables; namely, Tables 1, 2, 15, and 28. However, since all 28 tables were about the root extraction perhaps he did not feel that there was a need to translate all 28 tables. Luckey included all pages from the Arabic manuscript that he translated with one exception. He had a translation of Section 5, but Section 5 was missing from the Arabic pages that he included with the German translation.

Comment 1.2. Al-Kāshī used the following units of measurements in al-Risāla al-muhitīyya:

1. farsakh $\approx 12000$ adru’ $\approx 6$ km
2. dirāʿ $\approx 24$ qaṣabāt $\approx 50$ cm
3. isba = 1 finger $\approx 6$ times the width of a medium-size grain of barley
4. Width of a medium-size grain of barley $\approx 6$ shaʿrāt (plural of shaʿra)
5. shaʿra = The breadth of a horse (mane) hair

Adru’ is the plural of dirāʿ $\approx 50$ cm and it is different from the Iranian zar $\approx 104$ cm. We note that the value of dirāʿ varies from country to
country. For example, in Syria it is about 68 cm while in Egypt it is 58 cm.

Qasabāt is the plural of qasaba ≈ 3.55 m. In old Persian it used to be called qasab (or nāb). However, this measurement is no longer in use in modern day Iran.

Farsakh ≈ 6 km and its plural is farāsik in Arabic and farsakha in Persian. In Persian farsakh is also called farsang or pārsang, and in Arabic farsakh is also called farsak.

Comment 1.3. In both the Arabic and the Persian languages there is not a word which is equivalent to the word “circumference”. In both of these languages we use the word muhīt for both the circumference of a circle as well as the perimeter of a geometric figure in the plane. Thus, we see where the word al-muhīṭīyya (“On the circumference”) comes from. However some Persian scientists, including al-Bīrūnī, used the word “dour” which means “around” instead of muhīt for the circumference.

2. IN THE NAME OF GOD, MOST GRACIOUS, MOST MERCIFUL

Al-Kāshī started al-Risāla al-muhīṭīyya with the usual Islamic praise of the Almighty: “In the Name of God, Most Gracious, Most Merciful,” a common phrase used by devout Muslims before undertaking any task. Al-Kāshī discussed the calculations of π by Archimedes, Abu'l-Wafā' al-Būzjānī, and Abū Rayhān al-Bīrūnī. A complete English translation of the introduction of al-Risāla al-muhīṭīyya with commentaries was presented by the author in [2]. What follows is a summary of [2].

Al-Kāshī begins by writing, “Thanks be to the deserving God Who is aware of the ratio of the diameter to the circumference … ” and he asked for forgiveness and guidance from the Exalted God. Then, he stated that Archimedes has proven that the circumference of a circle is smaller than three times its diameter by less than \( \frac{1}{4} \) of the diameter and larger than three times of the diameter by less than \( \frac{10}{7} \) of the diameter. Hence, the difference of these two quantities is \( \frac{1}{497} \) of the diameter. Archimedes calculated the perimeter of the inscribed (circumscribed) regular polygons with 96 sides which is less (more) than the circumference of the circle.

Al-Būzjānī obtained an approximation for the half of the chord of one degree arc with the assumption that the diameter of the circle is 120 units. He multiplied this result by 720 to obtain the perimeter of the inscribed regular polygon with 720 sides. He then calculated the perimeter of the similar circumscribed regular polygon with 720 sides as well. He stated that the circumference of this circle would be 376 units and a fraction, where this fraction is more than 0; 59, 10, 59 and less than 0; 59, 23, 54, 12 and the difference of these two quantities is 0; 0, 12, 55, 12. However, Abū
Rayhān al-Bīrūnī calculated the chord of a two degree arc and he used it to calculate the perimeter of the inscribed regular polygon with 180 sides equal to 6; 16, 59, 10, 48, 0 and the perimeter of the similar circumscribed regular polygon with 180 sides equal to 6; 17, 1, 58, 19, 6. Then, he has taken the average of these two perimeters as the circumference of the circle.

Al-Kāshī concluded the introduction of al-Risāla al-muḥīṭiyyā by stating that the calculations of π by Archimedes, al-Būjānī, and al-Bīrūnī were confusing. Thus, he wanted to determine the circumference of a circle with a known diameter in such a way that he would be certain that in a circle in which its diameter is six hundred thousand times the diameter of the Earth, the difference between the result of his calculations and the actual value of the circumference would not reach the breadth of a hair: the breadth of which is one sixth of the width of an average grain of barley. In the last sentence of the introduction al-Kāshī wrote:

\[
\ldots \text{while I am asking for help from the dear bestower God, who is the guidance to the right path, I wrote this treatise on the determination of the circumference consisting of ten sections and a conclusion, and I named [entitled] it al-Muḥīṭiyyā.}
\]

3. THE FIRST SECTION
ON DETERMINATION OF THE CHORD OF AN ARC THAT IS THE SUM OF AN ARC OF KNOWN CHORD AND HALF OF ITS SUPPLEMENT TO THE HALF CIRCLE

In [3] we provided a complete English translation of the First Section of al-Risāla al-muḥīṭiyyā along with comments on German, Persian, and Russian translations. Also, we presented an alternative proof of al-Kāshī’s fundamental theorem. However, like all other sections, we present a summary of this section from [3] here. In this section of al-Risāla al-muḥīṭiyyā, al-Kāshī determined the chord of an arc which is the sum of an arc of a known chord and half of its supplement to the half circle. Although this was the foundation and the main ingredient of all of his calculations in this treatise, he did not state it as a theorem. He stated and proved the following seminal theorem which we call al-Kāshī’s fundamental theorem [3, p. 501]: Throughout the paper we use \(AB\) to represent the line segment connecting the points \(A\) and \(B\).

**Theorem 3.1 (Al-Kāshī’s Fundamental Theorem).** If on a semicircle with the diameter \(AB = 2r\) and the center \(E\) we consider an arbitrary arc \(AG\), and we call the middle of the arc \(GB\) the point \(D\), and we draw the chord \(AD\), then, \(r(2r + AG) = AD^2\).
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Al-Kāshī’s proof of his fundamental theorem was based on four different propositions from Euclid’s Elements [3]. However, Al-Kāshī did not burden the reader with mentioning Euclid or the Elements in his proof. This is because Euclid’s Elements had been well-known and widely used for geometrical proofs at that time. Al-Kāshī’s fundamental theorem gave him a distinct advantage over his predecessors by having found the most accurate value of π at that time.

![Figure 1](image)

He continued his discussion and he deduced that in a unit circle (Figure 1) if the arc $AG$ is equal to $2\theta$ radians, then, from the right triangle $AGB$ he would get $\overline{AG} = 2\sin \theta$. Similarly, from the right triangle $ADB$ he obtained

$$\overline{AD} = 2 \sin \frac{1}{2} \left( 2\theta + \frac{\pi - 2\theta}{2} \right) = 2 \sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right).$$

Next, he substituted these values of $\overline{AG}$ and $\overline{AD}$ as well as $r = 1$ in his fundamental theorem to obtain

$$2 + 2 \sin \theta = 4 \sin^2 \left( \frac{\pi}{4} + \frac{\theta}{2} \right).$$

That is,

$$\sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right) = \sqrt{\frac{1 + \sin \theta}{2}}, \quad \text{(where } \theta < \frac{\pi}{2}).$$

This is the trigonometric form of Al-Kāshī’s fundamental theorem, which is now known as the Lambert identity. It is intriguing to note that this identity appeared in Johann Heinrich Lambert’s work in 1770, while al-Kāshī proved it before 1424 (827 A.H.L.), some 346 years earlier than Lambert.

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It takes a brilliant mind to come up with such a simple concept and yet with remarkable applications.

In the alternative proof of al-Kāshī’s fundamental theorem we used modern notations and applied the Pythagorean theorem to right triangles.

4. THE SECOND SECTION
ON DETERMINATION OF THE PERIMETER OF AN ARBITRARY POLYGON INSCRIBED IN A CIRCLE AND OF THE PERIMETER OF THE SIMILAR POLYGON CIRCUMSCRIBED TO THE SAME CIRCLE

In the Second Section of al-Risāla al-muhīṭiya, al-Kāshī considered a circle with the diameter $AB = 2r$ and he chose the arc $AG$ equal to 60° (Figure 2). He called the middle of the arc $BG$ the point $D$, the middle of the arc $BD$ the point $Z$, and the middle of the arc $BZ$ the point $H$. He stated that with the help of his fundamental theorem one can calculate the chord $AD$ from the chord $AG$, the chord $AZ$ from the chord $AD$, the chord $AH$ from the chord $AZ$, and one can continue this process as needed.

![Figure 2](image)

In fact with modern notations and terminologies al-Kāshī has stated a procedure for calculating the chords corresponding to each of the following arcs:
arc $AG = \alpha_0 = 60^\circ$
arc $AD = \alpha_1 = 180^\circ - 60^\circ = 120^\circ$
arc $AZ = \alpha_2 = 180^\circ - \frac{60^\circ}{2} = 150^\circ$
arc $AH = \alpha_3 = 180^\circ - \frac{60^\circ}{2^2} = 165^\circ$

$\vdots$

$\alpha_n = 180^\circ - \frac{60^\circ}{2^{n-1}}$. \hspace{1cm} (4.1)

If we call the chord corresponding to the arc $\alpha_n$ by $c_n$ ($n \geq 0$), then the calculation of each chord $c_n$ can be obtained recursively from its preceding chord $c_{n-1}$ by applying al-Kâshî’s fundamental theorem. That is,

$$c_n^2 = r(2r + c_{n-1}),$$

and hence,

$$c_n = \sqrt{r(2r + c_{n-1})}.$$ \hspace{1cm} (4.2)

---

**Figure 3**

Next, in Figure 3, we let $c_n$ represent the chord $\overline{AM}$ and we let $a_n$ represent the chord $\overline{BM}$. Now, $c_n$ can be obtained from the above formula and since the triangle $AMB$ is a right triangle we have

$$a_n = \sqrt{4r^2 - c_n^2}.$$
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But, \( a_n \) is exactly the length of the side of the inscribed regular polygon with \( 3 \times 2^n \) sides. Hence, the arc \( MB \) can be obtained from the arc \( AM \) and Equation (4.1) as

\[
\text{arc } MB = 180^\circ - \text{arc } AM = 180^\circ - \left( 180^\circ - \frac{60^\circ}{2^{n-1}} \right) = \frac{60^\circ}{2^{n-1}} = \frac{360^\circ}{3 \times 2^n}.
\]
Thus, the arc \( MB \) is equal to \( \frac{1}{3 \times 2^n} \) of the circumference and the chord \( MB \) is equal to the length of the side of the inscribed regular polygon with \( 3 \times 2^n \) sides. Therefore, for any positive integer \( n \) one can calculate \( a_n \), the length of each side of the inscribed regular polygon with \( 3 \times 2^n \) sides where \( a_0 = \sqrt{3} \). For the calculation of the length of each side of the circumscribed regular polygon with \( 3 \times 2^n \) sides, al-Kāshī used a circle centered at the point \( O \) and radius \( r \) and he proceeded as follows (Figure 4).

![Figure 4](image)

Let the chord \( BM \) be the length of each side of the inscribed regular polygon with \( 3 \times 2^n \) sides. If we let the middle of the arc \( BM \) be the point \( T \) and we draw \( OT \) so that it intersects \( BM \) at the point \( I \) and we draw a tangent to the circle at the point \( T \) so that it intersects the continuation of \( OM \) at the point \( L \) and it intersects the continuation of \( OB \) at the point \( K \), then \( KL \) will be the side of the circumscribed regular polygon with \( 3 \times 2^n \) sides which will be similar to the inscribed regular polygon with \( 3 \times 2^n \).
sides. Al-Kāshī has proven that
\[
\frac{\overline{OT}}{\overline{OT} - \overline{OI}} = \frac{BM}{KL - BM} \tag{4.3}
\]
and he has said that since \(\overline{OT} = \frac{1}{2}MA\) and \(\overline{OT} = r\), if \(\overline{AM}\) and \(BM\) are known, then one can calculate \(KL\) from the Equation (4.3). Therefore, for any non-negative integer \(n\) one can calculate \(KL\), the length of each side of the circumscribed regular polygon with \(3 \times 2^n\) sides.

5. The Third Section

On How Many Sides [Equal Arcs] We Subdivide the Circumference and Up to What Sexagesimal Digit We Should Calculate so That the Circumference of a Given Circle Is Obtained with an Accuracy Not Larger Than the Breadth of a [Horse] Hair

In the Third Section of *al-Risāla al-muhīṭīyya*, al-Kāshī's goal was to determine the number of sides of a circumscribed regular polygon and to use precision in the calculation of its perimeter in such a way that, as he mentioned in the introduction of *al-Risāla al-muhīṭīyya*, in a circle where the diameter is six hundred thousand times the diameter of the Earth, the difference between the perimeter of the regular polygon and the circumference of the circle does not reach the breadth of a horse hair. Al-Kāshī has said that in a circle where the diameter is six hundred thousand times the diameter of the Earth, its circumference is also six hundred thousand times the circumference of the Earth (greatest circle of the Earth). And with the assumption that the circumference of the Earth is 8000 farāsikh, the circumference of the mentioned circle will be 600000 \(\times\) 8000 farāsikh. He multiplied this value by 12000 so that the circumference of the circle will be in terms of adru'. Then, he multiplied the result by 24, 6, and 6 to obtain the measurement of the circumference in terms of isba, the width of a medium-size grain of barley, and the width of a horse hair, respectively. After he obtained the circumference of the circle in terms of a horse hair, he has taken \(\frac{2}{600}^\text{th}\) of this circumference, namely the arc of one degree, and he has shown that \(\frac{1}{60^\text{th}}\) of this arc is almost equal to \(\frac{2}{5}\) of the width of a hair as follows
\[
\frac{600000 \times 8000 \times 12000 \times 24 \times 6 \times 6}{360 \times 60^8} = \frac{200}{243} \approx \frac{4}{5}.
\]
He concluded that if he calculates the perimeters of an inscribed and a circumscribed regular polygon in such a way that the difference of the two perimeters of these two polygons does not reach \(\frac{1}{60^\text{th}}\), then he would have accomplished his goal.
After applying very clever and skillful computational techniques for approximating the perimeters of the inscribed and the circumscribed regular polygons, he reached the conclusion that in a circle with a radius of 60 units, he must inscribe a regular polygon such that the length of each side of this regular polygon will not be more than $\frac{8}{60}$ units. Then, on the right hand side of a table he halved 120° twenty-eight times in sexagesimal system to get

$120^\circ, 60^\circ, 30^\circ, 15^\circ, \ldots, (0; 0, 0, 0, 5, 47, 36, 51, 26, 16, 45, 28, 7, 30)^\circ$.

On the left hand side of this table he started with an equilateral triangle and he doubled the number of sides of the polygon twenty-eight times in sexagesimal system to obtain

1, 2, 8, 16, 12, 48

which is equal to $3 \times 2^8 = 805306368$ in the decimal system. Finally, he concluded that in order for him to achieve his goal he must calculate the length of a side of a regular polygon with $3 \times 2^8$ sides inscribed in a circle whose radius is 60 units. He has argued in detail that in order for the results of his calculations to be sufficiently precise he must continue his calculations until the order $\frac{1}{60} \pi^9$ in the sexagesimal system.

6. THE FOURTH SECTION
ON THE CALCULATIONS

Section Four is the longest section of al-Risāla al-muhītiyya. In fact it is longer than half of the entire risāla and it contains more calculations than all other sections of al-Risāla al-muhītiyya combined. Al-Kāshī used the formula

$$c_n = \sqrt{r(2r + c_{n-1})}$$

to calculate successively the chords $c_1, c_2, c_3, \ldots, c_n$, corresponding to the arcs $120^\circ, 150^\circ, 165^\circ, \ldots$, and $180^\circ - \frac{360^\circ}{3 \times 2^n}$, respectively. We note that the chord of $60^\circ$, namely $c_0$ is equal to 1 ( = r). This section contains 28 ‘amāl (calculations or works)$^{10}$, where in each one of these calculations al-Kāshī used root extraction to determine each $c_n$, for $n = 1, 2, 3, \ldots, 28$. He organized the calculation of each $c_n$ in an intensive table, where each table took an entire page of al-muhītiyya. He has stated at the start of this section that in each calculation he did not go to the next step until he was completely certain that his calculations in each step were correct. And at the conclusion of the extraction of each root he has squared his result of each extraction of a root and added it to the remainder to make sure that his calculations were completely accurate. He calculated $c_{28}$ with 18
sexagesimal digits after the decimal point as
\[ c_{28} = 1, 59, 59, 59, 59, 59, 59, 59, 59, 59, 52, 12, 30, 48, 37, 49, 54, 40. \]

In a unit circle al-Kāshī used the above formula to obtain \( c_1, c_2, c_3, c_4, \ldots \), and \( c_{28} \) in the decimal system as
\[
\begin{align*}
  c_1 &= \sqrt{3} \\
  c_2 &= \sqrt{2 + \sqrt{3}} \\
  c_3 &= \sqrt{2 + \sqrt{2 + \sqrt{3}}} \\
  c_4 &= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}} \\
  \vdots \\
  c_{28} &= \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}},
\end{align*}
\]
where there are 28 nested radicals in the calculation of \( c_{28} \). In general, we note that if \( r \neq 1 \), then \( c_0 = r \), and consequently, \( c_n \) could be written with \( n \) nested radicals as
\[
  c_n = r \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}}.
\]

7. **The Fifth Section**

**On Determination of the Length of the Side of a Regular Polygon Inscribed in a Circle, Where the Number of Sides is 1, 2, 8, 16, 12, 48** [in Sexagesimal System]

The Fifth Section of *al-Risāla al-muhītiyya* is the shortest section. It is only one page long. Actually the entire section is just a single table consisting of the calculation of \( a_{28} \). With the help of the equation
\[
  a_n = \sqrt{4r^2 - c_n^2}
\]
al-Kāshī calculated the length of each side of an inscribed regular polygon in a unit circle. We note that in a unit circle \( c_0 = r = 1 \), and hence, \( a_0 = \sqrt{3} \). His main objective in this section was to calculate in a unit circle

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the length of each side of the inscribed regular polygon with $3 \times 2^{28}$ sides up to 14 digits after the decimal point in sexagesimal system, to obtain

$$a_{28} = 0; 0, 0, 0, 0, 6, 4, 1, 14, 59, 36, 14, 33, 36, 19, 25.$$ 

Also, al-Kāshī calculated $a_{28}$ in the decimal system using a unit circle to achieve

$$a_{28} = \sqrt{2 - \sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}},$$

where as in the calculation of $c_{28}$ there are 28 nested radicals in the calculation of $a_{28}$. In general, we note that if $r \neq 1$, then $a_0 = r$, $a_0 = r\sqrt{3}$, and consequently, $a_n$ could be presented with $n$ nested radicals as

$$a_n = r\sqrt{2 - \sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}}.$$ 

8. The Sixth Section


In the Sixth Section of *al-Risāla al-muhittīyya*, al-Kāshī has multiplied the value of $a_{28}$ that he calculated in Section Five by $3 \times 2^{28}$ to obtain the perimeter of the inscribed regular polygon with $3 \times 2^{28}$ sides. Similarly, he used the Equation (4.3) to calculate $KL$ (Figure 3), the length of one side of the circumscribed regular polygon to obtain the perimeter of the circumscribed regular polygon with $3 \times 2^{28}$ sides. He has taken the average of these two values as the circumference of the circle which in the sexagesimal system is

$$6, 16; 59, 28, 1, 34, 51, 46, 14, 50.$$ 

Then, he arranged (versified) this value of the circumference in sexagesimal system into a poem. Out of real concern that scribes might err in the recording of these numbers and thus waste the fruit of his labor, and since he also wanted the multiples of the circumference to be readily available for use, he has multiplied the circumference by integers from 1 to 60. He has recorded these multiples in a table in such a way that if one wants, for example 45 times the circumference, one could immediately obtain it from this table.

Al-Kāshī noted that if one takes the radius of the circle as the unit (instead of 60), the value of the circumference still will be the same, but one has to reduce all of its marātib once. This amounts to shifting the decimal point one digit to the left. Hence, according to al-Kāshī the value of $2\pi$ in sexagesimal system would be
$2\pi = 6; 16, 59, 28, 1, 34, 51, 46, 14, 50,$

where all 9 digits after the decimal point are *daqiq* (exact). We note that this value of $2\pi$ surpassed the practical needs of astronomy at that time: a remarkable achievement!

9. THE SEVENTH SECTION

On Ignoring [Rounding-off] the Last Digits of the Preceding Calculations in Additive and Subtractive Fractions

In the Seventh Section of *al-Risāla al-muhītiyya*, again al-Kāshī discussed in detail that ignoring (rounding off) the last digits of sexagesimal fractions in calculating the value of $2\pi$ will not cause any error in the final result of the calculation. For emphasis, at the conclusion of this section yet again, he wrote, that whatever he ignored in the last digits of the sexagesimal fractions in the calculation of $2\pi$, its value will not reach $\frac{1}{540}$ in the calculation of the circumference.

10. THE EIGHTH SECTION

On Conversion of the Size of the Circumference into Indian [Arabic] Numerals Assuming that the Half of the Diameter is One

In the Eighth Section of *al-Risāla al-muhītiyya*, al-Kāshī calculated the circumference of a unit circle, namely the value of $2\pi$ in the decimal system. He used decimal fractions very effectively and obtained the value of $2\pi$ as

$$2\pi = 6.2831853071795865$$

where all 16 digits after the decimal point are *daqiq* (exact). This indicates al-Kāshī's extreme care in calculating the value of $2\pi$. Again, as in the case of the sexagesimal system, to protect the accuracy of these digits from the negligence of scribes and for the reader to be able to readily use multiples of $2\pi$, al-Kāshī multiplied $2\pi$ by 1 through 10. Then, he recorded the 10 resulting numbers in a table in such a way that if one wants the value of, for example $9\pi$, one can obtain it instantly from this table. Al-Kāshī was well aware of the importance of his work in calculating the value of $2\pi$. Thus, in order to preserve the results of his calculations, he arranged (versified) the digits of $2\pi$ from left to right in a Persian *bait* (couplet) as well as an Arabic couplet.
11. THE NINTH SECTION
ON THE MANNERS OF CALCULATIONS WITH THE TWO TABLES

In the Ninth Section of *al-Risāla al-muhītiyya*, al-Kāshī explained the way to use the two tables that he obtained, in the seventh and eighth sections, consisting of multiples of $2\pi$ in the sexagesimal system as we do in the decimal system. He has shown how to calculate the circumference of a circle if its diameter is known, and the way to calculate the diameter provided the circumference is given. To clarify his explanation further he presented examples for the reader. One example is finding the circumference of a circle whose diameter is equal to $650844\frac{1}{8}$ *Adru*’ or *Parāsikh*, and the other example is finding the diameter of a circle whose circumference is $650844\frac{1}{8}$ *Adru*’.

12. THE TENTH SECTION
ON DETERMINATION OF THE DIFFERENCE OF WHAT IS IN COMMON USE AND FAMOUS AMONG PEOPLE AND WHAT WE HAVE OBTAINED

In the Tenth Section of *al-Risāla al-muhītiyya*, al-Kāshī compared the value that he himself obtained for $\pi$ with those values that were obtained by his predecessors. He has said that usually the value of $\pi$ is taken as $3\frac{1}{7}$ and he has calculated the difference between his calculation of $\pi$ and $3\frac{1}{7}$. He has shown that if the circumference of a circle with the radius 3600 *Adru*’ is calculated with his value of $\pi$ and $3\frac{1}{7}$, the difference between the two circumferences will reach $9\frac{1}{15}$ *Adru*’. Also, he has shown that if the radius of a circle is $7073\frac{1}{9}$ times the diameter of the Earth, then the difference between the two circumferences obtained by using his value of $\pi$ and $3\frac{1}{7}$ would be more than 177 times the diameter of the Earth. In effect, he has shown that the result of his calculations is much closer to the real value of $\pi$ than $3\frac{1}{7}$.

13. THE CONCLUSION [OF *AL-RISĀLA AL-MUHĪTIYYA*]
ON PROOF OF THE ERRORS OF ABU’L-WAFĀ’ [AL-BŪZJĀNĪ] AND ABŪ RAYḤĀN [AL-BĪRŪNĪ]

In the *al-Khāṣima* (conclusion) of *al-Risāla al-muhītiyya*, al-Kāshī continued his discussion concerning the errors of Abu’l-Wafā’ al-Būzjānī and Abu Rayḥān al-Bīrūnī that he started in the introduction of *al-Risāla al-Muhītiyya*. However, this time he presented proofs of how both al-Būzjānī and al-Bīrūnī erred in the calculation of $\pi$. At the beginning of the conclusion of *al-Risāla al-muhītiyya*, al-Kāshī applied theorems from Book I of *Almagest* to calculate the chord of a one degree arc and the chord of a thirty minute arc. Then he compared the value that he himself obtained for the chord of the thirty minute arc to the value which he attributed to
al-Būzhānī for the chord of a thirty minute arc, and he has shown that the
result of his calculation was more accurate (see Comment 13.1). Finally,
in the second part of the conclusion of al-Risāla al-muhīṭiyya, al-Kāshī has
calculated the chord of a two degree arc. This time he had compared his
result with the value that al-Bīrūnī had obtained for the chord of a two
degree arc. Again, as expected he has shown that his calculation of the
chord of a two degree arc was far more accurate than al-Bīrūnī’s incorrect
value (see Comment 13.2).

Comment 13.1. Al-Kāshī did not mention the work of al-Būzhānī that al-
Kāshī used as his source for pointing out al-Būzhānī’s error. According to
Qurbānī [12, p. 138], al-Kāshī most likely was referring to statements made
by Naṣr al-Dīn al-Tūsī (1201-1274) in his Tāṣr al-Dā’irā which is an
Arabic translation of Archimedes’ Division of the Circle. In the commentary
al-Tūsī stated:

... For example, let us assume that [the point] C is the
center of a circle and the arc AB is equal to \(\frac{1}{120}\) of the
circumference of that circle. We connect [draw] the chord
AB. Based on the mentioned principles\(^{38}\) and al-Būzhānī’s
calculations, the length of the chord AB is equal to 0; 31,
24, 55, 54, 55. Now, if we take the diameter as 120 parts,
then this value would be the chord of half of a degree [arc].
Next, if we take this chord as one side of the inscribed
[regular] polygon with 720 sides, then the perimeter of this
polygon would be 376; 59, 10, 59, ....

This is exactly what al-Kāshī attributed to al-Būzhānī in the introduction
of al-Risāla al-Muhīṭiyya. However, Woepcke [21] has shown that the num-
ber that al-Tūsī has taken as the chord AB is in fact the sine of one half
of one degree and not the chord of one half of one degree which al-Būzhānī
himself obtained in his Kitāb al-Kāmil (“Complete Book”), which is a sim-
plified version of Ptolemy’s Almagest. Therefore, it was al-Kāshī who erred
and not al-Būzhānī.

Comment 13.2. According to Qurbānī [12, p. 138], whatever al-Kāshī
stated in the introduction of al-Risāla al-muhīṭiyya about al-Bīrūnī and his
calculation of the circumference of a circle was obtained from al-Bīrūnī’s
Qānoūn al-Mas’ūdī. In Chapter Five of Book III of Qānoūn al-Mas’ūdī, al-
Bīrūnī recorded the chord of a two degree arc as 0; 2, 5, 39, 43, 36 which as al-
Kāshī pointed out in the introduction of al-Risāla al-muhīṭiyya is incorrect
and is larger than the actual value of the chord of a two degree arc. This
larger value of the chord of a two degree arc which al-Bīrūnī used as a side of
a regular inscribed polygon with 180 sides resulted in a larger perimeter for
this regular inscribed polygon. Also, since he used this value of the chord

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of a two degree arc to obtain the length of one side of a similar regular circumscribed polygon with 180 sides, again he obtained a larger value for the perimeter of the circumscribed polygon than its actual value. He used the perimeters of these inscribed and circumscribed regular polygons in a unit circle to find two approximations of $\pi$ in the sexagesimal system: namely, $3;8,29,35,24$ and $3;8,30,59,10$, respectively. In the decimal system these values are equal to $3.141742$ and $3.141744$ where both of these values are larger than $3.14166$, the known value of $\pi$ by Greeks and Indians at that time.

14. FOOTNOTES

1 In [3] the author was deceived by claims in some literature that the decimal fractions were discovered by al-Kâshî and he too erroneously stated that al-Kâshî was the discoverer of the decimal fractions.

2 That is, if $d$ represents the diameter of a circle, then $3d + \frac{10}{71}d < \text{circumference} < 3d + \frac{1}{7}d$, and hence the ratio of the circumference of a circle to its diameter is less than $3\frac{1}{7}$ and larger than $3\frac{10}{71}$. This is the famous Archimedes' bounds for $\pi$; that is, Proposition 3 of his treatise On the Measurement of the Circle.

3 It was customary to write the integral part of a number in decimal system and the fractional part in sexagesimal system as we see here.

4 Here by a hair al-Kâshî meant a horse hair.

5 Some mathematicians suspect that al-Kâshî may have been motivated by Ptolemy's theorem in stating and proving what we called al-Kâshî's fundamental theorem. This would be an interesting topic for a future investigation.

6 This can be obtained from the similar triangles $BOM$ and $KOL$. Or the similar triangles $BOI$ and $KOT$.

7 Al-Kâshî has taken the diameter of the Earth as 2485 farsîkhi. With this assumption the circumference of the earth would be $2485 \times \pi \approx 7805$ farsîkhi, where he has rounded it to 8000 farsîkhi.

8 We note that $3 \times 2^{28}$ is written in decimal system. This is one of the exceptions, for al-Kâshî performed his calculations in sexagesimal system.

9 That is, he needed to continue his calculations up to 18 sexagesimal digits after the decimal point. In fact he must have continued his calculations up to at least 19 digits after the decimal point in the sexagesimal system, for he used the round-off method.

10 'Amâl (the plural of 'amal) means "works" or "calculations".

11 This number has been recorded incorrectly as 80035168 in both the original manuscript and the manuscript used by Paul Luckey.
12 Marāṭīb is the plural of martaba’ which means position, and here of course it means the position of the decimal point. Thus, the marāṭīb of all digits will be reduced once if the decimal point is moved one digit to the left.

13 Using a modern electronic device to calculate $2\pi$ in the sexagesimal system [20] the ninth digit after the decimal point will be 49 and the tenth digit will be 55. However, as we mentioned before, Al-Kāshī used the rounded off method. Therefore, the ninth digit after the decimal point will be 50 and hence as he claimed, all 9 digits of $2\pi$ are exact.

14 Al-Kāshī used al-zā’id and al-bāqī fractions in the title of the Seventh Section. However, both Qurbani [12, p. 133] and Luckey [10, p. 85], translating from Arabic, used al-nāqīs instead of al-bāqī. Rosenfeld and Hogendijk [15, p. 37] have translated from Persian and used zā’id and nāqīs as “additive” and “subtractive” and stated that they play the role of the modern terms “positive” and “negative”, respectively.

15 That is, the error will be less than $\frac{1}{60^9}$. He used $60^9$ in the denominator because he has calculated $2\pi$ up to 9 sexagesimal digits after the decimal point.

16 Using a modern electronic device to calculate $2\pi$ in decimal system the sixteenth digit after the decimal point will be 4 and the seventeenth digit will be 7. But, since Al-Kāshī used the round-off method his sixteenth digit is 5 and not 4.

17 This approximation of $2\pi$ in 1424 was a remarkable achievement for that time, and it far surpassed all approximations of $\pi$ by all previous mathematicians, astronomers, and other scientists throughout the world including Archimedes, Ptolemy, and Muhammad al-Khwarizmi. After 186 years, Ludolf van Ceulen was the first to improve Al-Kāshī’s result by finding the value of $\pi$ correct to 35 decimal places in 1610.

18 That is, principles that are based on Ptolemy’s Almagest for calculating the length of the chord of an arc of a circle, or other similar principles used by astronomers.

15. ACKNOWLEDGEMENT

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Plate A: First page of al-Risāla al-muhādiya
Central Library of the Aslāne Quṣṣe Ṭubzawī
Plate B: Last page of al-Risāla al-muhīṭīyya
Central Library of the Āstāne Qudse Razawi

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REFERENCES


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