كتاب المفروضات لأقاطن

BOOK OF ASSUMPTIONS BY AQĀṬUN

text-critical edition

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor in de Wiskunde en Natuurwetenschappen aan de Universiteit van Amsterdam, op gezag van de Rector Magnificus Dr. G. den Boef, hoogleraar in de faculteit der Wiskunde en Natuurwetenschappen, in het openbaar te verdedigen in de aula der Universiteit (tijdelijk in de Lutherse Kerk, ingang Singel 411, hoek Spui) op woensdag 16 februari 1977 des namiddags te 15 uur precies

door

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Aan mijn ouders

Aan Albrecht

Acknowledgements.

In finishing this thesis I would like to thank all those who, in a direct or indirect way, have helped me to arrive at this point.
In the first place I want to thank my parents who gave me the opportunity to visit the university and, through the lively intellectual atmosphere at home, stimulated my scientific interests.
I am grateful to my teachers at the university of Amsterdam. In particular, I wish to mention professors E. M. Bruins, J. de Groot, A. Heyting, G. F. Pijper and J. Popken whose lectures impressed and influenced me so much. Prof. Bruins through his animating lectures on the history of mathematics aroused my interest in this field. He directed and guided me at many points, not the least by advising me to learn Arabic; I am much indebted to him for his help. To Prof. Pijper I am grateful for teaching me Arabic.
At Harvard University I had the privilege to attend the erudite lectures of Prof. J. Murdoch on the history of science. To them I owe a very substantial part of my education in this field. Many other scholars from whom I learned through lectures, discussions, or papers I can only thank in general.
Dr. H. Hermelink I wish to thank for fruitful collaboration through many years. I have particularly deep feelings of gratitude towards Prof. M. Schramm who was always willing to discuss with me the problems of my work and who answered many of my questions; moreover, he let me freely use the extensive resources of his library.
During the preparation of this thesis I received valuable help from Prof. M. Hamidullah, Paris, Prof. A. Hanna, Beirut and Dr. D. Goussous Heidelberg, especially in unscrambling the marginal notes of the Arabic manuscript. I am also much indebted to Prof. F. Sezgin who obtained for me in Istanbul printable photographs of the Arabic text.
Jane Vogt was so kind to read through the whole text and correct the language for which I can not thank her enough.
For advice and help in having this thesis printed, I want to thank Dr. K. Peters and Miss M. Doebl from Springer Verlag, Heidelberg.
Finally and forever, I thank Albrecht who encouraged me, who never complained about unmade beds or uncooked dinners; without his moral support I would never have finished this thesis.

Heidelberg, December 1976.

Contents.

Transliteration.

For the transliteration from Arabic into English the rules are used, laid down in Bulletin 49, November 1958, issued by the Cataloging Service of the Library of Congress.

ا - ʾ, ā	ر - r	ف - f
ب - b	ز - z	ق - q
ت - t	س - s	ك - k
ث - th	ش - sh	ل - l
ج - j	ص - ṣ	م - m
ح - ḥ	ض - ḍ	ن - n
خ - kh	ط - ṭ	ه - h
د - d	ظ - ẓ	و - w, ū
ذ - dh	ع - ^{c}	ى - y, ī
	غ - gh	

Fatḥa is represented by a , kasra by i , and ḍamma by u .

This is the same system as used by George Sarton in his Introduction to the History of Science [I, 46/47].

For the letters in the drawings I followed the recommandations laid down by H. Hermelink and E. S. Kennedy in "Transcription of Arabic letters in geometric figures" in Journal of the American Oriental Society, vol. 82 (1962) p. 204.

The following abbrviations are used:

A = treatise Aya Sofya 4830,5 (fol. 89v - 102v)
B = treatise Bankipore 2468,29 (fol. 141r - 144v)
E. = Euclid
K. = Kitāb (∼book), in Arabic titles
b. = ibn (son), in Arabic names.

Introduction and Summary.

For a better understanding of modern life and culture it is clearly indispensable to have some idea of the stream of events and historical connections that resulted in the present situation. The many factors that played a rôle in this development are partly found in "political" history, such as the history of wars, dynasties, social changes, partly in "cultural" history, such as the history of ideas in art, philosophy, and the sciences. Perhaps as a consequence of the Renaissance, the study of cultural history used to strongly emphasize the period from the 16th./17th. century onwards and the classical period ending with the decay of the Roman Empire. The intermediate "dark Middle Ages" were considered a period of standstill at best and were accordingly neglected. Only in more recent times it was generally accepted that the medieval period deserved equal attention, if only for the sake of continuity. But in that period the main stream flowed outside Europe through the Arab world, i.e. through countries in which Arabic was the scientific language. Although it may be controversial it is the opinion of this author that more scholars should therefore apply themselves to this period to arrive at a better understanding and appreciation of the Arabic contributions to our culture. She hopes that the present work may be accepted as a small contribution to this aim.

The purpose of this work is to edit an Arabic translation of a Greek mathematical treatise, of which the Greek text is not extant. Several years ago H. Hermelink drew my attention to two manuscripts dealing with this Arabic translation and to the many problems connected with them. The two manuscripts in question are, the Istanbul ms. Aya Sofya 4830,5 (fol. 89v - 102r) "Kitāb al-mafrūḍāt lī Aqāṭun" (A) and ms. Bankipore 2468,29 (fol. 141r - 144v) "Kitāb Arshimīdis fī'l-Uṣūl al-handasīya" (B). Treatise A consists of forty-three propositions dealing with plane geometry, whereas treatise B contains only nineteen propositions. All nineteen are also to be found in the first half of A, although not in exactly the same form and in a slightly different order. M. Schramm has apparently been the first to notice a correlation between these two treatises.

The first chapter of the present work describes the two treatises, compares their contents and the way in which they are expressed, and establishes the correlation between the two. We find that the general level of treatise A is higher than that of treatise B. The marginal

notes which appear all through treatise A are also discussed. As they date from the 13th. century A.D. they shed some light on the interests and, to some extent on the way of thinking of Arabic mathematicians during that period.
One of the problems we had to deal with, was to establish the title and a name for the author. Several possibilities are discussed in chapter II: they resulted in the acceptance of Aqāṭun as the author and of "Kitāb al-mafrūḍāt" as the title. This title is translated as "Book of Asumptions".
Chapter III treats the contents of the Book of Assumptions. Special care is taken to point out the connections which apparently exist with other mathematical works, either from a later or from an earlier period. The different influences are gathered in chapter IV. Through the intermediate of Pappus (Book VII), we have the impression that Apollonius' "Loci" has contributed the most. Among later mathematicians only Ibn al-Haytham is found to take up and extend material from the Book of Assumptions. Chapter IV summarizes our conclusions on the treatise.
In order to allow direct comparison the Arabic text is added at the end of the thesis; the American English translation of this text constitutes the contents of chapter V.

Chapter I.
Description and Comparison of the two Manuscripts.

1. Treatise A is part of the Istanbul ms. Aya Sofya 4830, which was discovered and described for the first time by Max Krause. This ms. consists of 235 pages, size 21 × 14 1/2 cm., 23 lines on a page, written in Neskhi. Except for a few rainspots and some of the marginal remarks, it is well readable. It contains various treatises, mostly of mathematical contents by well-known scholars. Some of the treatises are dated A.H. 626, 627 or 628 (= A.D. 1229 - 1231), sometimes with the addition "Damascus" or "Marāgha". On fol. 108v the name of the redactor reads as Muḥammad ibn Sartāq ibn Jawbān from Marāgha. (1) Treatise A, i.e. AS 4830,5 (fol. 89v - 102v) [Krause[I], 439] "Kitāb al-mafrūḍāt lī Aqāṭun", is placed between an anonymous treatise (AS 4830,4, fol. 86v - 89r) on "algebraic geometry"[Krause[I], 522], perhaps by al-Qūhī (see chapter II,1) and a fragment (half of fol. 102v) from Theodosius' Sphaerica. This fragment consists of a proof for a part of prop. III,11, and is different from the proof given in the Greek edition [Heiberg, 154 ff.]. AS 4830,6 (fol. 103r - 108v) is an anonymous treatise on "number theory", starting: Let us assume three proportional numbers ... '. In ms. A fol. 95, smaller than the other pages and in a different handwriting, although also in Neskhi, has wrongly been inserted. It does not belong to the treatise, which continues logically from fol. 94v on fol. 96r. Also the drawing belonging to prop. 23 is mainly situated on fol. 94v but spreads out on fol. 96r. On fol. 95r two stereometrical (astronomical?) figures are drawn. Fol. 95v contains a proposition by Muḥammad ibn Mūsā from the book K. fī'l-kura alladhī aḥāla ʿalāhu (On the Sphere which Turns around itself).

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(1) Nothing is known about the redactor.
Marāgha (مراغة), former capital of Azerbaijan. Probably the town existed already in Roman times. In A.H. 22 it was conquered by the Arabs, and during the following centuries was ruled by different dynasties. The Mongols captured Marāgha finally in A.H. 628, and Hulāgū made it his capital after the conquest of Baghdād in A.H. 656. This was the beginning of a time of prosperity, in which e.g. the famous observatory was built under the direction of Naṣir al-Dīn al-Ṭūsī. [Enzyklopaedie des Islam, III, 284-290, Leiden/Leipzig, 1936]

Treatise B is part of the Bankipore ms. 2468, described in the Catalogue Bankipore vol. XXII, p. 60-92. This ms. consists of 327 pages, size 24 ×15 and 20 × 12 1/2 cm., 31/32 lines on a page, written in Neskhi. It is a collection (majmūᶜah) of 42 treatises on mathematics, including practically applied mathematics and astronomy, by distinguished scholars. Some of the treatises are dated Moṣul A.H. 631/632 (= A.D. 1234/1235) (2). They were published by the Osmania Oriental Publications Bureau (Hyderabad, Deccan).

These treatises were probably collected by Abū'l Rayḥān Muḥ. b. Aḥmad al-Bīrūnī. A note at the end of the collection, indicating the date of composition, reads: "Abū'l-Rayḥān has finished ... with the compilation in Rajab (= the seventh month of the muslim year) of the year 418". Rajab A.H. 418 means December A.D. 1027 or January A.D. 1028. Between the years 1017 and 1030 al-Bīrūnī stayed in different parts of India, probably as a hostage [DSB II, 147-158]. Fifteen of the treatises are written by Abū Naṣr Manṣūr b. ᶜAlī b. ʾIrāq [DSB IX, 83-85], al-Bīrūnī's teacher, and eleven of these especially for al-Bīrūnī e.g. treatise 2468,19 (fol. 103v - 106v), which consists of answers to questions asked by al-Bīrūnī. Four of the treatises are by al-Bīrūnī himself. Also treatise 28 of this collection, i.e. ms. 2468,28 (fol. 134v - 141r), the unique copy of the Archimedean treatise On Touching Circles, was known to al-Bīrūnī. At any rate he uses the last proposition (prop. 15 in the German edition) (3) in his Qānūn al-Masᶜūdī, as a base for the third maqāla, calling it "Archimedes' Theorem". The three proofs by Archimedes are almost exactly reproduced by al-Bīrūnī

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(2) Moṣul (الموصل), city in N. Irāq, on the west bank of the Tigris, opposite ancient Nineveh. Until A.H. 527 (A.D. 1127/1128) the town was ruled by a succession of different dynasties. In that year the Seljuqs were overthrown by the Atābaks, and a period of prosperity began. [Enzyklopaedie des Islam, III, 656-658, Leiden/Leipzig, 1936]

(3) Three translations exist of this treatise: 1. A Russian translation by B. A. Rosenfeld, incorporated in the complete edition of Archimedes, edited by I. N. Veselovsky (Moscow, 1962); 2. J. Vernet and M. A. Catalá, Dos tratados del Arquímedes árabe: Trattado de los círculos tangentes y el libro de los triángulos, (Barcelona, 1972, Publicaciones del seminario de historia de la ciencia de la real academia de buenas letras) (Spanish); 3. Archimedes Opera Mathematica, IV, Über einander berührende Kreise. Aus dem Arabischen von Yvonne Dold-Samplonius, Heinrich Hermelink und Matthias Schramm (Stuttgart, 1975) (German).

in his Istikhrāj al-awtār fī'l-dā'ira (Chords), referring to Archimedes as the author [Dold, 34-39]. It may well be, that al-Bīrūnī collected these treatises during his stay in India, as the necessary literature was not there at his disposal.
The above described treatise ms. 2468,28 precedes our treatise B, i.e. Bankipore 2468,29 (fol. 141r - 144v), Kitāb Arshimīdis fī'l-Uṣūl al-handasīya. They were edited together in Hyderabad, Deccan in 1947. Treatise B is followed by ms. 2468,30 (fol. 144v - 145r), Faṣl fī takhṭīt al-sāʿat al-zamāniyya fī kull qubba aw fī qubba tustaʿmal lahā (On Drawing the Lines of Temporal [that is, unequal] Hours in any Hemisphere or in a Hemisphere Used for that Purpose), a "chapter" on the hemispherical sundial by al-Faḍl b. Ḥātim al-Nayrīzī [DSB X, 5-7].

So the copies of treatise A and treatise B date from the same period, i.e. the first half of the 13.th century A.D.

2. Treatise B seems to be translated in the ninth century A.D. From the title we are informed that "Thābit ibn Qurra translated the treatise from the Greek into the Arabic for Abū'l-Ḥasan ʿAlī b. Yaḥyā, companion to the caliph". Thābit b. Qurra translated many scientific treatises and was an outstanding scholar. He lived from A.H. 221 - 288 (= A.D. 836 - 901) [DSB XIII, 288-295]. Abū'l-Ḥasan ʿAlī b. Yaḥyā, the son of the astronomer Yaḥyā b. Abī Manṣūr [DSB XIV, 537/538], was the companion to the caliph al-Mutawakkil (A.H. 232 - 247), he died A.H. 275 (= A.D. 888/889). Thus the translation must have been made before A.D. 889. In treatise A no information about the translator is given.

Comparing the text of treatise A and treatise B the Arabic is rather similar, but some systematic differences in the translation exist: Throughout treatise A consequently سطح (saṭḥ) is used, where treatise B has مسطح (musaṭṭaḥ), both meaning "product". In treatise A also the word ضرب (ḍarb), beating, multiplication occurs for product. This happens in prop. 6, in the part only found in A (fol. 91r, 12/13) and in prop. 26 (fol. 96v, 5-10). To distinguish the use of "ḍarb", "ḍarb of A and B" is consequently translated as "A times B". In many places the conjuctions in A and B are different, and the words for comparing ratios, triangles, lines etc., i.e. مساو , ك , مثل differ often in the two.
In prop. 4 A(fol. 90r, 22) writes فرض (faraḍa), to assume, where B (fol. 141v, 29) has جعل (jaʿala), to make. This difference only oc-

curs once. In prop. 10 A(fol. 92r, 5) also writes "faraḍa", but here B(fol. 143r, 2) writes علم V (taʿallama), to learn, study. In prop. 9, a special case of prop. 10, where a similar sentence occurs, both treatises write "to study". Another difference in prop. 10 is, A(fol. 92r, 7) uses جنى (janā), to bring about, cause, whereas B(fol. 143r, 4) has خرج (kharaja), which is generally used for "to draw". In A the verb "janā" occurs in several places (props. 10, 23, 27, 34, 36, 40, 41,1, 41,2) for "to bring about (a line parallel to a given line)" → "construct". Here the different translation also gives a somewhat different meaning.

3. Sometimes the reading of the text in A and B differs also. Those differences are only mentioned when they are significant for the meaning or the understanding of the text. In most cases A has the more logical version:
Title and author are different in A(fol. 89v, 1) and B(fol. 141r, 8-10), see Chapter II.
Prop. 1: A(fol. 89v, 7) "the product of GB and BE equals the product of BD and DG" is missing in B(fol. 141r, 16). After the first part of prop. 1 only B(fol. 141r, 22) remarks "And this is what we wanted to prove", which corresponds with the splitting up of prop. 1 into two propositions in treatise B. After the first case of prop. 1 A(fol. 89v, 11-14) contains an explanatory remark, in which the distinction between the possible cases is exposed (cf. translation). This passage is not clear in B(fol. 141r, 22/23), where it reads: "By this argument is completely proved what we have asserted in this construction, if we say". Also the drawings in A represent two cases, whereas in B the difference is not clear, as in both drawings a semicircle has been taken. These differences make the understanding of prop. 1 in treatise B rather vague. Additional precise information is given in B in l. 25: (extend the lines) "at the sides Z and H", and in l. 28: "thus the ratio of EZ to ZY is like the ratio of DH to HY".
Prop. 3: No drawing is made in A, and the room left free for this purpose is not sufficient. B contains superfluous information in (fol. 141v, 18/19): "But angle DAG equals angle EZG", and also in l. 23: "therefore line TZ is perpendicular to line HZ". In A "And this is what we wanted to prove" is missing.
Prop. 4: A(fol. 90r, 21) has "isosceles", where B(fol. 141v, 27) uses "equilateral". In both treatises the drawings correspond to the text; "isosceles" is sufficient and necessary. In the proof both treatises (A fol. 90v, 4; B fol. 142r, 4) write "isosceles". In A "And this is

what we wanted to prove" is again missing.
Prop. 6: A(fol. 90v, 21/22) "which meet at point Z" has been left out in B(fol. 142r, 11), although it forms necessary information. A(fol. 91r, 5) (two angles are equal) "because the base of both is the same arc". This argument does not occur in B. The proposition is on the whole expressed differently in A and B. Also B gives only an indirect proof in the form of the analysis of the proposition. A, on the contrary, gives the same analysis, but followed by a proof.
Prop. 7: A(fol. 91r, 14) "which meet at point Y" has been left out in B(fol. 142r, 24). A(fol. 91v, 1/2) "and the squares of TD and DZ equal the square of TZ, because angle TDZ is right" has been left out in B(fol. 142v, 4). In both cases the left out sentences contain needed information. B(fol. 142r, 27) gives as extra, useful information: "As has been proved in the preceding ⌊proposition]".
Prop. 9: A(fol. 91v, 23/fol. 92r, 1) "Line EH is perpendicular to line TB, and so line BK equals line EH", has wrongly been left out in B (fol. 142v, 30). B(fol. 142v, 30/31) "figure KEZY is a parallelogram", whereas A(fol. 92r, 1/2) writes only "line KY is parallel to line EZ"; the additional information "AG parallel to TE", is an assumption made at the beginning of the proof, A(fol. 91v, 20) = B(fol. 142v, 26).
Prop. 10: Points K and Y are interchanged in the drawings of B and A, here the drawing in B is according to the text. The argument in B is less concise than the argument in A.
Prop. 12: B(fol. 143r, 21/22) "and so line ZH equals line ZG" is superfluous, A(fol. 92v, 5). Also A(fol. 92v, 11) "Therefore the triangles [margin: i.e. triangles DAG and GAH] are similar" is superfluous, B(fol. 143r, 29). "And this is what we wanted to prove"is left out in A.
Prop. 13: A(fol. 92v, 19) "The product of GD and GE is then twice the amount of the product of DG and GB". This extra remark, which facilitates the understanding, does not occur in B(fol. 143v, 15)).
Prop. 14: A(fol. 92v, 23/fol. 93r, 1) ~ B(fol. 143v, 19/20): this passage is in B more complicated, somewhat distorted, perhaps due to an overzealous copyist.
Prop. 15: A(fol. 93r, 10/11) "Angle e equals angle z, and angle h equals angle t. Consequently angle e equals angle h and angle z equals angle t". This passage is missing in B(fol. 144v, 12), although necessary for an understanding of the proof. B gives in the line above the superfluous remark (fol. 144v, 11): "Angles e and z are already equal to a right ⌊angle]". The drawing is correct in treatise A but not in treatise B (cf. chapter III).

Prop. 16: A(fol. 93r, 19/20) "Thus line AD equals line DB" is superfluous and does not occur in B(fol. 143v, 30).
Prop. 19: A(fol. 93v, 21) "with equal sides AB and AG", this necessary information has been left out in B(fol. 144r, 1). B(fol. 144r, 7/8) "because angle AEG is right", has rightly been left out in A(fol. 94r, 5), as this information is already given two lines above. Due to these two differences the proposition is more to the point in A than in B.
Prop. 20: A(fol. 94r, 7) "with equal sides AB and AG", is left out in B(fol. 144r, 9). As in prop. 19 this information is necessary: prop. 20 is an application of prop. 19.
In prop. 22 B makes several explanatory remarks, which can also be left out.
The end of B(fol. 144v, 14-16) reads: "Finished is the book by Archimedes on the Elements of Geometry which consists of twenty propositions. Praise be to Allah, and his benedictions on his prophet Muḥammad and his family." The amount of twenty propositions results from the fact that here the two cases of prop. 1 are numbered as prop. 1 and prop. 2. A(fol. 102v, 9/10) ends with: "Finished are the propositions. Praise be to Allah."

The differences listed in par. 3 could mean that the copyist of A was a more accurate copyist. But the differences in par. 2 point more towards two different translators.

4. It may even be that these translators started with two different Greek texts, as A consists of forty-three propositions, but B only of nineteen. According to the last marginal remarks(fol. 102v, top), however, the redactor has also corrected and enlarged treatise A. The numeration and the sequence of the propositions contained in both treatises is not the same in A and B. The propositions numbered in A as 5, 17, 18, 23-43 occur only in A. Some of these twenty-four propositions may have been inserted by later revisors or by the redactor, but some are clearly of Greek origin. To the latter belong prop. 27, literally the same as Pappos VII,189, prop. 33 and prop. 34, Menelaus' Theorem for the plane, and prop. 36, literally the same as Pappos VII, 145. For the other propositions the origin is not so obvious. Here I took as a criterion the sequence of the letters in the drawings. If in the drawing belonging to a proposition the letter A was placed on the left, the letter B more to the right etc., i.e. the letters were placed according to the system we are used to in our and in Greek geometrical drawings, the proposition was taken to be of Greek origin. If,

on the contrary, the letter A was placed on the right, the letter B more to the left etc., i.e. the drawing was made according to the Arabic custom, the proposition could be either of Arabic origin or of Greek origin, but in an Arabic adaptation. The latter group consists of the propositions 5, 25, 28, 32, 35, 39, 40, 43 (all A numeration). Of these, prop. 25 is most probably Greek, as it is applied in prop. 26, which has the Greek letter-sequence; prop. 39 is probably Greek, as it shows a strong connection with prop. 11 and Pappos VII,119 (cf. chapter III). Prop. 35 might very well be an Arabic try at demonstrating Postulate 5 of Euclid; also prop. 5, prop. 28 and prop. 32 might well be Arabic insertions. Prop. 40 and prop. 43, which are the same proposition with different proofs, pose a problem. As they form the conclusion of the treatise, we should assume at least one of them to be Greek; because prop. 43 is connected with Apollonius' circle (cf. chapter III), it could be Greek. In the drawing of prop. 29 the letters run from top to bottom, which leaves the sequence undecidable.

The following scheme gives the connection between the propositions in A and B, and a survey of the propositions only occurring in A. The numeration in A is the order of the propositions, whereas in B the numbers are written in the margin.

A: prop. 1	=	B: prop. 1 and prop. 2	
prop. 2	=	prop. 3	
prop. 3	=	prop. 4	
prop. 4	=	prop. 5	
prop. 5		---	, might be Arabic
prop. 6 - 10	=	prop. 6 - 10	
prop. 11	=	prop. 12	the order in A is more logical, as
prop. 12	=	prop. 11	A,11 is used in the proof of A,12
prop. 13 - 14	=	prop. 13 - 14	
prop. 15	=	prop. 20	, this proposition is neither logically placed in A, nor in B; in both cases the connection with the context seems lost.
prop. 16	=	prop. 15	
prop. 17		---	, Greek
prop. 18		---	, Greek
prop. 19 - 22	=	prop. 16 - 19	

The remaining propositions belong only to treatise A.

prop. 23 , Greek

prop. 24 , Greek

A:	prop. 25	, Arabic letter sequence, either Arabic or Greek
	prop. 26	, Greek
	prop. 27	, Greek ~ Pappus VII,189
	prop. 28	, might be Arabic
	prop. 29	, undecidable
	prop. 30	, Greek
	prop. 31	, probably Greek
	prop. 32	, might be Arabic
	prop. 33/34	, Greek ~ Menelaus' Theorem
	prop. 35	, might be Arabic
	prop. 36	, Greek ~ Pappus VII,145
	prop. 37	, Greek
	prop. 38	, Greek, perhaps with Arabic proof
	prop. 39	, Arabic letter sequence, but probably Greek
	prop. 40	, Arabic letter sequence
	prop. 41	, Greek
	prop. 42	, Greek
	prop. 43	, Arabic letter sequence, but presumably Greek.

In the following chapters the numbers of the propositions will always be those used in treatise A, unless otherwise stated.

5. An interesting feature of treatise A are the marginal remarks. As often happened, this treatise may have gone through the hands of two men, one who redacted the contents, and another one with a good handwriting, the copyist. It looks as if after the copying the redactor, Muḥ. b. Sartāq al-Marāghī, went again through the treatise adding his remarks, and thereby trying to elucidate the contents of the treatise. This consists of filling in words left out by the copyist, adding explanatory remarks, and giving existence-proofs, e.g. demonstrating why two lines really meet (prop. 37, 43). Especially the last aspect is very interesting as we do not find this in the classical Greek mathematical text-book, i.e. Euclid. The amount and thoroughness of the marginal remarks indicate a still lively interest for this treatise in the 13.th century A.D. among Arabic mathematicians.

Chapter II.
On Author and Title.

1. The author's name.

The treatises give two possibilities: A has Aqāṭun and B Archimedes. Furthermore there is a note in a more recent handwriting on the backside of the first page of A, i.e. fol. 89r. This reads (4): "These propositions are extremely nice. Allah be merciful to the one who obtained it. All of them are correct except the second proposition from it. The fakīr, who needs Allah, has redacted it, Muḥammad ibn Sartāq [ibn Jawbān] (5) from Marāgha (cf. chapter I). .... in making use of it, with the date the second of Dhū-lḥijja, the blessed as to pilgrimage, sixhundert and twenty seven. And he has studied it, and it belongs to the discovery of Waijan b. Wustam al-Qūhī, the geometer, Allah, who is sublime, be merciful to him." In our treatise proposition two is correct. This makes it possible for this remark to belong to the treatise before. This one (AS 4830,4, fol. 86v - 89r) is listed in Krause [I, 522] as an anonymous treatise without a title. It starts: "If two lines have been divided in a similar way, then the ratio of the product of one of the two [lines] and one of its parts to the square of its other part is like the ratio of the product of the other line [ms: qism] and its corresponding part to the square of its other part." In proposition two of this treatise is asked [fol. 86v, 21/22]: "to divide an assumed line such that the ratio of the product of the whole line and one of its parts to the square of its remaining part is an assumed ratio." This is solved by means of a "Neusis" construction. The redactor calls this a "discussion without evidence" and does not accept the solution. Thus the above cited remark refers to the preceding treatise. This means that al-Qūhī is said to be the author of the anonymous treatise (AS 4830,4) but not of the treatise "Book of Assumptions" (AS 4830,5). On the last page [fol. 102v] of the latter treatise two marginal remarks read: "And the amount of propositions by Aqāṭun is 43, counting improvements and additions", and "By Aqāṭun. He has explained this book, he has ascertained it, and he has corrected

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(4) I am much indebted to Prof. M. Hamidullah, who checked this passage for me in the original manuscript while visiting Istanbul. On the fotocopy the text is extremely difficult to read, even in the original text, one line is not certain. (5) This addition is made on fol. 108v, according to Hamidullah.

it, may Allah, who is sublime, compensate him, the slave, the fakīr, who needs Allah, Muḥammad ibn Sartāq al-Marāghī, in the madrasah al-Niẓamiyyah ... the meeting in the year 628. And he has solved it all in Marāgha in one night before this approximately. And a year follows. Thus let his intelligence be admired and the independence of time in it be praised. The End." [cf. translation]. Thus the treatise has been enlarged and corrected by a redactor named Muḥammad ibn Sartāq al-Marāghī. According to the second remark, Muḥ. b. Sartāq lived in the first half of the 13th century A.D. and worked or studied at the madrasah (~ religious college) al-Niẓāmiyyah. From the description of Baghdad by Ibn Jubayr in 581/1185 we learn "There were about thirty madrasas, all housed in excellent buildings with plenty of "wakf" and endowments for their upkeep and for the students expenses. The most famous madrasa was the Niẓāmiyya which was rebuilt in 1110. [Encyclopaedia of Islam, vol. I, 901 (Leiden, 1960)] The fame of the school points towards an intelligent scholar. This impression is confirmed by the contents of the marginal remarks.

Which mathematician is meant with Aqāṭun is not known. Repeating the name Aqātun twice at the end of the treatise might indicate that he was known, but not well-known. The only other place in Arabic literature where I have found the name is in Ibn al-Qifṭī"s Taʾrīkh al-ḥukamā. In his article on Sinān b. Thābit a list of Sinān's work is given (p. 195). Among these is (l. 14) His Improvement of the Book by Aflāṭun on the Elements of Geometry, and He Extended this Book Extremely. In the printed edition Aflāṭun is written as the authors name, but Lippert remarks in a footnote that all codices have Aqāṭun instead. Ibn Abī Uṣaibiʿa (vol. I, 224, l. 17) has a gap instead of this name. Sezgin (V, 291) has Yāqūt's version: Euclid. If Ibn al-Qifṭī's reading is the correct one, the assertion could be made, that treatise B, the Elements of Geometry, is by Aqāṭun, and, that Sinān's extension of this is contained in treatise A. This would mean, however, changing the author's name given in B, and changing in A the given title and the given author's name. As at the moment none of Sinān's mathematical works are extant, no comparison can be made based on contents. So we have no evidence whatsoever to justify our conjecture.

Kapp [II, 91] thinks Aqātun (أقاطن) to be a corrupted version of Aflātūn (افلاطون), the Arabic form for Plato. His arguments are, that according to the Fihrist (p. 245) and Ibn al-Qifṭī (p. 18) Plato has written a treatise on the elements of geometry; and secondly, that Sinān is said to have studied Plato's Republic [DSB XII, 447/448]. For this conjecture also too many names would have to be changed.

Krause [I, 440 footnote], however, believes Aqāṭun as "lectio difficilior" more original. Sezgin, who, contrary to these two, knew both treatise A and treatise B, considers Aqātun a corruption of Archimedes (see further chapter II,3).
Regarding Archimedes: If this treatise is really by Archimedes, it has come to us in a distorted version. Probably additions and omissions occurred; also the quality went down: e.g. proposition two in our treatise is the same as proposition twelve in On Touching Circles, almost certainly by Archimedes. A comparison of the two makes the difference in quality apparent. In On Touching Circles, the proposition is formulated in an abstract way, after which the three possible cases are considered. In our treatise only one case is formulated, and this in a rather slipshod way (cf. chapter III). A plausible explanation for Archimedes' reputed authorship lies in the commercial point of view. A treatise by a famous author, even translated by a famous translator, will find a more general interest.

2. The title "Kitāb fī'l-Uṣūl al-handasīya".
Sezgin [V, 503] mentions:
(1) Euclid, K. al-Uṣūl; (2) Ibn Waḥshīya, K. al-Uṣūl; (3) Menelaus, Uṣūl al-handasa; (4) Ps. Plato, K. Uṣūl al-handasa; (5) Abū Jaʿfar al-Khāzin, K. al-Uṣūl al-handasīya; (6) Archimedes, K. fī'l-Uṣūl al-handasīya; (7) Serenus, K. al-Uṣūl al-handasīya; (8) Ibn al-Haytham, K. fīhi'l-Uṣūl al-handasīya wa-l-ʿadadīya min Kitāb Uqlīdis wa-Abulūniyūs.

On (1): Euclid's Elements are of a different size and scope than the present treatise. This makes a further discussion and comparison of the two superfluous.
On (2): The full title of this treatise is given in Sezgin [IV, 282], as K. al-Uṣūl al-kabīr or Uṣūl al-ḥikma. It is said to be [Plessner, 548/549] a book on alchemy of a strong polemical character, with many quotations.
On (3): According to the Fihrist (p.267) Menelaus composed three books on the elements of geometry which were edited by Thābit ibn Qurra. The work is neither extant in Greek, nor in Arabic. However, some quotations are found [Sezgin V, 163/164]:
Proclus [344/345; ed. Morrow, 269/270] gives for Euclid I, 25 an alternative proof by Menelaus, which is direct, whereas Euclid's proof is by reductio ad absurdum [Heath, E. vol. I, 297/298]. This proof was probably taken either from Menelaus' Book on the Triangle, which is

likewise not extant, or from his Elements of Geometry.
The Banū Mūsā demonstrate in their Liber trium fratrum de geometria [Clagett, 334-341] the proposition, "how two quantities are placed between two quantities, so that the four quantities are in continued proportion", an operation for extracting the side of a cube. For this they use "the method of one of the ancients called Menelaus, who wrote a Book of Geometry". This turns out to be Archytas" solution.
In his Istikhrāj al-awtār (p.49) al-Bīrūnī cites Menelaus' Elements of Geometry and Thābit b. Qurra's commentary to it. He relates how Menelaus in the second proposition of the third book wanted to draw in an assumed half-circle a bent line equal to an assumed line.
From this gathered information there is no reason to assume that our treatise should have Menelaus as its author.
On (4): This lost treatise is listed by Ibn al-Nadīm (p. 246) and by Ibn al-Qifṭī (p. 18). It was translated into Arabic by Qusṭā b. Lūqā. Its influence on Arabic geometry is not yet known. As mentioned before, it has been discussed whether Aqāṭun (آقاطن) could be a distortion of Aflāṭūn (افلاطون). I think the differences in the writing considerable, but as long as the contents of Plato's treatise are not known, this remains an open question.
On (5): Abū Naṣr b. ʿIrāq cites in his Taṣḥiḥ zīj al-ṣafāʾiḥ this work by Abū Jaʿfar al-Khāzin, in which Menelaus is critisized. However, the probability for our author to have been an Arabic mathematician is very small: Treatise B has added to the title that the work was translated from Greek into Arabic by Thābit b. Qurra. Also, the propositions 8 - 10 are quoted by Ibn al-Haytham (see chapter IV) and attributed by him to the geometers of antiquity (literally: the ones being in front - al-mutaqaddimūn).
On (6): This is our treatise B.
On (7): Serenus' treatise is not extant. In his Istikhrāj al-awtār (p. 4) al-Bīrūnī treats the theorem known as Archimedes' Premise, and gives among the proofs three by Archimedes from the Book of the Circle [Archimedes IV, 54-60]. In the Hyderabad edition of al-Bīrūnī's work also the authorship of Sārīnūs is added to these three proofs (p. 7, 18, 20), the first two times with the title of his work Elements of Geometry. The Hyderabad edition is based on the manuscript in Bankipore. Other copies of the manuscript are found [Sezgin V, 381] in Leiden, Istanbul and Cairo. Suter's translation Das Buch der Auffindung der Sehnen im Kreise uses the Leiden ms. As there is no reference to Sārīnūs authorship in this translation, a comparison with the available mss. was made. The Leiden ms. (Or. 513 (5)) and the Istan-

bul ms. (Murat Molla 1396/4) do not mention Sārīnūs (6). These two mss. seem to have about the same text, which is shorter than the text of the Hyderabad edition. In the case of these three proofs the text was exactly the same. As the Hyderabad edition of the Istikhrāj al-awtār contains many insertions (7), the name Sārīnūs might be an addition. Until this name is also found in another copy of the treatise, we should regard it with scepticism. Schramm thinks, however, that both versions are by al-Bīrūnī himself.

In the third maqāla of his Qānūn al-Masʿūdī (p. 300) al-Bīrūnī proves and uses a proposition by Sārīnūs, which he also uses in maqāla 10 (p. 1283). The name of the work is not given. The proposition reads: "In a half-circle the ratio of a greater arc to a smaller arc is greater than the ratio of the chords belonging to them". The only propositions in our treatise dealing with inequalities are the propositions 24 - 26 (see chapter III). There is no direct connection between these propositions and the one quoted by al-Bīrūnī.

Of the work by Serenus [DSB XII, 313-315] only two treatises are extant in Greek and none in Arabic. On the Section of a Cylinder is written to display that the oblique sections of a cylinder and of a cone are both ellipses. On the Section of a Cone deals mainly with the various triangular sections made by planes passing through the vertex of a cone and either through the axis or not through the axis. Sezgin [V, 186] suggests a connection between the propositions 45 and 57 of this treatise and the above mentioned quotation by al-Bīrūnī.

Based on these two treatises there is no reason to consider Serenus as the author of our treatise. Moreover it is not even sure he really wrote an Elements of Geometry.

On (8): The treatise is not extant. Ibn Abī Uṣaibiʿa gives Ibn al-Haytham's description of it [II, 93, l. 24-27]: "A work, in which I collected the elements of geometry and arithmetic from the works by Euclid and Apollonius; in it I classified the elements, arranged them

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(6) I am much indebted to Drs. J. J. Witkam of the University Library, Leiden and to Prof. M. Hamidullah, Paris, who made the original texts available to me. I did not succeed in obtaining information about the Cairo ms.. (7) Saidan gives a critical analysis as to which pages of the Istikhrāj al-awtār, in the Hyderabad edition can be removed such that the book remains complete. Probably a few lines between the two sections so formed are then missing, but can easily be interpolated. [A.S. Saidan, The Rasā'il of Bīrūnī and Ibn Sinān, in Islamic Culture, 34(1960), 173-175].

and proved them by means of proofs which I took successively from mathematics, from calculus, and from logic; I also transposed the order as found in the treatises by Euclid and Apollonius".
The reasoning that our author was not an Arabic mathematician applies here even more strongly than in (5). Ibn al-Haytham lived more than a century later than Thābit b. Qurra (cf. chapter IV).

3. The title "Kitāb al-mafrūḍāt".
Sezgin [V, 491] lists two works with this title: (1) by Thābit ibn Qurra, and (2) by Ps. Archimedes.

On (1): This treatise is extant [Sezgin V, 271]. One copy is known in the original version by Thābit ibn Qurra, and many exist in the redaction by Naṣīr al-Dīn al-Ṭūsī. In the last form the treatise belongs to al-Ṭūsī's redaction of the Kutub mutawassiṭāt (the intermediate books). This collection was studied after Euclid's _Elements_ and prior to Ptolemy's _Almagest_ (8), and hence was widely spread. The books contained in the collection vary somewhat, and this treatise is not always included. In the opening lines to his redaction of Thābit's _Kitāb al-mafrūḍāt_ Naṣīr al-Dīn mentions that some of the copies have 36 propositions and some 34. In the last case propositions 4 and 23 are missing. To compare this treatise with the one by Aqāṭun I used the original edition (AS 4832,4, fol. 35v - 39r) [Krause I, 453/454] (9), the edition redacted by al-Ṭūsī, i.e. the mss. in Oxford Arch. Seld. A. 45 (fol. 159v - 164v) and Paris B.N. Fonds arabe 2467 (fol. 68v - 72v), and the printed al-Ṭūsī edition (Hyderabad-Dn., 1940). The original version contains 36 propositions (3-9 missing) and is not identical with the version redacted by al-Ṭūsī. It then appeared that the contents of Thābit's treatise differ clearly from our treatise (cf. chapter IV).

On (2): This is our treatise A.
In our treatise the author is called Aqāṭun, but Sezgin [V, 135] thinks Aqāṭun to be a deformation of Archimedes, and this treatise to be identical with the one mentioned by Ibn al-Nadīm in the _Fihrist_

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(8) See M. Steinschneider, Die mittleren Bücher der Araber und ihre Bearbeiter, in Zeitschrift für Mathematik und Physik, 10 (1865), 456-498. (9) I am very grateful to Prof. M. Schramm, Tübingen, who lent me his microfilms both of this manuscript (AS 4832) and of the manuscript AS 4830.

(p. 266). Also Ibn al-Qifṭī and Haji Khalfa list in their articles on Archimedes the Kitab al-mafrūḍāt among his works. The K. fī'l-Uṣūl would then be an extract of the K. al-mafruḍat. Sezgin's reason is that the author starts every problem with "li-nafriḍ". My objection to this reason is that in Thābit b. Qurra's K. al-mafruḍat, as well in the original version as in its redaction by al-Ṭūsī, none of the problems start with "li-nafriḍ", but this has of course not been translated from the Greek. However, the treatise On Touching Circles, supposedly by Archimedes has been translated from the Greek. This treatise gives, in its more rigid structure, first the general proposition and then a practical example. These examples all start with "li-nafriḍ".

Nevertheless I favor the title "Kitāb al-mafrūḍāt" on the following grounds:

1. The contents of the K. al-mafrūḍāt are of a higher and more logical quality than the contents of the K. al-Uṣūl (see chapter I), although the redactor is partly responsible for this. According to the last marginal notes [fol. 102v, top] the present copy of the Kitāb al-mafrūḍāt is an improved and enlarged version.

2. Our treatise seems to be something like Pappus, Book VII, i.e. working out known mathematics. In the next chapter we will see that the author gives proofs for assumptions made, but not proved, in other mathematical works [e.g. props. 8, 22, 27, 42], from which the title K. al-mafrūḍāt could originate.

As for the translation of "Kitāb al-mafrūḍāt", Flügel [Haji Khalfa V, 154] gives Liber datorum sive definitorum. Liber datorum or Data is misleading as this is usually, that is, in the case of Euclid, used for K. al-muʿṭayāt. Dodge (II,636) translates Things determined. I propose Book of Assumptions, which connected with the beginning of the propositions "let us assume", covers the same meaning as "K. al-mafrūḍāt" and "li-nafriḍ".

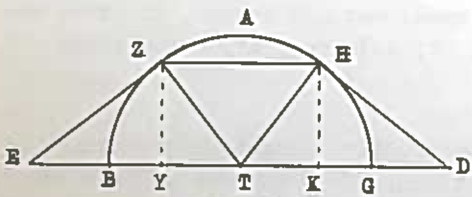
Chapter III.

The Contents of the Book of Asumptions.

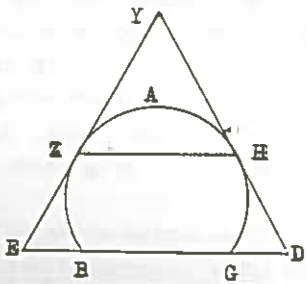
The forty-three propositions of the treatise are of a varying level and of different interest. Some belong together in groups, while others seem to have no connection with the rest of the text. On the whole the work appears to be more the result of someone who studied hard to understand the existing mathematics, and not the development of a new theory.

Together with the propositions, their connections with other mathematical works are treated. The proofs are here only given in a concentrated form, but can be read in full length in the translation.

Prop. 1: If the base BG of the circle-segment ABG be extended in either direction with pieces of equal length, and from the endpoints D and E the tangents EZ and DH be drawn to the circle-segment, then the line ZH connecting the tangential points is parallel to the line ED .



(fig. 1)



(fig. 2)

First the special case of a semi-circle is proved:

T center of circle ABG , join ZT and TH ,

$$GE.BE = BD.DG \rightarrow \left.\begin{array}{l} EZ^2 = DH^2 \\ ZT = HT \\ TE = TD \end{array}\right\} \rightarrow \measuredangle ZTE = \measuredangle HTD$$

$$\rightarrow \text{chord } ZB = \text{chord } HG \rightarrow ZH // ED .$$

Then follows a discussion on what happens if the circle-segment is smaller or greater than a semi-circle, with the possibility of having two parallel tangents in the last case. After this the proposition is proved for a circle-segment greater than a semi-circle:

$$GE.BE = BD.DG \rightarrow EZ^2 = DH^2 \rightarrow EZ = DH$$

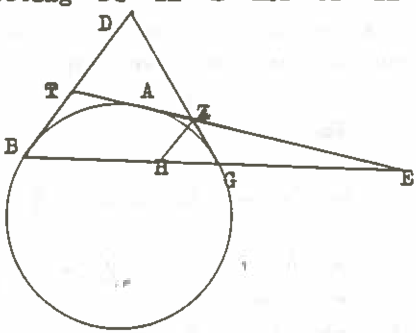
extend EZ and DH till Y $\rightarrow YZ = YH$

$$\rightarrow EZ : ZY = DH : HY \rightarrow HZ // DE .$$

This proposition resembles Pappos VII, 162 [Hultsch II, 914-917], where the following is proved : If the line ZH is parallel to the base BG of a semi-circle and the perpendiculars ZY and HK are drawn from Z and H to the base, then BY = KG . [fig. 1]
The proof also bears a resemblance, starting out as in the first case: T center of circle ABG , join ZT and TH ,

$$\left.\begin{array}{l} ZT^2 = HT^2 \\ ZY^2 = HK^2 \end{array}\right\} \rightarrow YT^2 = TK^2 \rightarrow YT = TK \rightarrow BY = KG . \quad \text{q.e.d.}$$

Prop. 2: Assume the two lines DB and DG are tangents to a circle. Let the chord BG connecting the tangential points be extended till E , and let from E a third tangent be drawn, touching at A and meeting DG in Z and BD in T , then TE : EZ = TA : AZ .



(fig. 3)

Remark: If point E lies in infinity, TZ will be parallel to BG and may touch the circle either at side D or at the opposite side. If it touches at the opposite side, we have a special case of Prop. 1: the base of the circle-segment is concentrated at this one tangential point.

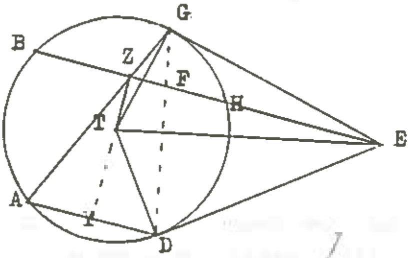
Proof: draw ZH// TB $\rightarrow$ BD : DG = HZ : ZG

$$BD = DG \rightarrow \left.\begin{array}{l} HZ = ZG \\ TE : EZ = TB : ZH \end{array}\right\} \rightarrow TE : EZ = TB : ZG$$

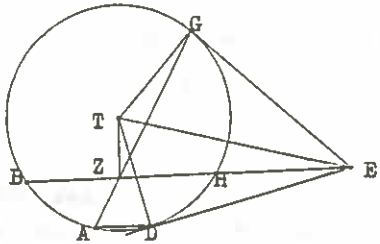
$$BT = TA , \quad GZ = ZA \rightarrow TE : EZ = TA : AZ. \qquad \text{q.e.d.}$$

This proposition occurs as proposition 12 in "On Touching Circles" by Archimedes, where it is more generally formulated and treated. [cf. Chapter II, 1] There the proof starts with the remark, that the proof is different, depending on whether DB and DG are parallel or not parallel to each other. The proofs for both cases are then given. Here, as shown above, only the not parallel case is treated. The letters used in the drawings of the two treatises are not the same, but the proofs are similar. The only difference is that in "On Touching Circles" equal angles are used to prove ZH = ZG .

Prop. 3: Assume the two lines EG and ED are tangents to a circle, while EB cuts it at H and B. Let DA be the chord through D parallel to EB, and let AG meet BH in Z. Then BZ = ZH.



(fig. 4)



(fig. 5)

Proof: (The different configuration makes no difference for the proof. The drawings are taken from treatise B, as they have been left out in treatise A.)

T center of circle, join TD, TE, TG, TZ

$$\left.\begin{array}{l} TD = TG \\ TE \text{ common} \\ ED = EG \end{array}\right\} \rightarrow \measuredangle DTE = \measuredangle GTE \rightarrow \measuredangle DTG = 2\measuredangle ETG$$

$$\measuredangle DTG = 2\measuredangle DAG \rightarrow \left.\begin{array}{l} \measuredangle DAG = \measuredangle ETG \\ \measuredangle DAG = \measuredangle EZG \end{array}\right\} \rightarrow \measuredangle ETG = \measuredangle EZG$$

$$\rightarrow E, G, Z, T \text{ concyclic} \rightarrow \measuredangle EGT = \measuredangle EZT$$

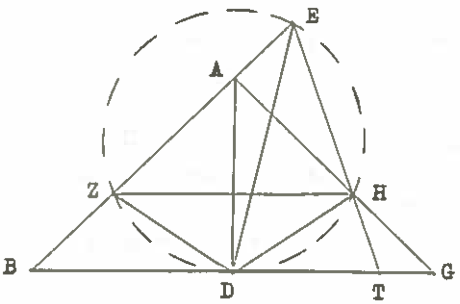
$$\measuredangle EGT = 90^\circ \rightarrow \measuredangle EZT = 90^\circ \rightarrow TZ \perp HB \rightarrow BZ = ZH.$$

q.e.d.

If we replace the last phrase of the proposition "Then BZ = ZH" by "Then, if ZY be drawn perpendicular to AD, it will bisect it at Y", we have Archimedes' Lemmata prop. 10 instead. The propositions are equivalent, and having proved one, the other follows immediately. Although the problems are related, neither the wordings nor the proofs of the two propositions are very similar.

Heath [Archimedes, 312] has remarked that the figure of Lemmata, prop. 10, recalls the figure of Pappus VII, 108, one of the lemmas to the first book of Apollonius' treatise "On Contacts" (περὶ ἐπαφῶν) [Hultsch II, 836]. This lemma is [fig. 4]: "Given a circle and two points Z, F, to draw through Z, F respectively two chords AG, DG having a common extremity G and such that AD is parallel to ZF". As the figure of prop. 3 is very similar to the figure of Lemmata, prop. 10, the same remark might be made regarding prop. 3 and Pappus VII, 108, although the connection in the last case is less obvious.

Prop. 4: Let ABG be an isosceles triangle and AD the perpendicular to the base BG . Assume on AB the points Z , E such that $BD^2 = BE.BZ$. Let ZD be joined, ZH be drawn parallel to BG , and EH be joined, then $\measuredangle EHG = 2 \measuredangle AZD$.



(fig. 6)

Remark: The drawings are neither correct in ms. A nor in ms. B: in both cases the circle through E , Z , D , H is not and can not be drawn. Also: ms. A assumes the triangle ABG to be isosceles, whereas B assumes it to be equilateral. As A does not indicate which sides are assumed equal, equilateral might be the more original version, being replaced in A by isosceles, the latter condition being necessary and sufficient.

Proof: join DE , DH

$EB.BZ = DB^2 \xrightarrow{\text{E. VI, 5}} \measuredangle ZDB = \measuredangle BED$; $\measuredangle ZDB = \measuredangle HZD$ } $\rightarrow \measuredangle HZD = \measuredangle ZED$

$\triangle$ HDZ isosceles $\rightarrow \measuredangle HZD = \measuredangle ZHD \rightarrow \measuredangle ZED = \measuredangle ZHD$

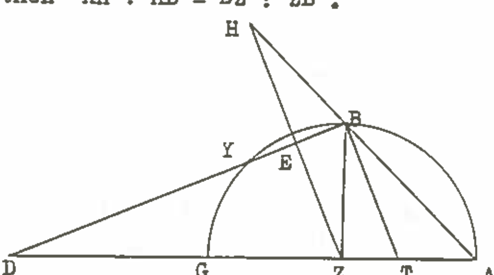
$\rightarrow$ EZDH in a circle.

extend EH to T [on BG] $\rightarrow \measuredangle DHT = \measuredangle EZD$; $\measuredangle DZE = \measuredangle AHD$ } $\rightarrow$

$\rightarrow \measuredangle AHT = 2 \measuredangle AHD$; $\measuredangle AHT = \measuredangle EHG$, $\measuredangle AHD = \measuredangle AZD$ } $\rightarrow \measuredangle EHG = 2 \measuredangle AZD$.

q.e.d.

Prop. 5: If AG be the diameter of a semicircle and B the middle of the arc AG ; let from D , on the extension of AG , DB be joined cutting the circle at Y . If YE = EB and from the center Z, ZE is drawn until it meets the extension of AB in H , then AH : HB = DZ : ZB .



(fig. 7)

Proof: draw $BT \parallel HZ \rightarrow AH : HB = AZ : ZT$

$YE = EB$, Z center of circle $\rightarrow \angle ZED = 90^\circ$ [E. III,3]

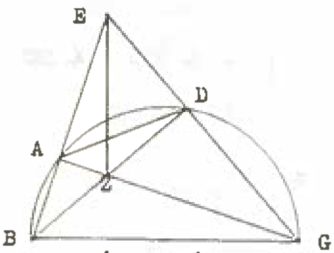
$\angle ZED = \angle TBD \rightarrow \angle TBD = 90^\circ$

$\rightarrow \triangle DZB \sim \triangle BZT \rightarrow DZ : ZB = BZ : ZT$

$AZ = ZB$, $\rightarrow AH : HB = DZ : ZB$. q.e.d.

Written in modern symbols both prop. 4 and prop. 5 represent a hyperbola . Prop. 4 gives $xy = p^2$, by keeping $DB = p$ constant and varying BZ and BE . Prop. 5 gives $rt = -\sqrt{2}$, in which DG and HB vary; the minus-sign results from the Arabic sequence. This way of thinking is, however, too far away from classic plane geometry. But it might be possible that an ancient Greek or Arabic methematician was inspired by prop. 4 to set up prop. 5 by a link of thoughts we do not see now.

Prop. 6: If BG be the diameter of a semicircle, and the chords AG , BD meet in Z , and if BA , GD be drawn meeting in E , then $BD.DZ = GD.DE$.



(fig. 8)

Remark: In both the drawings of A and B the line AD is drawn parallel to the line BG , which is not necessary for the proof.

In B only an indirect proof is given by means of an analysis of the proof. In A the direct proof follows the analysis.

Proof: $\angle GDB = 90^\circ \rightarrow \angle BDE = 90^\circ$; $\angle GAB = 90^\circ \rightarrow \angle ZAE = 90^\circ$ $\Big\} \rightarrow$ ZDEA in a circle

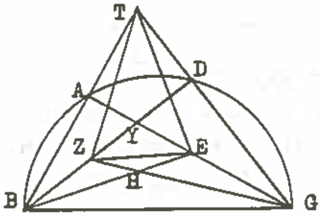
$\angle DBG = \angle DAG$; $\angle DAZ = \angle DEZ$ $\Big\} \rightarrow \angle DEZ = \angle DBG \rightarrow$

$\rightarrow \triangle BDG \sim \triangle EDZ \rightarrow BD.DZ = GD.DE$. q.e.d.

This proposition is in the spirit of Archimedes' Lemmata, e.g. prop. 2, but there is no direct resemblance. Proposition 6 is needed to prove proposition 7 :

Prop. 7: Let BG be the diameter of a semicircle, and the chords AG , BD meet in Y . Take Z on BD and E on AG so that $BD.DY = DZ^2$

and GA.AY = AE^2 , and let EB , ZG be joined meeting in H , then ZH = HE .



(fig. 9)

Remark: As in prop. 6, line AD is drawn parallel to line BG, though proposition and proof are more general.

Proof: draw TZ and TE

BD.DY = GD.DT (prop. 6) $\xrightarrow{\text{assumption}}$ $\left.\begin{array}{l} GD.DT = DZ^2 \\ \measuredangle TDZ = 90^\circ \end{array}\right\} \rightarrow \measuredangle TZH = 90^\circ$

likewise $\measuredangle$ TEH = 90°.

BT.TA = BA.AT + AT^2 = GT.TD = GD.DT + TD^2 $\rightarrow$

$\rightarrow$ $AE^2 + AT^2 = DZ^2 + TD^2$ $\rightarrow$ $TE^2 = TZ^2$ $\rightarrow$ TE = TZ ,

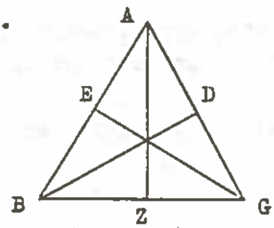
draw EZ , $\rightarrow$ $\measuredangle$ TZE = $\measuredangle$ TEZ

$\rightarrow$ ($\measuredangle$ TZH - $\measuredangle$ TZE) = $\measuredangle$ EZH = ($\measuredangle$ TEH - $\measuredangle$ TEZ) = $\measuredangle$ ZEH

$\rightarrow$ ZH = HE . q.e.d.

The next three propositions belong together. The first step is prop. 8, used in the proof of prop. 9. Although a basic proposition, it is not found in Euclid, so had to be treated here. The second step is prop. 9, a special case of prop. 10. After this, prop. 10 is proved by bringing it back to prop. 9.

Prop. 8: In the equilateral triangle ABG the heights AZ , BD , GE , are of equal length.



(fig. 10)

Proof: in $\triangle$ ABG: AB = AG , AZ $\perp$ BG , $\rightarrow$ BZ = ZG $\Big\}$ $\rightarrow$

in $\triangle$ GAB: GA = GB , GE $\perp$ AB , $\rightarrow$ AE = EB

$\rightarrow$ GZ = AE $\Big\}$ $\rightarrow$ AZ = GE

AG = AG , $\measuredangle$ AGZ = $\measuredangle$ GAE

Likewise is proved GE = BD $\rightarrow$ GE = AZ = BD . q.e.d.

The proposition can be proved in an easier way by comparing areas:

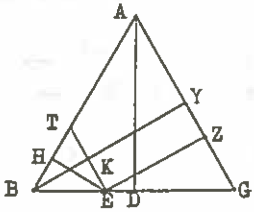
area $\triangle$ ABG = area $\triangle$ GAB = area $\triangle$ BGA

1/2 AZ.BG = 1/2 GE.AB = 1/2 BD.AG

BG = AB = AG $\rightarrow$ AZ = GE = BD .

This asks, however, for a completely different way of thinking. In this treatise "area-thinking" occurs only in prop. 22. But prop. 22 is almost literally the same as Pappos VII, prop. 134. Van Schooten uses "area-thinking" in his proof for prop. 10, in this way he does not need prop. 8 and prop. 9.

Prop. 9: Let ABG be an equilateral triangle, and AD its height. Let from a point E on BD the perpendiculars EZ and EH be drawn on the sides GA and AB , then AD = EZ + EH .



(fig. 11)

Proof: draw ET // AG and BY $\perp$ AG $\rightarrow$

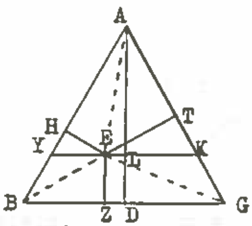
$\rightarrow$ { $\triangle$ TBE equilateral ; BK $\perp$ TE ; EH $\perp$ TB } $\xrightarrow{\text{prop. 8}}$ BK = EH

KY // EZ $\rightarrow$ KY = EZ [on B's version see chapter I,3]

$\rightarrow$ BY = AD = EH + EZ . q.e.d.

By taking for the point E an arbitrary interior point of the equilateral triangle ABG we arrive at proposition 10 :

Prop. 10: Let ABG be an equilateral triangle, and AD its height. Let from an interior point E the perpendiculars to the sides EZ , EH , ET , be drawn, then AD = EZ + EH + ET .



(fig. 12)

Remark: In the drawing of ms. A the points K and Y are interchanged, and the intersecting point L of AD and KY is forgotten.

Proof: EZ // AD
draw YEK // BG } → EZDL parallelogram → LD = EZ
YEK // BG → △ AYK equilateral
AL ⊥ YK , EH ⊥ AY , ET ⊥ AK } —prop. 9→ AL = EH + ET
⇒ AD = EZ + EH + ET . q.e.d.

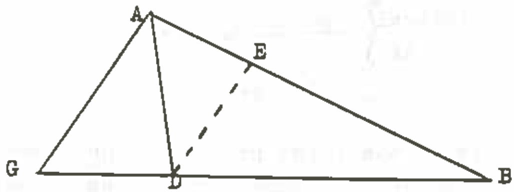
In the Western world this proposition was first proved in the seventeenth century by Frans van Schooten in his Commentary to Descartes' Geometria. Van Schooten, however, was looking for the locus of the points inside an equilateral triangle with the property, that the sum of the perpendiculars from a point on the sides is equal to the height of the triangle. He analyses, that any interior point of the equilateral triangle has this property, and proves this [fig. 12] by dividing triangle ABG into the triangles ABE , BGE , GAE [cf. prop. 8]:

area △ ABG = area △ ABE + area △ BGE + area △ GAE
all bases equal → AD = EH + EZ + ET .

After this, van Schooten explains the relation in the case when point E lies outside the equilateral triangle: the height of the triangle is equal to the sum of the two perpendiculars on the sides of the angle, in which the point lies, minus the perpendicular on the side which subtends this angle. The paragraph is closed with the remark that similar relations can be found for rectilinear figures.
As van Schooten's teacher Jacob Golius was an eminent Arabist who travelled in Spain to collect Arabic treatises, van Schooten might have known about this Arabic proposition. The context and proof, however, point to a genuine achievement. [Hofmann, p. 16/17 ; van Schooten, p. 228-230]

In the Arabic world Ibn al-Haytham around A.D. 1000 begins his treatise Fī khawāṣṣ al-muthallath min jihat al-ʿamūd (On the properties of the triangle with regard to the height) with this proposition, mentioning that it was known to the ancient geometers. He then continues "what we have discovered about this, that we report now" and develops a similar relation for isosceles triangles. He even thinks to have found a relation for any kind of triangle, but here a small error occurs. [see Hermelink]

Prop. 11: If in triangle ABG the line AD is drawn meeting BG at D, such that angle BAD is equal to angle AGD, then $GB.BD = AB^2$.



(fig. 13)

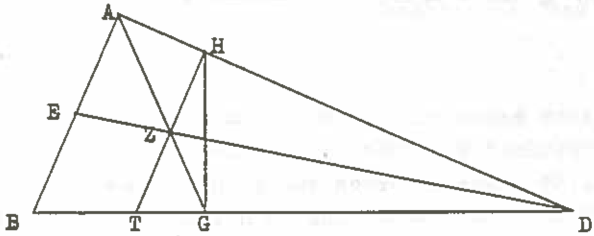
Remark: The drawing in ms. A is not correct.

Proof: $\measuredangle AGB = \measuredangle BAD$
$\measuredangle ABG \in \triangle ABG$ and $\in \triangle ABD$ $\Big\} \rightarrow \measuredangle BAG = \measuredangle BDA$
$\Rightarrow \triangle ABG$ and $\triangle DBA$ equiangular $\xrightarrow{E.VI,4}$
$\rightarrow GB : AB = AB : BD \rightarrow GB.BD = AB^2$. q.e.d.

This proposition, a consequence of Euclid VI,4, is in a practical form the direct converse of Euclid VI,5 [Heath, E. vol. II, p.202]: If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal to which the corresponding sides subtend. Prop. 11 is easy and handy to apply, and this happens in the proofs of the propositions 12, 13, 24, 28, 40, whereby prop. 12 and prop. 13 seem to be treated here mainly for this reason. Euclid VI,5 is used in the proofs of the propositions 4, 18, 24 and 25. Prop. 11 together with E. VI,5 form the "key-proposition" of this treatise, and indicate the interest and the level of the original Greek author.

There exists a close connection between the propositions 11, 39 and Pappos VII,119:
Prop. 39 [fig. 13]: If in triangle ABG the line AD is drawn meeting BG in D, such that angle BAD is equal to angle AGB, then $GB : BD = GB^2 : BA^2 = GA^2 : AD^2$. This proposition also follows from E. VI,4
The combination of prop. 11 and prop. 39 results immediately into Pappos VII,119 [fig. 13]: Given triangle AGD, if the line AB is drawn such that $GA^2 : AD^2 = GB : BD$, then $GB.BD = AB^2$. The condition in Pappos VII,119 is more theoretical, and not as easy to construct as the condition in the propositions 11 and 39. Pappos with the aid of DE // AG proves angle DAB to be equal to angle AGD and concludes from this directly the wanted relation. To the proof is added "the converse is obvious".

Prop. 12: Let in triangle ABG , with AB equal to AG , AD be drawn perpendicular on AB meeting the extension of BG in D . If AB is bisected at E and ED is joined cutting AG at Z , and if through Z HZ is drawn parallel to AB , then $DA.AH = AG^2$.



(fig. 14)

Proof:

$$\left.\begin{array}{l}\triangle ABG \text{ isosceles} \\ ZT \parallel AB\end{array}\right\} \rightarrow ZT = ZG$$

$$AE = EB, \quad EB \parallel HT \rightarrow HZ = ZT$$

$$\rightarrow ZH = ZT = ZG \rightarrow$$

$$\text{join } HG \rightarrow \measuredangle HGT = 90^\circ$$

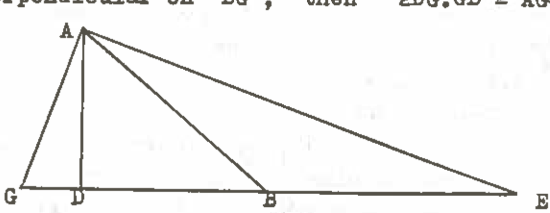
$$\left.\begin{array}{l}\measuredangle ZHG + \measuredangle ZTG = 90^\circ \\ \measuredangle ZTG = \measuredangle ABG\end{array}\right\} \rightarrow \left.\begin{array}{l}\measuredangle ABG + \measuredangle ZHG = 90^\circ \\ \measuredangle ABG + \measuredangle ADB = 90^\circ\end{array}\right\} \rightarrow$$

$$\left.\begin{array}{l}\rightarrow \measuredangle ADB = \measuredangle ZHG \\ \measuredangle ZHG = \measuredangle ZGH\end{array}\right\} \rightarrow \measuredangle ADB = \measuredangle ZGH \xrightarrow{\text{prop. 11}} DA.AH = AG^2 .$$

q.e.d.

In treatise B the order of prop. 11 and prop. 12 is reversed. From the use of prop. 11 in the proof of prop. 12 the sequence in treatise A appears more logical. The use of prop. 11, however, seems no longer clear in later times:in the proof given in treatise A an extra line before the last one is inserted maybe by the redactor, maybe already by an earlier scribe. This superfluous remark reads: "so the triangles are similar" , to which is added in the margin "i.e. the triangles DAG and GAH ".

Prop. 13: If in triangle ABG , with AB equal to AG , AD is drawn perpendicular on BG , then $2DG.GB = AG^2$.



(fig. 15)

Proof: draw $AE \perp AG$, with E on BG

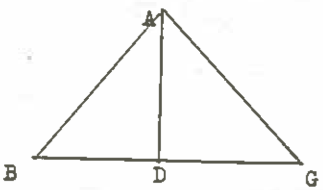
$$\left.\begin{array}{l}\measuredangle\, EAG = 90^{\circ} \\ GB = BA\end{array}\right\} \rightarrow \quad EB = BA = BG \quad \rightarrow \quad EG = 2GB$$

$$\left.\begin{array}{l}\measuredangle\, GAE = 90^{\circ} \\ DA \perp BG\end{array}\right\} [\rightarrow \measuredangle\, GAD = \measuredangle\, AEG] \xrightarrow{\text{prop. 11}} EG.GD = GA^2$$

$$\rightarrow \quad 2DG.GB = GA^2 \; . \qquad \text{q.e.d.}$$

The line within the brackets makes this proposition clearly an application of prop. 11. Without it, this could also be an application of Euclid, Lemma before X,33. In some Greek texts this lemma is found as a porism to E. VI,8 , but according to Heiberg as an interpolation [Heath, E. vol. II, p.211]. By means of Pythagoras' Theorem the proof would have been more direct and shorter, but also more algebraical.

Prop. 14: If in triangle ABG the perpendicular AD on BG is drawn, then $BD^2 - DG^2 = BA^2 - AG^2$.



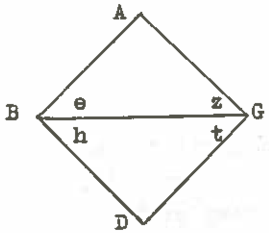
(fig. 16)

Remark: The marginal note in mirror-script does not belong to this proposition.

Proof: $BD^2 - DG^2 = (BD^2 + DA^2) - (AD^2 + DG^2) = BA^2 - AG^2$. q.e.d.

The proof is a direct application of Pythagoras. This proposition is equal to the first part of Pappos VII,120. In Pappos the proposition is considered to be obvious and its proof is better formulated than the proofs given in our treatises. These are similar, but the one in B is expressed in a more complicated way than the one in A .
It is interesting to note that the phrase "the difference of AB^2 and BG^2 " is denoted by "the excess of the square of AB over the square of BG "; namely in the Arabic text as: ziyāda murabbaʿ AB ʿalā murabbaʿ BG [A,fol. 92^v l.22 ff.], while in the Greek text a similar expression is used: ἡ τῶν ἀπὸ ΑΒ ΒΓ ὑπεροχὴ... [Hultsch, Pappos II,854 l.6 ff.]. The word ὑπεροχὴ covers the same meaning as the word زيادة : excess, surplus → the amount by which one (geometrical) quantity surpasses another one [Mugler, 438].

Prop. 15: Let the line AB be equal to the line AG and the line BD equal to the line DG , and let both angle BAG and BDG be right, then $\measuredangle$ ABD = $\measuredangle$ AGD .



(fig. 17)

Remark: The drawing in treatise B is not correct: AB = AG and DB = DG , but $\measuredangle$ GAB = $90^\circ + \varepsilon$ and $\measuredangle$ GDB = $90^\circ - \varepsilon$, so that no square is drawn.
The designation for the angles e, h, z,t comes from the Arabic treatise.

Proof: join BG

$$\left.\begin{array}{l}\measuredangle BAG = 90^\circ \rightarrow \measuredangle e + \measuredangle z = 90^\circ \\ \measuredangle BDG = 90^\circ \rightarrow \measuredangle h + \measuredangle t = 90^\circ\end{array}\right\} \rightarrow \measuredangle e + \measuredangle z = \measuredangle h + \measuredangle t$$

$$\measuredangle e = \measuredangle z \quad , \measuredangle h = \measuredangle t \implies \measuredangle e = \measuredangle h \quad \text{and} \quad \measuredangle z = \measuredangle t$$

$$\rightarrow \measuredangle e + \measuredangle h = \measuredangle z + \measuredangle t \; .$$

q.e.d.

As an exception this proof is given in full detail,to show, that it does too much. The following is already sufficient:

$$\left.\begin{array}{l}\measuredangle e = \measuredangle z \\ \measuredangle h = \measuredangle t\end{array}\right\} \rightarrow \measuredangle e + \measuredangle h = \measuredangle z + \measuredangle t \; .$$

The author examines what happens if two right angles are put together to a quadrilateral, whereby each angle has two equal legs. Without the last condition the result would be a quadrilateral in a circle, according to the converse of Euclid,III,22, with the diagonal BG as the diameter of the circle [E.III,20]. Taking also the second condition imto account, the two triangles at different sides of the diameter have to be isosceles too, which results in the square. The author could have weakened the condition "let both angle BAG and BDG be right" into "let angle BAG be equal to angle BDG ", in which case the proposition is still valid. Then a parallelogram would be obtained with all sides equal, and the proof follows immediately from E,I,34.

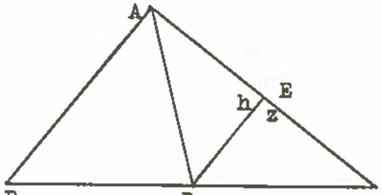
Prop. 16:Let A be the right angle of the rectangular triangle ABG . If BG is bisected at D and AD is joined, then BD = DG = DA .

Proof: draw DE // AB , BD = DG $\rightarrow$ AE = EG

$$\left.\begin{array}{l}AE = EG \\ \measuredangle BAG = 90^\circ \rightarrow \measuredangle h = 90^\circ , \measuredangle z = 90^\circ \\ ED \text{ common}\end{array}\right\} \rightarrow AD = DG \rightarrow$$

$$\rightarrow AD = BD = DG \; .$$

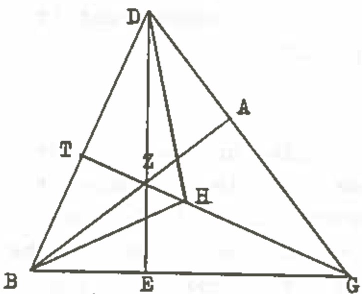
q.e.d.



(fig. 18)

This proposition is a consequence of E.III,31, first part: "In a circle the angle in the semicircle is right". [Heath, E. vol. II, p.61] As D is the middle of BG , it is therefore the center of the circle circumscribed around triangle ABG → AD = BD = GD .

Prop. 17: Let in the rectangular triangle ABG , with angle BAG right, on the extension of AG a point D be taken, from which DE is drawn perpendicular on BG and cutting AB at Z*. Let GZ be joined and on it H be taken, such that $BH^2 = AB.BZ$, and let DH be joined, then $DH^2 = DE.ZD$.



(fig. 19)

Remarks: To the assertion is remarked * (in mirror-script on opposite page): Because of the assumption DE meets AB between points A and B.
The redactor has inserted by hand line AE. He needs this line in his proof [margin 93v] that line GZT is perpendicular to line DB. This is once proved by means of EGZ TDZ, and a second time by means of AGZ TBZ.

Proof: draw DB , extend GZ to T :

BA , DE perpendiculars in △ DBG →

→ GT perpendicular [margin:

[prop. 14] → $\left\{\begin{array}{l} DH^2 - HB^2 = DT^2 - TB^2 \\ DT^2 - TB^2 = DZ^2 - ZB^2 \end{array}\right\}$ $DH^2 + BZ^2 \overset{*}{=} HB^2 + DZ^2$

∡ A = 90° , ∡ E = 90° → E , B , A , D in circle

→ AZ.ZB = DZ.ZE →

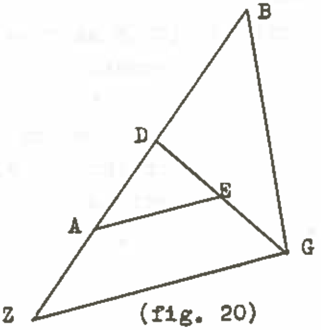
$BZ.(AZ + ZB) + DH^2 = DZ(DZ + ZE) + HB^2$

$AB.BZ = BH^2$

→ $DZ.ED = DH^2$. q.e.d.

* margin fol. 93v left: Because the sum of the two extremities in this numerical relation is the same as the sum of the two mediums.

Prop. 18: Let the lines AB and BG meet at B, and on AB the point D be taken, such that $AB^2 = AD^2 + BG^2$. Let DG be joined and bisected at E, and AE be joined, then $\measuredangle DAE = \measuredangle DGB$.



(fig. 20)

Remark: The drawing in the treatise is not correct: $\measuredangle BGD \neq \measuredangle DGB$. In the treatise is written " ZD > DB ", this has to be " AB > DA ", as this is actually used in the proof.

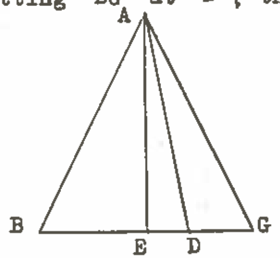
Proof: extend BA to Z with ZA = AD, AB > DA, join ZG

$$\left.\begin{array}{l}(AB + AD)(AB - AD) + AD^2 = AB^2 \\ AB^2 = AD^2 + BG^2\end{array}\right\} \rightarrow \quad ZB.BD = BG^2$$

$$\left.\begin{array}{l}\xrightarrow{E.VI,5} \measuredangle BGD = \measuredangle BZG \\ AE \,//\, ZG \rightarrow \measuredangle BZG = \measuredangle DAE\end{array}\right\} \rightarrow \measuredangle DAE = \measuredangle DGB \,. \qquad \text{q.e.d.}$$

This proposition is in the same spirit as the propositions 11 - 13.

Prop. 19: If in the isosceles triangle ABG, with AB equal to AG, an arbitrary line AD is drawn, cutting BG at D, then $BD.DG + DA^2 = AG^2$.



(fig. 21)

Remark: In treatise A $\measuredangle BAG > 90^\circ$, in treatise B $\measuredangle BAG < 90^\circ$.

Proof: draw $AE \perp BG$

$$\left.\begin{array}{l}BE = EG \\ BD \neq DG\end{array}\right\} \rightarrow \quad BD.DG + ED^2 = EG^2$$

$$\underline{\qquad\qquad + AE^2 \quad + AE^2}$$

$$BD.DG + AE^2 + ED^2 = AE^2 + EG^2$$

$$\rightarrow \quad BD.DG + DA^2 = AG^2 \,. \qquad \text{q.e.d.}$$

This proposition is closely connected with the group of propositions 11 - 14 : As we have seen, prop. 11 is related to Pappos VII,119 and prop. 14 to Pappos VII,120. Pappos VII,119 and Pappos VII,120 belong to the eight lemmata given by Pappos to facilitate the study of the two Books by Apollonius "On Plane Loci" (τόποι ἐπίπεδοι). Ver Eecke (Pappos II,p. 669) treats in a footnote a remark by M.Chasles from the Aperçu Historique, which reads that these eight lemmata can all be considered as consequences of the second of the general theorems by Stewart: Given three points A , C , B on a straight line and another point D outside or in the direction of this straight line, then $DA^2.BC + DB^2.AC - DC^2.AB = AB.AC.BC$ (Ver Eecke writes wrongly DB^2 instead of DC^2). Here the four points are B , D , G on a straight line and A outside:

$$AB^2.DG + AG^2.BD - AD^2.BG = BG.BD.DG$$

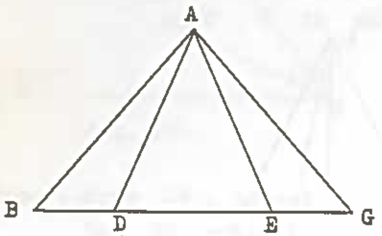
as here $AB = AG$

$$AG^2(DG + BD) = AD^2.BG + BG.BD.DG \longrightarrow AG^2 = AD^2 + BD.DG \ .$$

So the same relation is valid in prop. 19, which means there is a strong connection between prop. 19 and prop. 11 - 14.

The next proposition is a direct application of prop. 19, which seems to have been the reason to take it into this treatise.

Prop. 20: If in the isosceles triangle ABG , with AB equal to AG , the lines AE and AD are drawn, meeting BG at E and D such that $BD.DG : DA^2 = GE.EB : EA^2$, then $DA = AE$.



(fig. 22)

Remark: In treatise B the points D and E in the drawing are interchanged. Because of the symmetry this does not make any difference to the proof.

Proof: $BD.DG : AD^2 = GE.EB : AE^2$

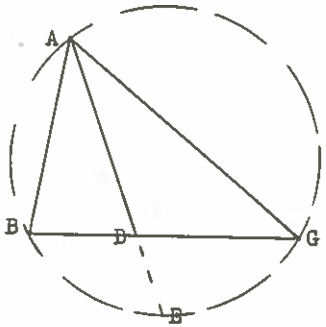
$$\xrightarrow{\text{composition}} (BD.DG + AD^2) : AD^2 = (GE.EB + AE^2) : AE^2$$

$$\xrightarrow{\text{prop. 19}} AB^2 : AD^2 = AG^2 : AE^2$$

$$AB = AG \longrightarrow AD = AE \ .$$

q.e.d.

Prop. 21: Let in triangle ABG the angle BAG be bisected by the line AD , meeting BG at D , then $(BA + AG) : GB = AB : BD$.



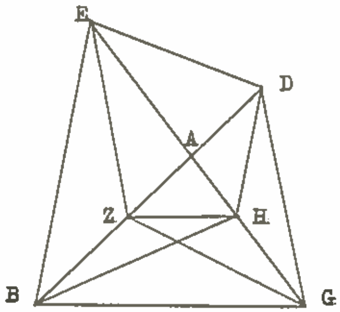
(fig. 23)

Proof: ∡ BAD = ∡ DAG → BA : AG = BD : DG [marg. remark]
alternately AB : BD = AG : GD
whole : whole = single : single [E. V,12]
→ (BA + AG) : GB = AB : BD . q.e.d.

Marginal remark: This step is based on E. VI,3. In the margin a later commentator, probably even later than our redactor, explains which proposition has been used here. As the writing has somewhat faded out and the text is cut off at the edges, the reference is not clear [cf. C. V]. In prop. 43, where Euclid VI,3 is also applied, a marginal reference to the same proposition in the same handwriting occurs.
The proposition E. V,12 is also quoted by Aristotle in the shortened form "the whole is to the whole what each part is to each part (respectively)" [Heath, E. vol. II,p. 160].

Proposition 21 is in the same spirit as the previous propositions. It also appears as an in-between result in the proof of Euclid's Data, prop. 93 [fig. 23]: If the angle BAG of a triangle ABG be bisected by the straight line AE meeting the circle circumscribed about the triangle in E , and if BE be joined, then (BA + AG) : AE = BG : BE. [Heath, E. vol. II,p. 227; Thaer,p. 62].
In Euclid's demonstration AE is supposed to meet BG in D , then is proved in exactly the same way as in proposition 21, that (BA + AG) : BG = AG : GD .

Prop. 22: Let from triangle ABG AB be extended to D and AG to E , let DH be drawn parallel to EB and EZ parallel to DG, then ZH will be parallel to BG .



(fig. 24)

Remark: In both treatises A and B the line ED is drawn parallel to the line BG, which is superfluous.

The proof consists of applying twice E. I,37 [Triangles which are on the same base and in the same parallels are equal to one another.], and then the partial converse of E. I,37, i.e. E. I,39 [Equal triangles which are on the same base and on the same side are also in the same parallels.], to find the wanted relation:

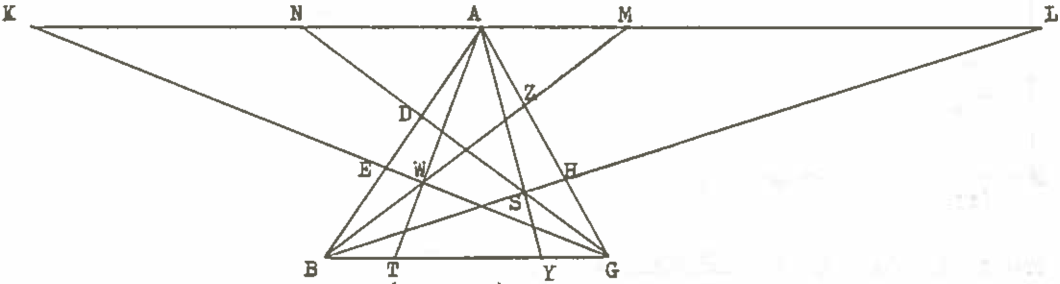
join ZG, HB, ED

DG common base, DG // EZ $\xrightarrow{E.I,37}$ [area] △ DEG = [area] △ DZG; △ DAG common [area] →
→ [area] △ DAE = [area] △ GAZ

EB common base, EB // DH $\xrightarrow{E.I,37}$ [area] △ DEB = [area] △ EHB; △ EAB common [area] →
→ [area] △ DAE = [area] △ ABH; [area] △ DAE = [area] △ AGZ →
→ [area] △ ABH = [area] △ AGZ; △ AZH common [area] →
→ [area] △ BZH = [area] △ HZG; ZH common base $\xrightarrow{E.I,39}$ ZH // BG. q.e.d.

The same constellation occurs in Pappos VII,134: Given the bomisque ABGZHED, if ZH is parallel to BG and DH parallel to BE, then EZ is parallel to DG. [βωμίσκος = altar, is Heron's expression for this geometrical figure; see Ver Eecke, Pappos II,p. 680]. The proof by Pappos is, mutatis mutandis, exactly the same as the one given here.

Prop. 23: Let in triangle ABG AD be equal to BE and AZ equal to GH, let GE, GD, BZ, BH be joined, GE and BZ intersecting in W and GD and BH in S. If AS and AW are joined and extended meeting BG in T and Y, then BT is equal to GY.



(fig. 25)

Proof: draw through A KL // BG , and extend BZ , BH , GD , GE

$$\left.\begin{array}{l} AZ = HG \\ HZ \text{ common} \end{array}\right\} \rightarrow AH = GZ \rightarrow AH : HG = GZ : ZA$$

$AH : HG = AL : GB$

$GZ : ZA = BG : AM$

$\rightarrow AL : GB = BG : AM \rightarrow AL.AM = BG^2$

similarly $KA.AN = BG^2$

$\rightarrow LA.AM = KA.AN \longrightarrow LA : AN = KA : AM$

$KA : AM = GT : TB$

$LA : AN = BY : YG$

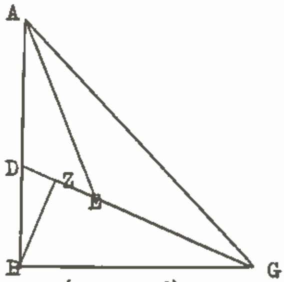
$\rightarrow GT : TB = BY : YG \xrightarrow{\text{composition}} GB : BT = BG : GY \rightarrow$

$\rightarrow BT = GY$. q.e.d.

If the intersection points $AT \cap GD$ and $AI \cap BZ$ are taken instead of the points W and S, the proposition will still be valid.

The next three propositions deal with inequalities. In Euclid I,21 it is proved [Heath, E. vol. I,p.289]: If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle. This proposition does not say anything about the relation between the length of the side of the triangle from which extremities the two lines are constructed, and the length of one of these two lines. To be able to decide this relation, extra conditions are necessary.

Prop. 24: Let angle AGB from the rectangular triangle ABG , with angle ABG right, be bisected by line GD and angle DAE be equal to either angle AGD or angle BGD , then GD > GE .



(fig. 26)

Proof: $\angle DGA = \angle DAE \xrightarrow{\text{prop. 11}} GD.DE = DA^2$

$\left.\begin{array}{l} AD : DB = AG : GB \\ AG > GB \end{array}\right\} \rightarrow AD > DB \rightarrow AD^2 > DB^2$

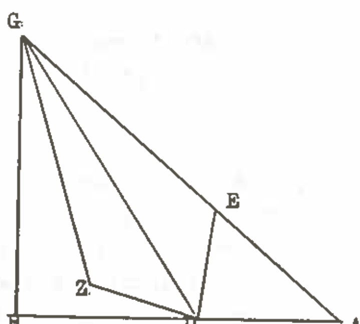
assume $GD.DZ = DB^2$ $[\rightarrow DZ < DE \rightarrow GZ > GE]$

$\left.\begin{array}{l} \text{join } ZB, \xrightarrow{\text{E. VI,5}} \angle ABZ = \angle ZGB \\ \angle ZDB \text{ common to } \triangle BDZ \text{ and } \triangle BGD \end{array}\right\} \rightarrow \angle DBG = \angle DZB$

$\rightarrow \angle DZB = 90^\circ \rightarrow BG > GZ \rightarrow BG \gg GE$. q.e.d.

Prop. 24 is needed for the proof of prop. 25; when the author applies prop. 24, he even remarks "because we have proved it previously" [fol. 96^r, 22].

Prop. 25: Let angle AGB from the rectangular triangle ABG , with angle ABG right, be divided by line GD such that angle BGD is twice angle DGA , then $BG.GA > GD^2$.



(fig. 27)

Proof: assume $AG.GE = GD^2$, join ED $\xrightarrow{\text{E. VI,5}} \angle DAG = \angle EDG$

$\rightarrow \angle GDB = \angle EDG + \angle AGD \rightarrow \angle GDB > \angle EDG$

make $\angle GDZ = \angle GDE$ and $\angle BGZ = \angle DGZ$

$\rightarrow EG = GZ$

$\angle BDG = \angle EGD + \angle EDG$

$\underline{\angle GDZ = \qquad\qquad \angle EDG}$ $-$

$\left.\begin{array}{l} \angle BDZ = \angle EGD \\ \angle EGD = \angle DGZ \end{array}\right\} \rightarrow \angle BDZ = \angle ZGD \xrightarrow{\text{prop. 24}} BG > ZG$

$$GZ = GE \rightarrow BG > GE$$

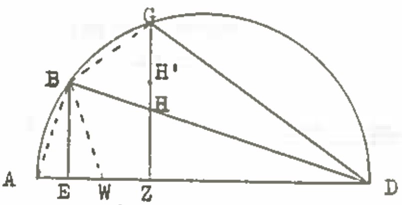
$$\left.\begin{array}{l}\rightarrow BG.AG > AG.GE \\ AG.GE = GD^2\end{array}\right\} \rightarrow BG.GA > GD^2 \quad .$$

q.e.d.

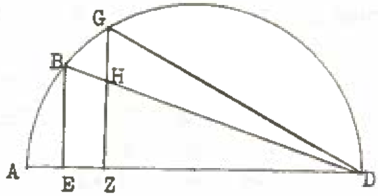
In modern notation we have $\cos 2\alpha = \frac{BG}{GD}$, $\cos 3\alpha = \frac{BG}{GA}$, thus we find the relation $\cos^2 2\alpha > \cos 3\alpha$ $(0 < \alpha < 30^\circ)$

Prop. 25 is used in the proof of prop. 26, in both the version of the treatise and in the "original" version.

Prop. 26: Let in the semicircle with diameter AD arc AB be equal to arc BG , let GD be joined and BE be drawn perpendicular to AD , then $GD < ED$.



(fig. 28)



(fig. 29)

Remark: The drawing in the treatise is not according to the conditions of the proposition, as arc AB ≠ arc BG : G has been taken in the middle of arc AD , and arc AB is equal to half the diameter -- arc AB = 2 arc BG . I suppose "arc AB twice arc BG" to be the condition in the original form of the proposition, as this is what is actually proved with prop. 25 as an auxiliary. As the proposition stands here with the condition "arc AB equal to arc BG", the proof can be much easier with a more exact result, as has been shown by Ptolemy in the first Book of his Syntaxis [fig. 28]:

make $DW = GD$, join AB , BG , BW

$\measuredangle GDB = \measuredangle BDA \rightarrow \triangle BGD \cong \triangle BWD$

$$\left.\begin{array}{l}\rightarrow BG = BW = BA \\ BE \perp AD\end{array}\right\} \rightarrow AE = EW \rightarrow GD + AE = ED \quad .$$

This relation, written as $AE = 1/2(AD - GD)$, together with the relation $AD.AE = AB^2$ are the two basic geometric propositions used by Ptolemy for the preparation of his Table; they are equivalent to $\sin^2\alpha = 1/2\,(1 - \cos 2\alpha)$. [Heath II, 405-409; Ptolemaeus, vol. I, 39/40]

For the "original" version of prop. 26 we have to make one change in the proof, according to the change in the conditions i.e. $\measuredangle ADB = \underline{2}\ \measuredangle BDG$, as a result we also have exactly the required conditions for applying prop. 25. In the version of the treatise we need an extra consideration [fig. 28]:

"it is necessary" : according to prop. 25

$$\left.\begin{array}{l}\triangle DGZ \text{ rectangular} \\ \measuredangle H'DZ = 2\ \measuredangle H'DG\end{array}\right\} \rightarrow \quad ZD.DG > H'D^2$$

$$\measuredangle HDZ = HDG \quad \rightarrow \quad DH < DH' \quad \rightarrow \quad ZD.DG > DH^2 \ .$$

[Prop. 26]: Let in the semicircle with diameter AD arc AB be <u>twice</u> arc BG , let GD be joined and BE be drawn perpendicular to AD , then $GD < ED$.

Proof [fig. 29]: join BD , draw GZ AD $\rightarrow$ intersectionpoint H

$$\rightarrow \quad AD.DZ = BD.DH \qquad \text{[margin: as } \triangle ADB \backsim \triangle ZDH]$$

$$AD.DZ = DG^2 \quad \rightarrow \quad DG^2 = BD.DH$$

$$\left.\begin{array}{l}\measuredangle ADB = \underline{2}\ \measuredangle BDG \\ \triangle DGZ \text{ rectangular}\end{array}\right\} \xrightarrow{\text{"it is necessary" [prop. 25]}} DG.DZ > DH^2$$

$$\left.\begin{array}{l}\rightarrow \ (DG^2 : DG.DZ) < (DG^2 : DH^2)) \\ DG^2 = BD.DH\end{array}\right\} \rightarrow$$

$$(DG^2 : DG.DZ) < (BD.DH : DH^2) \qquad \text{[marginal remark]}$$

$$\Rightarrow \quad DG : DZ < BD : DH$$

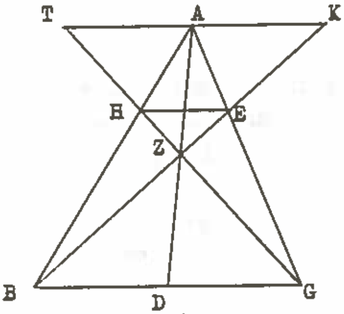
$$BD : DH = DE : DZ \quad \rightarrow \quad DG : DZ < DE : DZ \quad \rightarrow \quad DG < DE \ .$$

q.e.d.

In the last marginal remark is explained that DG^2 and $DG.DZ$ have the same term DG , so that we can divide by it, similarly for the second part of the equation.

This proposition and proof have a more algebraïc outlook, which also shows in the use of the word ضرب ḍarb - multiplication instead of سطح saṭḥ - product.

Prop. 27: If, in triangle ABG, BG be bisected at D and AD be joined, from B an arbitrary line be drawn cutting AD in Z and AG in E , and GZ be joined and extended to H on AB , then, if HE be joined, it is parallel to BG .



(fig. 30)

Proof: construct through A TK // BG , extend BE to K , GH to T

$GD = DB \rightarrow TA = AK \rightarrow GB.TA = GB.AK$

(TA // BG , AB and GT between both $\xrightarrow{[E.VI,4]}$)

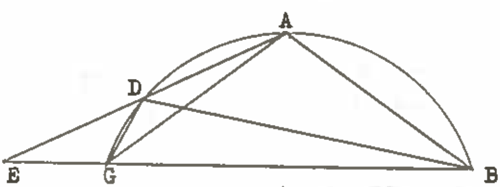
$BG : TA = BH : HA$

$GB : AK = GE : EA$

$\rightarrow BH : HA = GE : EA \xrightarrow{[E.\ VI,2]} HE \,//\, BG$. q.e.d.

This proposition is the same as Pappos, VII,189, which by Pappos is formulated [fig. 30]: Given the figure ABGZHED , if line BD is equal to line DG , then line HE is parallel to line BG [cf. prop. 22]. The proof by Pappos is the same but more compact, especially the line put in brackets is left out by Pappos.

Prop. 28: If in the circlesegment, standing on line BG , arc BG be bisected at A , E be taken on the extension of BG , and AG and AE , cutting the segment in D be joined, then $EA.AD = AG^2$.



(fig. 31)

Remark: In the treatise this proposition is enunciated for a semicircle. As the property, angle BAG equal to 90° , is not used in the proof, the proposition is valid for any circle-segment.

Proof: join AB , DG , DB

arc AB = arc AG $\xrightarrow{[E.\ III,27]} \angle ABG = \angle AGB$

$\angle ADB = \angle DBE + \angle DEB$

$\angle ADB = \angle AGB$, both on arc AB [E. III,21]

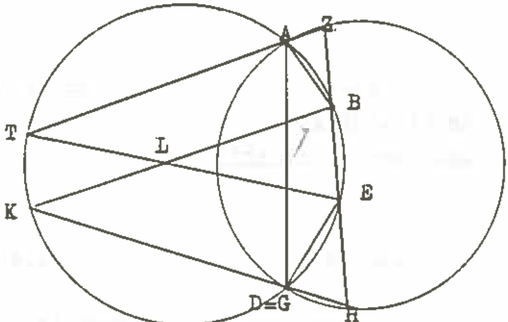
$\rightarrow \angle ABG = \angle DBE + \angle DEB$

$\underline{\angle DBE \quad \angle DBE}$

$\angle ABD = \angle DEB \xrightarrow{prop.\ 11} EA.AD = AB^2$

$AB = AG \rightarrow EA.AD = AG^2$. q.e.d.

Prop. 29: If through two circles intersecting in A and D , an arbitrary line be drawn cutting one circle in Z and H and the other in B and E , and ZA , AB , ED , DH be joined, then $\angle ZAB = \angle EDH$.
Secondly, if HD and ZA be extended to K and T on circle ABED , and BK and ET be joined intersecting in L , then BL = LE.



(fig. 32)

Proof: join AD

$$\text{ADEB in circle} \xrightarrow{\text{E. III,22}} \angle BAD + \angle BED = 180^\circ$$
$$\text{AZHD in circle} \xrightarrow{\text{E. III,22}} \angle ZAD + \angle ZHD = 180^\circ$$
$$\rightarrow \angle ZAD + \angle ZHD = \angle BAD + \angle BED$$
$$\underline{\angle BAD \qquad\qquad \angle BAD} \; -$$
$$\left.\begin{array}{l} \angle ZAB + \angle EHD = \angle BED \\ \angle BED = \angle EHD + \angle EDH \end{array}\right\} \rightarrow$$
$$\rightarrow \angle ZAB + \angle EHD = \angle EHD + \angle EDH \rightarrow \angle ZAB = \angle EDH \ . \qquad \text{q.e.d.}$$

Secondly, proof 1.: $\angle ZAB = \angle EDH$

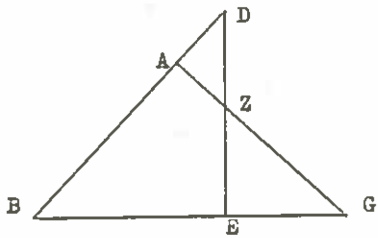
$$\left.\begin{array}{l} \angle ZAB \text{ exterior angle of ABET} \rightarrow \angle ZAB = \angle TEB \\ \angle EDH \text{ exterior angle of BKDE} \rightarrow \angle EDH = \angle KBH \end{array}\right\} \rightarrow$$
$$\rightarrow \angle TEB = \angle EBL \Rightarrow BL = LE \ .$$

proof 2.: $\angle ZAB = \angle EDH \xrightarrow{\text{E. I,13}} \angle BAT = \angle EDK$

$$\text{ABET in circle} \xrightarrow{\text{E. III,22}} \angle BAT + \angle TEB = 180^\circ$$
$$\text{EDKB in circle} \xrightarrow{\text{E. III,22}} \angle EDK + \angle EBK = 180^\circ$$
$$\rightarrow \angle BAT + \angle TEB = \angle EDK + \angle EBK$$
$$\underline{\angle BAT \qquad\qquad = \angle EDK} \; -$$
$$\angle LEB = \angle EBL \rightarrow BL = LE \ . \qquad \text{q.e.d.}$$

The two proofs for the second part of this proposition are very similar. Both use Euclid I,13 [If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles. Heath, E. vol. I,p.275] and Euclid III,22 [The opposite angles of quadrilaterals in circles are equal to two right angles. Heath, E. vol. I, p.51]. Only in proof 1. the use of E. I,13 and E, III,22 is two times combined, whereas these propositions form two separate steps in proof 2..

Prop. 30: If from triangle ABG , with angle BAG right, BA be extended with AD , and from D DE be drawn perpendicular on BG , cutting AG in Z , then $BD.DA = GZ.ZA + ZD^2$.
Conversely: If $BD.DA = GZ.ZA + ZD^2$, then $\measuredangle DEB = 90^o$.



(fig. 33)

Proof: $\left.\begin{array}{l}\measuredangle BAG = 90^o \\ \measuredangle ZEB = 90^o\end{array}\right\} \rightarrow$ B, A, E, Z on circle circumference [E. III,22 conv., not proved by Euclid]

$\rightarrow BD.DA = ED.DZ$

$ED.DZ = EZ.DZ + ZD^2$

$EZ.DZ = GZ.ZA$ [margin: for $\triangle GEZ \backsim \triangle DAZ$]

$\rightarrow BD.DA = GZ.ZA + ZD^2$. q.e.d.

This last line giving the conclusion is missing in the treatise.

Conversely, proof:

$\left.\begin{array}{l}BD.DA = GZ.ZA + ZD^2 \\ ZD^2 = DA^2 + AZ^2\end{array}\right\} \begin{array}{l}BD.DA = GA.ZA + AD^2 \\ \underline{DA^2 \qquad\qquad DA^2} \; - \\ BA.AD = GA.AZ\end{array}$

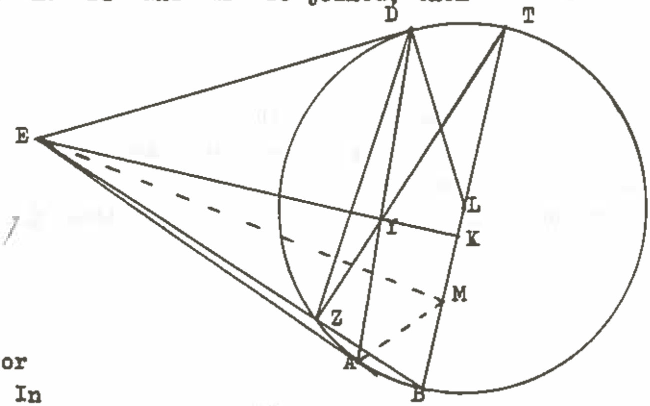
$\rightarrow BA : AG = ZA : AD \rightarrow \triangle BAG \backsim \triangle DAZ$

$\rightarrow \left.\begin{array}{l}\measuredangle BDE = \measuredangle AGB \\ \measuredangle EZG = \measuredangle AZD\end{array}\right\} \rightarrow \measuredangle DAZ = \measuredangle ZEG$

$\measuredangle DAZ = 90^o \rightarrow \measuredangle ZEG = 90^o \rightarrow \measuredangle DEB = 90^o$. q.e.d.

Prop. 30, converse is needed in the proof of prop. 31. Also prop. 19 is again applied.

Prop. 31: If the tangents ED and EA at the circle ABTD be drawn, and an arbitrary line EZ cutting the circle at Z and B ; if AD be joined, on EZ the perpendicular ZT be drawn cutting AD in Y and the circle in T , and EY and ZD be joined, then
$\measuredangle$ DEY = 2 $\measuredangle$ DZT .



(fig. 34)

Remark: Line AM has been drawn in later without the use of a ruler, probably by the redactor. In the second part of the proof the distinction is made between K either falling in L or in between L and B. In the second case K is then called M . This is correct, but rather formalistic.

Proof: extend EZ to B on circle, join BT ,
draw DL ⊥ DE, extend EY to K

$\measuredangle$ BZT = 90° → BT diameter
DL ⊥ DE → DL diameter
} ⇒ L center circle →
→ $\measuredangle$ DLT = 2 $\measuredangle$ DZT

So it is requested that $\measuredangle$ DLT = $\measuredangle$ DEY , or K , D , E , L on a circle circumference, or $\measuredangle$ EDL = $\measuredangle$ EKL = 90° :

△ AED isosceles, EY in it
—prop. 19→ AY.YD + EY^2 = AE^2
AE^2 = BE.EZ
AY.YD = TY.YZ
} ⇒ TY.YZ + EY^2 = BE.EZ
—prop. 30 conv.→ $\measuredangle$ TKY = 90°

Objection: the extension of EY falls either in point L , or in between L and B .

Case 1. extension in L :

$\measuredangle$ ELT = 90°, according to the proof above
$\measuredangle$ ELT = $\measuredangle$ ELD + $\measuredangle$ DEL
$\measuredangle$ ELD $\measuredangle$ ELD   −
$\measuredangle$ DLT = $\measuredangle$ DEL

Case 2. extension not in L , but between L and B in M :
join AM

$90° = \measuredangle LME = \measuredangle LAE \rightarrow$ A , E , L , M on circle circumference

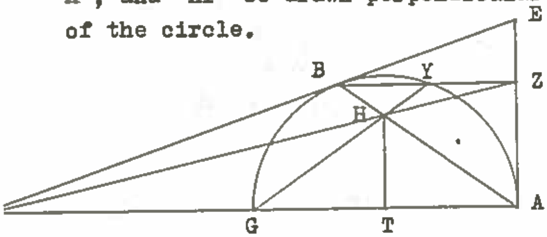
$\left.\begin{array}{l}\measuredangle AED + \measuredangle ALD = 180° \\ \measuredangle ALB + \measuredangle DLT + \measuredangle ALD = 180°\end{array}\right\} \rightarrow$

$\rightarrow \measuredangle AED = \measuredangle ALB + \measuredangle DLT$

$\measuredangle AEM = \measuredangle ALB$ – , on same arc ⌊E. III,21⌋

$\measuredangle DEM = \measuredangle DLT$. q.e.d.

Prop. 32: If at the semicircle with diameter AG the tangents EA and EB be drawn, EB and AG be extended until the intersection point D, BZ be drawn parallel to AD , DZ and AB be joined intersecting in H , and HT be drawn perpendicular to AG , then T is the center of the circle.



(fig. 35)

Remark: GH and HY are joined, Y being the intersection point of circle and BZ , and proved to form a straight line. In the treatise the fact "GHY a straight line" is used in the proof, which is therefore not correct. This error has been repaired in a marginal note by the redactor, who crossed out the erroneous part of the main text. The redactor makes also one remark in the enunciation of the proposition and three remarks in the first part of the proof. Of these the last one is a necessary improvement, written in between the lines as the square of ZA^2 has been forgotten. The first one and the one in the enunciation are formally necessary, i.e. to state "join GH and HY ", and to state "DZ and AB intersect in H ". But the remaining one is superfluous, i.e. to change DA into DB and add an extra sentence, written in ⌊], so that the proof reads DE : EB = DB [: ZB , DE : EB = DA]: ZB . The change, made in the last part of the proof, is not necessary, but makes the proof somewhat more elegant. All this shows that the redactor had difficulties with this proposition as well as worked on its understanding.

Proof: ⌊Y intersectionpoint BZ ∩ circle⌋

join GH and HY [margin⌋

AE , EB tangents $\rightarrow$ AE = EB $\rightarrow$ DE : EB = DE : EA

DE : EA = DB : ZA , DE : EB = DA* : ZB ⌊* marg. remark⌋

$\rightarrow$ DA : BZ = DB : ZA $\xrightarrow{\text{alternate}}$ AD : DB = BZ : ZA $\rightarrow$

$\rightarrow AD^2 : DB^2 = BZ^2 : ZA^{2*}$ [* ^{2} in margin⌋

$BD^2 = AD.DG$, $ZA^2 = BZ.ZY$

$\rightarrow \quad AD^2 : AD.DG = BZ^2 : BZ.ZY \Rightarrow AD : DG = BZ : ZY$

Marginal text:

$\xrightarrow{\text{alternate}} \left.\begin{array}{l} AD : BZ = DG : ZY \\ AD : BZ = DH : HZ \end{array}\right\} \rightarrow$

$\rightarrow \left.\begin{array}{l} DG : YZ = DH : HZ \\ \measuredangle\ GDH = \measuredangle\ YZH \end{array}\right\} \rightarrow \triangle GDH \sim \triangle YZH$

$\rightarrow \quad \measuredangle\ GHD = \measuredangle\ YHZ$

Main text again: $\dfrac{\measuredangle\ YHD \qquad \measuredangle\ YHD}{\measuredangle\ YHG = \measuredangle\ ZHD}$ +

$\measuredangle\ ZHD = 180^o \rightarrow \measuredangle\ YHG = 180^o \rightarrow$ YHG straight line

$\measuredangle\ BYH = \measuredangle\ HGA$

$\measuredangle\ BYH = \measuredangle\ BAG$, on same arc BG [E. III,27]

$\rightarrow \measuredangle\ BAG = \measuredangle\ HGA$

alternative text in margin: BY // AG

$\rightarrow$ arc AY = arc BG

$\rightarrow \measuredangle\ BAG = \measuredangle\ HGA$

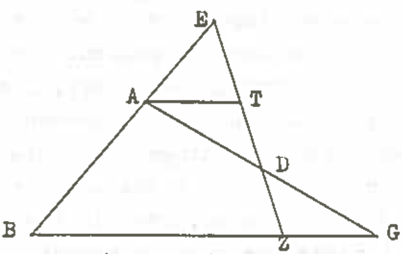
$\rightarrow$ AH = HG

HT $\perp$ AG [assumption] $\rightarrow$ AT = TG

AG diameter circle $\Rightarrow$ T center circle . q.e.d.

Prop. 33: If line AG of triangle ABG be bisected at D , line BA be extended to E , and ED be joined and extended to Z on line BG , then BE : EA = BZ : ZG .

And conversely, if BE : EA = BZ : ZG , then AD = DG .



(fig. 36)

Proof: $\left.\begin{array}{l} \text{draw } AT \ // \ BG \\ AD = DG \end{array}\right\} \rightarrow AT = ZG$

$\triangle$ EBZ : AT // base BZ $\rightarrow$ BE : EA = BZ : AT

$\Rightarrow$ BE : EA = BZ : ZG . q.e.d.

Conversely, proof:

$\left.\begin{array}{l} AT \ // \ ZG \\ BE : EA = BZ : AT \end{array}\right\}$ [$\rightarrow$ AT = ZG] $\rightarrow$ AD = DG . q.e.d.

This proposition is a special case of Menelaus' theorem in plane geometry. This theorem is expressed in the equations [fig. 36]:

$\frac{GZ}{BZ} = \frac{GD}{AD} \cdot \frac{EA}{BE}$ [1] and $\frac{GB}{BZ} = \frac{GA}{AD} \cdot \frac{ED}{EZ}$ [2]

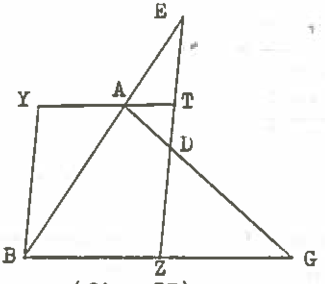
[Braunmuhl, 16]

As we have in prop. 33 the extra condition "AD = DG" , the required relation follows immediately from Menelaus' theorem [1]. It is obvious, that prop. 33, converse: $\frac{BE}{EA} = \frac{BZ}{ZG} \rightarrow AD = DG$, is a direct consequence of Menelaus' theorem [1].

Braunmühl notes that neither Menelaus nor Ptolemy cut a triangle by a transversal, but each studied the relation between the lines AG and ZE on the one hand and the lines EB and GB on the other.

Also in prop. 34, another special case of Menelaus' theorem, a triangle is examined and not the relation between intersecting lines. As in prop. 33 Menelaus' name is not mentioned and the proof is by different means, although applying Menelaus' theorem would be shorter.

Prop. 34: If line BA of triangle ABG be extended with AE , line BG be bisected at Z and line AG be divided at D such that BE : EA = GD : DA , then E , D , Z are collinear.



(fig. 37)

Remark: proof with Menelaus:

$\frac{BE}{EA} = \frac{GD}{DA} \rightarrow 1 = \frac{GD}{DA} \cdot \frac{EA}{BE}$

$BZ = ZG \rightarrow \frac{GZ}{BZ} = 1$

$\rightarrow \frac{GZ}{BZ} = \frac{GD}{DA} \cdot \frac{EA}{BE}$, this is Menelaus [1]

$\Rightarrow$ EDZ a straight line .

Proof, according to the treatise:

construct AT // BG , join ETD

BE : EA = GD : DA
GD : DA = GZ : AT [ZT straight]
GZ = BZ
} → BE : EA = BZ : AT

draw BY // ET
extend TA → Y
} → BE : EA = YT : TA

→ YT = BZ
YT // BZ , constructed
ED // BY , assumed
} ⇒ EZ a straight line . q.e.d.

The proof would be more precise by changing "join ETD" into "join ET and TDZ" and "ED // BY , assumed" into "ET // BY , constructed".

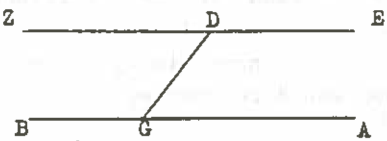
The redactor remarks in the margin at the end of the proof:

$$\left.\begin{array}{l} ZT \parallel BY \\ ED \parallel BY \end{array}\right\} \rightarrow \quad ZT \parallel ED \quad \text{or} \quad ZTED \text{ collinear}$$

$$ZT \text{ meets } ED \text{ in } T \quad \rightarrow \quad EZ \text{ a straight line .}$$

This is actually an application of prop. 35, where the redactor very rightly puts a note in the margin: "This proposition should precede the preceding proposition".

Prop. 35: Let through point D outside line AB the lines ED and DZ be drawn, both parallel to line AB , then EDZ is a straight line.



(fig. 38)

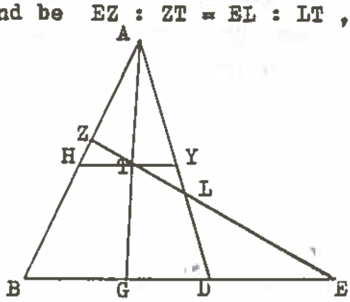
Proof: assume G on AB , join GD

$$\begin{array}{lcl} ED \parallel AB & \xrightarrow{E.\ I,29} & \measuredangle EDG + \measuredangle DGA = 180^{\circ} \\ AB \parallel DZ & \xrightarrow{E.\ I,29} & \underline{\measuredangle DGB + \measuredangle ZDG = 180^{\circ}} \; + \\ & & 4 \text{ angles} \qquad = 360^{\circ} \\ & & \underline{2 \text{ angles [margin: G]} = 180^{\circ}} \; - \\ & & 2 \text{ angles [margin: D]} = 180^{\circ} \end{array}$$

$\rightarrow$ EDZ a straight line . q.e.d.

The theory of parallels, and especially postulate 5, has from ancient times on attracted many mathematicians. Euclid in Book I, def. 23 defines parallel lines as: "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction" [Heath, E.vol.I p. 190]. To this Proclus remarks: "The basic propositions about parallels and the attributes by which they are recognized we shall learn later, but what parallel straight lines are is defined in the words above" [Morrow, 137]. These propositions, i.e. E. I, 27-31, contain the properties "A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles" [E. I,29] and the converse [E. I,27 and 28], moreover "Straight lines parallel to the same straight line are also parallel to one another" [E. I,30] and "Through a given point to draw a straight line parallel to a given straight line" [E. I,31]. From the proof of prop. 35 we can deduct the criterion for parallelism used here, i.e. the property that two interior angles on the same side are equal to two right angles. This means that prop. 35 does not bring something new, but is an immediate consequence of E. def. 23 and E,30. As Proclus remarks, through the same point two parallels to the same straight line cannot be drawn. It seems therefore clear that, as the redactor already noticed, prop. 35 has been inserted to justify the last step in the proof of prop. 34. However, looking at the proof [see above] this connection was no longer apparent.

Prop. 36: Let from point E the lines EZ and ED be drawn cutting the lines AB , AG , AD respectively in Z , T , L and in B , G , D and be EZ : ZT = EL : LT , then EB : BG = ED : DG .



(fig. 39)

Remark: In the proof some lines [i.e. fol. 99^v, 14 (last two words), 15, 16 and 17 (first four words)] have been crossed out, probably by the redactor. They contain the same argument as had already been given in the lines before.

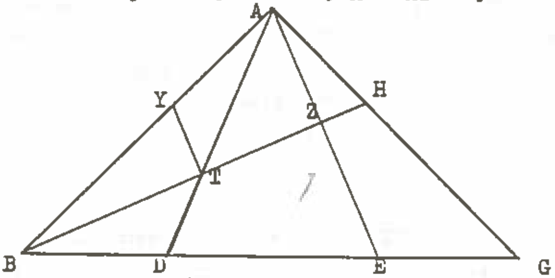
Proof: construct in T HTY // BE → EZ : ZT = EB : HT
EZ : ZT = EL : LT [assumption]
TY // DE → EL : LT = ED : TY } →
→ EB : HT = ED : TY $\xrightarrow{\text{alternate}}$ EB : ED = HT : TY
△ ABD : AG arbitrary, BD // HY → HT : TY = BG : GD
→ EB : ED = BG : GD $\xrightarrow{\text{alternate}}$ EB : BG = ED : DG . q.e.d.

This proposition is identical with Pappus VII,145, a special case of Pappus VII,129. The proofs are also the same, only that in our treatise the details are worked out more elaborately.
The proposition deals with the anharmonic property. Heath [II, 401] thinks it probable that both the anharmonic property and Menelaus' Theorem for the sphere were already included in some earlier text-book than Menelaus' Sphaerica, and that they were known to Hipparchus. The corresponding planar theorems appear in Pappus among his lemmas to Euclid's Porisms, and it is reasonable to infer that they were assumed by Euclid as known. However, the importance of this proposition lies in the fact that it proves the projective invariance of the anharmonic ratio.

Prop. 37: Let in the rectangular triangle ABG , with angle BAG right, the lines AD and AE be drawn, such that angle DAE be equal to angle ABG , and from point B the perpendicular BZ on AE [,cutting AD in T,] be drawn and extended to H on AG , then

$$DA.AT + GB.BD = HB.BT + GA.AH \quad ;$$

and secondly $GB.BD + GA.AH = AB^2$.



(fig. 40)

Remark: The proof of the proposition shows that one circle goes through A , H, T , Y , one through B , Y, T , D and one through G , D , T , H ; so the triangle is divided into three quadrilaterals, every one of them lying in a circle. All three circles go through point T , which yields an interesting configuration.

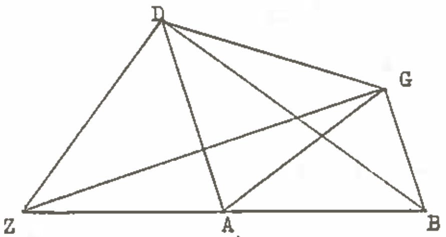
Proof: $\measuredangle$ DAE = $\measuredangle$ ABG
$\measuredangle$ BAD $\quad$ $\measuredangle$ BAD $\quad$ +
$\measuredangle$ BAE = $\measuredangle$ ADG
$\measuredangle$ AZH = 90° $\longrightarrow$ $\measuredangle$ ZAH + $\measuredangle$ AHZ = 90° } $\longrightarrow$
$\measuredangle$ BAH = 90° [assumption]
$\measuredangle$ ZAH + $\measuredangle$ AHZ = $\measuredangle$ BAG
$\measuredangle$ ZAH $\quad$ $\measuredangle$ ZAH $\quad$ −
$\measuredangle$ AHZ = $\measuredangle$ BAE $\quad$ $\longrightarrow$ $\measuredangle$ AHZ = $\measuredangle$ ADG
[margin: $\longrightarrow$ $\measuredangle$ GHT + $\measuredangle$ GDT = 180°]
$\longrightarrow$ H , T , D , G on circle circumference
$\longrightarrow$ DA.AT = GA.AH , HB.BT = GB.BD
$\longrightarrow$ DA.AT + GB.BD = HB.BT + GA.AH .

Secondly, proof:
assume GB.BD = AB.BY and join TY
GB.BD = HB.BT [proved above]
$\longrightarrow$ AB.BY = HB.BT $\longrightarrow$ A , H , Y , T on circle circumference
$\longrightarrow$ $\measuredangle$ BYT = $\measuredangle$ AHT } $\longrightarrow$ $\measuredangle$ BYT = $\measuredangle$ ADG
$\measuredangle$ AHT = $\measuredangle$ ADG [proved above]
$\longrightarrow$ B , Y , D , T on circle circumference
$\longrightarrow$ BA.AY = DA.AT
DA.AT = GA.AH , GB.BD = AB.BY
$\longrightarrow$ GA.AH + GB.BD = AB.BY + AB.AY
$\longrightarrow$ $GA.AH + GB.BD = AB^2$. $\qquad$ q.e.d.

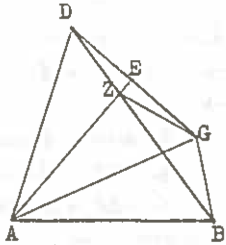
The beginning of the second part of the proof is changed by the redactor. His version starts "draw in T TY perpendicular on BH", this is more intelligent and easier to do than "assume GB.BD = AB.BY and join TY" as written in the main text. But this brings the redactor more difficulties in proving A, H, Y, T on a circle circumference. The easiest would have been to draw in D DY perpendicular on BG, which means A, G, D, Y on a circle circumference, hence the assumption in the proof, GB.BD = AB.BY, follows immediately. The rest of the marginal remark, as far as readable, does not seem shorter or better expressed than the main text.

The next six propositions belong together: Prop. 40 and prop. 43 are the same proposition, but with different proofs. The propositions 38 and 39 are needed in the proof of prop. 40, and the propositions 41 and 42 in the proof of prop. 43.

Prop. 38, 1): If AB = AG = AD and DG, GB, BD be joined, then
$\measuredangle$ GBA + $\measuredangle$ GDB = 90°.
2): Moreover, if AB = AG and $\measuredangle$ GBA + $\measuredangle$ GDB = 90°, then
AB = AG = AD.
3): Also, if AD = AB and $\measuredangle$ GBA + $\measuredangle$ GDB = 90°, then AD = AG = AB.
4): Finally, if AD = AG and $\measuredangle$ GBA + $\measuredangle$ GDB = 90°, then
AD = AG = AB.



(fig. 41)



(fig. 42)

Proof 1) [fig. 41]: extend BA with AZ = AB, join GZ

ZA = GA = BA → $\measuredangle$ ZGB = 90°

→ $\measuredangle$ GZB + $\measuredangle$ GBZ = 90° and Z, G, B on circle circumference

DA = GA = BA → D, G, B, Z on circle circumference

→ $\measuredangle$ BZG = $\measuredangle$ GDB

→ $\measuredangle$ GDB + $\measuredangle$ GBZ = 90°.

Remark: This proof is built up as the proofs for the other assertions of prop. 38, though in this special case the proof could be shorter and more concise:

from the first line follows $ZA = GA = BA = DA$
$\rightarrow$ Z , G , B , D on circle circumference with center A
$\rightarrow \measuredangle GDB + \measuredangle GBZ = 90^\circ$.

Proof 2) [fig. 41]: extend BA with $AZ = AB$, join GZ

$$\left.\begin{array}{l}\rightarrow \measuredangle ZGB = 90^\circ \rightarrow \measuredangle GZB + \measuredangle GBA = 90^\circ \\ \measuredangle GBA + \measuredangle GDB = 90^\circ \quad \text{[assumption]}\end{array}\right\} \rightarrow$$

$\rightarrow \measuredangle GDB = \measuredangle GZB \rightarrow$ G , D , Z , B on circle circumference
$ZA = BA = GA \rightarrow$ A center circle
$\rightarrow AD = AG = AB$. q.e.d.

Proof 3) [fig. 41]: extend BA with $AZ = AB$, join DZ

$$\left.\begin{array}{l}\rightarrow \measuredangle BDZ = 90^\circ \rightarrow \measuredangle DBZ + \measuredangle DZB = 90^\circ \\ \measuredangle GBA + \measuredangle GDB = 90^\circ \quad \text{[assumption]}\end{array}\right\} \rightarrow$$

$$\rightarrow \underbrace{\measuredangle GBA - \measuredangle DBZ}_{\measuredangle BGD} + \measuredangle GDB = \measuredangle GBD + \measuredangle GDB = \underbrace{\measuredangle DZB}_{\measuredangle BGD}$$

$$\overline{\text{angles } \triangle BGD \quad = \measuredangle BGD + \measuredangle DZB}^{\;+}$$

$\rightarrow \measuredangle BGD + \measuredangle DZB = 180^\circ \rightarrow$ B , Z , D , G on circle circumference
$ZA = DA = BA \rightarrow$ A center circle
$\rightarrow AG = AD$. q.e.d.

Proof 4) [fig. 42]: draw $AE \perp DG$, join ZG
$\rightarrow DE = EG$ and $\measuredangle DZE = \measuredangle GZE$

$$\left.\begin{array}{l}\measuredangle DEZ = 90^\circ \rightarrow \measuredangle EDZ + \measuredangle EZD = 90^\circ \\ \measuredangle GBA + \measuredangle GDB = 90^\circ \quad \text{[assumption]}\end{array}\right\} \rightarrow$$

$\rightarrow \measuredangle EZD = \measuredangle GBA \rightarrow \measuredangle GZE = \measuredangle GBA$
$\rightarrow$ Z , G , B , A on circle circumference

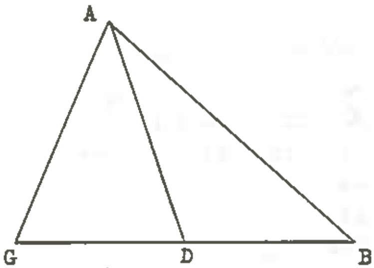
$$\left.\begin{array}{l}\rightarrow \measuredangle AGB = \measuredangle AZB \\ \measuredangle AZB = \measuredangle DZE \\ \measuredangle DZE = \measuredangle GBA\end{array}\right\} \rightarrow \measuredangle AGB = \measuredangle GBA$$

[margin: $\rightarrow AG = BA$] . q.e.d.

Prop. 38 is needed in the proof of prop. 40.

Prop. 39: If in triangle ABG line AD is drawn meeting BG in D such that angle DAG is equal to angle ABG , then
$BG : GD = BG^2 : GA^2 = BA^2 : AD^2$.

$$\text{Proof: } \left.\begin{array}{l}\measuredangle ABG = \measuredangle GAD \quad \text{[assumption]} \\ \measuredangle AGD \text{ common to } \triangle ABG \text{ and } \triangle DAG\end{array}\right\} \rightarrow \measuredangle BAG = \measuredangle ADG$$



(fig. 43)

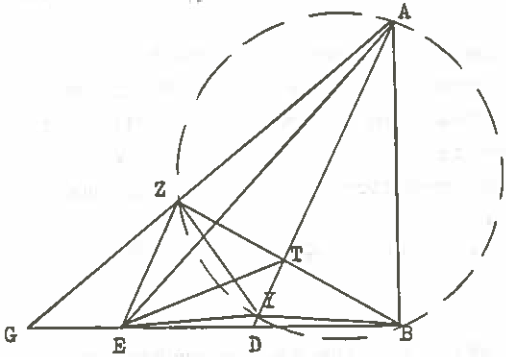
Remark: The drawing in the treatise is not correct.

$\xrightarrow{\text{E. VI,4}}$ BG : AG = AG : GD = AB : AD

$\rightarrow$ BG : GD = BG^2 : AG^2 = AB^2 : AD^2 . q.e.d.

On the connection between prop. 39, Pappus VII,119 and prop. 11 see prop. 11.
Prop. 39 is needed in the proof of prop. 40.

Prop. 40 [= prop. 43]: If in the rectangular triangle ABG , with angle ABG right, angle BAG be bisected by line AD , line AE be drawn at random, in point E line EZ be constructed parallel to line AD , ZB be joined, cutting AD in T , and ET be joined, then
AZ : ZT = AE : ET .



(fig. 44)

Remark: The drawing in the treatise is not correct: In the treatise Y is only assumed on the extension of ATD , and not, as in fig. 44, constructed by means of the circle through A , B , Z . The redactor makes a note on the position of point Y in the margin: "So point Y , either it falls below point D or on it or beyond it". This is correct: when E moves nearer towards G the circle enlarges and Y moves towards D and passes it.

Proof: make AT.TY = ZT.TB [margin: on position of Y , see above] :
join ZY , EY , BY [BY added by redactor]

$\xrightarrow{\text{[E. III,35 conv. + E. III,21]}}$ $\measuredangle$ ZAT = $\measuredangle$ ZBY

$\measuredangle$ ZAT = $\measuredangle$ TAB $\rightarrow$ $\measuredangle$ TAB = $\measuredangle$ ZBY crossed out in treatise:

ZT.TB = AT.TY

$\rightarrow \measuredangle$ BZY = $\measuredangle$ BAT $\Rightarrow$ = $\measuredangle$ YBZ

$\xrightarrow{[\text{prop. } 11]}$ AY.YT = BY^2

(ZT.TB = AT.TY

$\rightarrow \measuredangle$ BAY = $\measuredangle$ Y)

$\measuredangle$ BAY = $\measuredangle$ GAY

$\rightarrow \measuredangle$ GAY = $\measuredangle$ BZY

$\xrightarrow{[\text{prop. } 11]}$ AY.YT = ZY^2 = BY^2

instead of this in margin:

$\left.\begin{array}{l}\measuredangle \text{ TAB} = \measuredangle \text{ TBY} \\ \measuredangle \text{ BYT common}\end{array}\right\} \rightarrow$

AY : YB = BY : YT $\rightarrow$

$\rightarrow$ AY.YT = BY^2

AT : TB = ZT : TY

the 2 angles T are opposite to each other

$\rightarrow \measuredangle$ TZY = $\measuredangle$ BAT = $\measuredangle$ TAZ

$\measuredangle$ TYZ common

$\rightarrow$ AY : YZ = YZ : YT

$\rightarrow$ AY.YT = YZ^2

main text again:

$\rightarrow$ BY = YZ

$\measuredangle$ ABD = 90° $\rightarrow \measuredangle$ ADB + $\measuredangle$ DAB = 90°

$\measuredangle$ DAB = $\measuredangle$ ZBY , $\measuredangle$ ADB = $\measuredangle$ ZED $\rightarrow$

$\left.\begin{array}{l}\rightarrow \measuredangle \text{ ZBY} + \measuredangle \text{ DEZ} = 90^\circ \\ \text{BY} = \text{ZY}\end{array}\right\} \rightarrow$ BY = ZY = YE [margin: by prop. 38, 4)]

$\rightarrow$ AY.TY = YE^2 [$\xrightarrow{\text{E. VI,5}}$ $\measuredangle$ YET = $\measuredangle$ YAE]

$\rightarrow$ AY : YT = AE^2 : ET^2 [margin: by prop. 39]

AY : YT = AZ^2 : ZT^2 [margin: also by prop. 39]

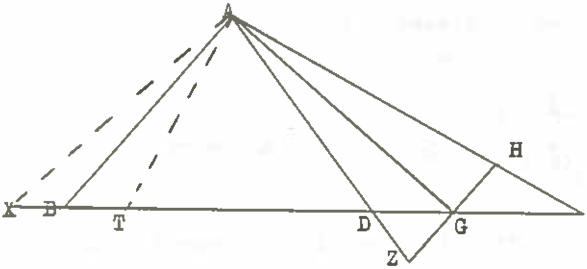
$\rightarrow$ AE : ET = AZ : ZT . q.e.d.

Again with prop. 40 the redactor seems to have worked hard on its understanding. He makes several useful remarks, given in the text. The remark on the position of point Y shows his mathematical insight, it is not clear, however, whether he realized the drawing to be wrong. The part of the text crossed out by the redactor is not so clear, because of some superfluity, put in (), and the many mistakes. But the solution of the main text, twice applying prop. 11, is more elegant than what the redactor offers instead.

In prop. 43 a different proof is given for the same proposition.

Prop. 41,1): Let in the rectangular triangle ABG , with angle BAG right, from point A to line BG the lines AD and AE be drawn, such that angle DAG be equal to angle GAE , then BE : EG = BD : DG .
2) Conversely: If BE : EG = BD : DG [and angle BAG be right], then $\measuredangle$ DAG = $\measuredangle$ GAE

[margin 3): Let angle EAG be as angle GAD and BD : DG = BE : EG , then $\measuredangle$ GAB = 90° .]



(fig. 45)

Remark: The lines AT and AX have been drawn in later without the use of a ruler. They are needed for the proof of 41,3), which is added later in the margin of fol. 101^v.

Proof 1): construct HGZ // AB , <join DZ> [<> crossed out and replaced in margin by: extend AD to Z on HGZ]

$\sphericalangle$ BAG = 90° → $\sphericalangle$ AGH = 90° → its supplement $\sphericalangle$ AGZ = 90°
$\sphericalangle$ DAG = $\sphericalangle$ GAE , ⌊AG common⌋ } →
△ AGZ ≅ △ AGH → GZ = HG
→ AB : HG = BE : EG = AB : GZ
AB : GZ = BD : DG } → BE : EG = BD : DG .

q.e.d.

Proof 2): construct HGZ // AB
BE : EG = BA : HG
BD : DG = AB : GZ } → AB : HG = AB : GZ → GH = GZ
$\sphericalangle$ BAG = 90° → $\sphericalangle$ AGH = 90° → $\sphericalangle$ AGZ = 90°
AG = AG , GH = GZ
$\sphericalangle$ AGH = $\sphericalangle$ AGZ } → < → AH = AZ → △ GAH ≅ △ AGZ >
[<> crossed out by redactor⌋
→ $\sphericalangle$ ZAG = $\sphericalangle$ GAH . q.e.d.

The redactor adds to these two cases a third case of which the enunciation [see above] is given on fol. 102^r and the proof on fol. 101^v, both in the margin. As the edge of this page is mutilated the proof is difficult to read.

Proof 3) [fig. 45]: draw AT ⊥ AG
so either AT falls on AB or on the inside or on the outside
TD : DG = TE : EG
BD : DG = BE : EG } —interchange first and last→ EG : DG = ET : TD = EB : BD
→ ED : DT = ED : BD → DT = BD → AT on AB →
→ $\sphericalangle$ GAB = 90° .

q.e.d.

This addition to prop. 41 is also proved by Pappus in Book VI,52 , a proposition belonging to the section Εἰς τὰ ὀπτικὰ Εὐκλείδου (On Euclid's Optics). His proof reads [fig. 45]:

construct HGZ // AB , extend AD to Z

$$\left.\begin{array}{l} BD : DG = BE : EG \\ AB : HG = EB : EG \end{array}\right\} \rightarrow \quad BD : DG = AB : HG$$

$$AB : GZ = BD : DG \quad \rightarrow \quad HG = GZ$$

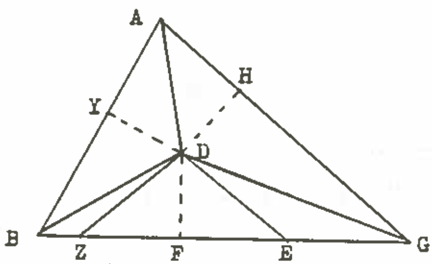
$$HA : AZ = HG : GZ \quad \rightarrow \quad HA = AZ$$

$$HG = GZ \quad , \quad AG = AG \quad , \quad HA = AZ \quad \rightarrow \quad \measuredangle HGA = \measuredangle AGZ \rightarrow \quad = 90^\circ$$

$$HZ \,//\, AB \quad \rightarrow \quad \measuredangle GAB = 90^\circ \quad .$$

This proof is in the spirit of our treatise, and it is clearer than the marginal proof by the redactor. Hultsch [Pappus Vol. II, p. 587] notes that the existence of two converses to this proposition, to which Commandino has added the proofs, is manifest. They are used by Pappus in resp. Book VI,53/54 and Book VII,156. These two converses are the same as prop. 41 and its converse, as given in this treatise. For the proof of prop. 43 prop. 41 is needed.

Prop. 42: Let in triangle ABG angle B be bisected by line BD and angle G by line GD , then angle A is bisected by line AD .



(fig. 46)

Remark: On the position of the points Z and E the redactor remarks: "So point E*, either it falls between B and G , or on G , or away from it towards the opposite of B ; and in this manner is the remark on point Z in relation with B ." This is correct: the proof is also true when $BG < AB$ and/or $BG < AG$. (* written D)

Proof: assume BE = BA and GZ = GA [marginal remark see above]

join DZ , DE

$$\left.\begin{array}{l} AB = BE \\ BD \quad \text{common} \end{array}\right\} \rightarrow \quad \measuredangle BAD = \measuredangle BED \quad , \quad DA = DE$$

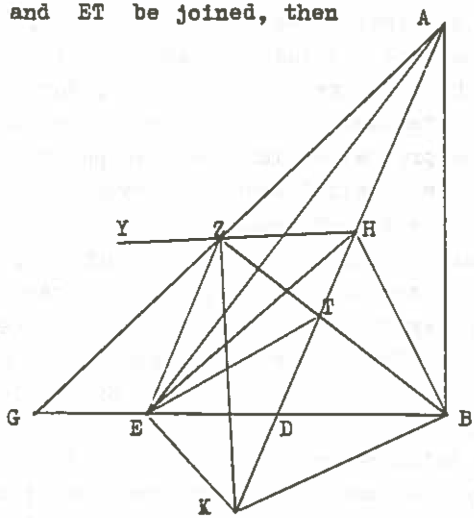
$$\left.\begin{array}{l} AG = GZ \\ GD \quad \text{common} \end{array}\right\} \rightarrow \quad \measuredangle DAG = \measuredangle DZG \quad , \quad ZD = DA$$

$$\left.\begin{array}{l} \rightarrow \quad DZ = DE \quad \rightarrow \quad \measuredangle EZD = \measuredangle ZED \\ \measuredangle ZED = \measuredangle DAB \; , \; \measuredangle EZD = \measuredangle DAG \end{array}\right\} \rightarrow \quad \measuredangle BAD = \measuredangle GAD \quad .$$

q.e.d.

This proof is a nice and original achievement, different from what is found in Euclid. In Euclid the proposition "the bisectors of the three angles of a triangle meet in a point" is neither enunciated nor proved in this form. However, it follows immediately from the proof of IV,4 :"In a given triangle to inscribe a circle". Euclid constructs this circle [fig. 46] by bisecting angle ABG by line BD and angle AGB by line GD . He then draws from their meeting point D the perpendiculars DF , DH , DY to the three sides, and shows that the circle with center D and radius DF = DH = DY is the required circle. Prop. 42 is needed in the proof of prop. 43.

Prop. 43 [= prop. 40]: If in the rectangular triangle ABG , with angle ABG right, angle BAG be bisected by line AD , line AE be drawn at random, in point E line EZ be constructed parallel to line AD , ZB be joined, cutting AD in T , and ET be joined, then AZ : ZT = AE : ET .



(fig. 47)

Proof: bisect $\measuredangle$ ABZ by line BH , join ZH $\xrightarrow{\text{[prop.42]}}$ ZH bisects $\measuredangle$ AZB

draw BK $\perp$ BH ,

join ZK , EH , EK

[margin:

$\measuredangle$ ABZ + $\measuredangle$ BAZ $< 180^\circ$ $\xrightarrow{\text{halving}}$

$\measuredangle$ ABH + $\measuredangle$ BAH $< 90^\circ$

$\rightarrow$ $\measuredangle$ AHB $> 90^\circ$

$\rightarrow$ $\measuredangle$ BHT $< 90^\circ$

$\rightarrow$ AD extended meets BK in K]

$\left.\begin{array}{l}\measuredangle \text{ HBK} = 90^\circ \\ \measuredangle \text{ ABH} = \measuredangle \text{ HBT}\end{array}\right\}$ $\xrightarrow{\text{[prop. 41,1)]}}$

$\left.\begin{array}{l}\text{KA : AH = KT : TH} \\ \measuredangle \text{ AZH} = \measuredangle \text{ HZT}\end{array}\right\}$ $\xrightarrow{\text{[prop. 41,3)]}}$ $\left.\begin{array}{l}\measuredangle \text{ HZK} = 90^\circ \\ \measuredangle \text{ HBK} = 90^\circ\end{array}\right\}$ $\rightarrow$

$\rightarrow$ B , H , Z , K on circle circumference

$\measuredangle$ ABG = 90° = $\measuredangle$ HBK

$\measuredangle$ HBE $\qquad$ $\measuredangle$ HBE $-$

$\measuredangle$ HBT = $\measuredangle$ ABH $\qquad$ = $\measuredangle$ DBK

$\measuredangle$ TBD $\qquad$ $\measuredangle$ TBD $+$

$\measuredangle$ HBD $\qquad$ = $\measuredangle$ KBZ = $\measuredangle$ KHZ

extend HZ to Y

EZ // KH $\xrightarrow{\text{[E. I,29]}}$ $\measuredangle$ KHZ = $\measuredangle$ EZY

[$\measuredangle$ DBH]* = $\measuredangle$ DHZ $\rightarrow$ $\measuredangle$ YZE = $\measuredangle$ DBH $\qquad$ [* not readable]

$\rightarrow$ [$\measuredangle$ EZH + $\measuredangle$ HBE = 180° $\xrightarrow{\text{E. III,22 conv.}}$]

E , Z , H , B on circle circumference

[margin: B , H , Z , K on circle circumference $\rightarrow$ B , H , Z , E , K on <u>one</u> circle, [E. III,10]: because the two circles intersect in more than two points, i.e. in B , H , Z]
[$\rightarrow$ KH diameter]

$$\left.\begin{array}{l} \rightarrow \measuredangle HEK = 90^\circ \\ KA : AH = KT : TH \end{array}\right\} \xrightarrow{[\text{prop. } 41,2)]} \measuredangle AEH = \measuredangle HET$$

$\xrightarrow{[E.\ VI,3]}$ AH : HT = AE : ET [marginal remark]
[$\measuredangle$ AZH = $\measuredangle$ HZT $\rightarrow$] AH : HT = AZ : ZT
$\rightarrow$ AE : ET = AZ : ZT . q.e.d.

The marginal remarks raise the quality of the proof. They illustrate the scientific attitude and skill of the redactor.
In the first remark [fol. 102^r, bottom left] is proved that BK will meet the extension of AD . Often in Greek geometry, also in Euclid, existence proofs for intersection points are left out. It seems the mathematicians relied upon the drawings.
Also the second remark [fol. 102^v, right middle] is a necessary addition: Proving the circle through E , Z , H , B is the same as the circle through B , H , Z , K gives "KH is a diameter", from this it is clear that angle HEK is right. There is a curious mistake in this remark: The points belonging to both circles are B , H , D . However, B , K , D have been written and it looks as if "K" is a later "correction".
The third remark [fol. 102^v, middle] explains which proposition underlies the conclusion: This because of the third proposition in the second chapter in the third classification in a Book by al-Kamāl.
I know nothing neither about al-Kamāl, the al-Kamāls found in Sezgin all live rather late, nor about his book. The proposition in question is E. VI,3 [cf. prop. 21].
The other two marginal remarks refer to the treatise as a whole, to the author Aqāṭun and to the redactor [cf. chapter II].
The fact that so many remarks are made to these last propositions, and the fact that two proofs are given for prop. 40/ 43 show the importance of this proposition in ancient Greek and also in later Arabic times.
The circle through the points B , H , Z , E , K is <u>Apollonius' circle</u>. The circle of Apollonius is a locus [Heath, E. vol. II, p. 198]: "The locus of a point such that its distances from two given points are in a given ratio (not being a ratio of equality) is a circle".
Heath adds that the construction apparently was not discovered by Apollonius, but earlier, since it appears in exactly the same form in Aristotle, <u>Meteorologica</u> III. 5, 376 a 3 sqq. Ptolemy, who needs this

circle in his chapter on Retrograde Motion (Almagest, Book XII, Chapter I) calls it explicitely the circle of Apollonius of Perga. The application of this circle in optics and astronomy counts for its importance. Perhaps prop. 40/43 is important for a similar reason.

Chapter IV.

Influences on and from the Book of Asumptions. Conclusion.

As we have seen in the previous chapters different influences are noticeable in this treatise. A connection with Archimedes is found in the beginning. Prop. 2 and prop. 3 are directly linked with Archimedes. Prop. 6 is in the spirit of Archimedes' Lemmata, which also means that prop. 7, based on prop. 6, is loosely connected with Archimedes. Prop. 4 could be Archimedean.

Menelaus' Theorem is found in the propositions 33 and 34. However, we can not deduce from this that the author has been influenced by Menelaus, as these two propositions are in the more primitive planar form of Menelaus' Theorem. Euclid is used all over the treatise, as is to be expected. A direct link with Euclid's Data is apparent in prop. 21. An indirect connection between Euclid, i.e. Euclid's Porisms, a treatise now lost, and the propositions 1, 22, 36 is brought to our attention by Pappus. Prop. 41 consists of two converses to a lemma given by Pappus in Book VI among the additions to Euclid's Optics. Thus a connection with Euclid's Optics might exist.

Through Pappus we also learn about the influence of Apollonius, e.g. Apollonius' Conics in prop. 27. Ver Eecke [Pappus II, 739/740] explains that this proposition is justified as it proves the only step in prop. III,8 which is left unproved by Apollonius. In prop. 3 we may see a loose association with Apollonius' On Contacts, a non-extant treatise.

A clear connection exists again between Apollonius' Loci and the propositions 11, 14, 39. This can be extended to the propositions 12 and 13 which are both based on prop. 11, to prop. 17 based on prop. 14, to prop. 18 which is in the spirit of prop. 11 - 13, to prop. 19 connected with the propositions 11 and 14, and to prop. 20 based on prop. 19. The propositions 24, 28, 40 using prop. 11 in their proof, also prop. 31 where prop. 19 is used could perhaps in a wider sense be included in this sphere of influence. A more direct relation with Apollonius is again found in prop. 43 where Apollonius' circle occurs.

In Book VII of the Collectiones quae supersunt Pappus gives a summary of the contents of Apollonius' Loci. This treatise by Apollonius, now lost, is said to have been divided into two books containing 147 propositions and eight lemmas. Among the properties treated in the second book one finds: "If from two given points two intersecting lines be drawn and these lines have to each other a given ratio, then the intersection point lies either on a straight line or on a given circle circumference". This circle is referred to in later literature as Apollonius' Circle. The definition of this circle is again found in the treatise Maqāla fī'l-maʿlūmāt (On Discoveries) by Ibn al-Haytham. This treatise is divided into an introduction and two books. The first book, containing 24 propositions is claimed to consist of wholly new discoveries, of a type not even known to ancient geometers, which is clearly exaggerated. The second book, containing 25 propositions is said to form a sequence of propositions analogous to what is treated in the Data, but which are not found in Euclid's work. Book I, prop. 9 enunciates [Sédillot, 445]: "If from two given points two straight lines be drawn which meet in a point, and if the ratio of these two lines, i.e. of the largest to the smallest be known, the meeting point lies on a given circle circumference". Although Ibn al-Haytham had a good knowledge of older mathematical literature, this proposition may also be his own achievement.

In another treatise, Maqāla fī khawāṣṣ al-muthallath min jihat al-ʿamūd (On the Broperties of the Triangle with Regard to the Height) Ibn al-Ḥaytham first relates the results acquired by the mathematicians before him, namely the contents of the propositions 8 - 10 ; these state that "in an equilateral triangle the sum of the perpendiculars from an interior point to the three sides is equal to the height of the triangle". Extending this result Ibn al-Haytham develops similar relations for isosceles triangles and even thinks to have found a generalization for any triangle, which is not quite correct [see Hermelink, 247 and chapter III]. This is the only direct influence we have found from our treatise on later mathematicians.

Although Thābit b. Qurra is said to have translated treatise B, Uṣūl al-handasīya from Greek into Arabic he might still have known or heard of treatise A, K. al-mafrūḍāt. This title may have inspired him to write a treatise, equally entitled K. al-mafrūḍāt. However, in Thābit's case, the title K. al-muʿṭayāt (Data) would have been more proper, as the contents of Thābit's treatise are in the spirit of Euclid's Data.

Other connections have not been found, but as treatise A and treatise B are the only two extant copies, their sphere of influence can not have been very large.

Conclusion:

From the exposition in the previous chapters we conclude that

1. Kitāb al-mafrūḍāt, Book of Assumptions could well be the original title of the treatise. (10)

2. Archimedes was probably named as the author from a commercial point of view. Neither the subject nor the composition of the treatise point toward Archimedes. One might ask whether commercialism also accounts for the title Uṣūl al-handasīya, but for this I have no evidence.

3. The name of the author of the treatise may well be Aqāṭun. As this name also turns up in Ibn al-Qifṭī it is not likely to be a misreading. (11) The author, i.e. Aqāṭun seems to have been a man with a good knowledge of the mathematical literature, judging from the different influences on his treatise. He may have lived around the same time as Pappus. (12)

4. The treatise contains interesting propositions. However, the overall impression which the treatise offers is more that of an occupation with existing mathematics than that of the opening of new horizons.

5. The treatise has still been of value and interest to Arabic mathematicians in the thirteenth century, according to the extensive marginal notes. Yet the sphere of influence has apparently not been very large. (13)

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(10) Cf. chapter II,3. (11) Cf. chapter II,1.

(12) On the one hand Aqāṭun gives two converses to a lemma by Pappus. On the other hand both Pappus and Aqāṭun prove a lemma related to Apollonius' _Conics_ (prop. 27) and a lemma connected with Euclid's _Porisms_ (prop. 22). These proofs by Aqāṭun and Pappus are similar but not equal. (13) Cf. chapter IV, above, and chapter I,5.

Chapter V.

Translation of the Book of Assumptions.

This chapter contains the translation of treatise A. Treatise B is only mentioned insofar as it essentially differs from A, whereas minor differences are passed over. Also the many misprints or variations in H, the printed edition of B, remain uncited. From treatise A and B each folio and line has been marked in the translation, whereas from H only the pages have been indicated.

The following symbols are used:

[A:]	text between brackets only occurs in A
[B:]	idem for B
[ ]*	text badly damaged, not readable
[ ]	my addition
( )	superfluous.

[A: $89^{v},1$ In the name of Allah, the compassionate Merciful. The Book of Assumptions by Aqāṭun.]
[B: $141^{r},8$ In the name of Allah, the compassionate Merciful. ^{9} The Book by Archimedes on the Elements of Geometry, translated from the Greek language ^{10} into the Arabic language for Abū'l-Ḥasan ʿAlī Ibn Yaḥyā companion to the Caliph; Thābit Ibn Qurra.]
[Prop. 1, fig. 1]$^{22}_{HA}$ $^{11}_{B}$ Let us assume semicircle ABG, extend line BG (rectilinearly) in both directions $^{3}_{A}$ to points D and E, $^{12}_{B}$ assume lines BE and GD to be equal, draw from points E and D $^{4}_{A}$ tangents to semicircle $^{13}_{B}$ ABG, namely lines EZ and DH, and let us join ZH; Then I assert that line ZH is parallel $^{5}_{A}$ to line ED. .˙.
Proof: $^{14}_{B}$ Let us specify the center of circle ABG, let it be point T, and let us join it to H and Z. $^{6}_{A}$ Because line EB equals $^{15}_{B}$ line GD and we add BG common [to both], the [B: total] line EG equals the [B: total] line HD. Line BE $^{7}_{A}$ equals $^{16}_{B}$ line GD. Therefore [A: the product of GE and BE equals the product of BD and DG. But] the product of EG and BE equals $^{8}_{A}$ the square of EZ $^{17}_{B}$ and the product of HD and DG equals the square of DH. Thus the square $^{18}_{B}$ of EZ equals the square of DH [B: and thus line DH equals $^{19}_{B}$ line EZ]. Because the two lines $^{9}_{A}$ HT [B: GT] $^{20}_{B}$ and TD equal the two lines ZT and TE and base EZ equals base DH, angle $^{10}_{A}$ $^{21}_{B}$ ZTE equals angle HTD, and so chord $^{3}_{H}$ ZB equals chord HG. Hence line ZH is parallel to line $^{11}_{A}$ $^{22}_{B}$ ED. .˙.

[B: And this is what we wanted to prove.]
[A: Likewise, when the segment is less than a semicircle. When it is greater than a semicircle $^{12}_{A}$ then it is possible for line DH to meet line EZ at side A or at the other side, and the proof is the same; or it is $^{13}_{A}$ possible for them to be parallel to each other. If they are parallel to each other and equal, then the lines joining (what is between)) $^{14}_{A}$ their endpoints are parallel to each other and equal .˙.]
[B: By this argument is completely proved what we have asserted in this $^{23}_{B}$ construction, if we say]
[fig. 2] Because the product of GE and EB equals the product of BD and $^{15}_{A}$ DG, the product of GE and EB equals the square of EZ, the product of BD and DG $^{24}_{B}$ equals the square $^{16}_{A}$ of DH, the square of DH then equals $^{17}_{A}$ $^{25}_{B}$ the square of EZ. Therefore line EZ equals $^{18}_{A}$ line DH. We now extend $^{19}_{A}$ lines DH and EZ [B: at the sides Z and H] until they meet $^{26}_{B}$ in point Y. Thus line YZ equals line YH, because both [B: of them] issue from the same point, $^{20}_{A}$ $^{27}_{B}$ namely point Y, and touch the circle [B: ABG]. It has already been proved that line EZ equals $^{28}_{B}$ line DH $^{21}_{A}$ [B:,thus the ratio of EZ to ZY is like the ratio of DH $^{29}_{B}$to HY]. Hence line HZ is parallel to line DE [B: GB]. And this is $^{30}_{B}$ what we wanted to prove. .˙. .˙.

[Prop. 2, fig. 3] Let us assume $^{31}_{B}$ a circle through A, B, G, and let the lines $^{22}_{A}$ $^{141v}_{B}$ DB and DG touch it. Let us join BG, extend it (rectilinearly) to point E, and draw $^{2}_{B}$ from point $^{23}_{A}$ E a line, which touches the circle [B: ABG] [in point A], meets line DB in point T [and line DG in point Z], namely line EZT;
$^{3}_{B}$ Then I assert that the ratio $^{90r}_{A}$ of TE to EZ is the same as the ratio of TA to AZ.
$^{4}_{H}$ Proof: Let us draw from point $^{4}_{B}$ Z $^{2}_{A}$ a line parallel to line TB, namely line ZH [meeting BG in H]. The ratio of HD to DG $^{3}_{A}$ is then the same as the ratio of HZ to ZG. But $^{5}_{B}$ line BD equals line DG, thus line HZ $^{4}_{A}$ equals line ZG. Because the ratio of TE to EZ $^{6}_{B}$ is the same as the ratio $^{5}_{A}$ of TB to ZH, and line ZH equals line ZG, the ratio of TE $^{6}_{A}$ to EZ is the same as the ratio of TB to ZG. $^{7}_{B}$ But line BT equals line TA, $^{7}_{A}$ because both touch the circle, and line GZ equals line AZ. Hence the ratio of TE $^{8}_{B}$ to $^{8}_{A}$ EZ is the same as the ratio of TA to AZ. And this is what we wanted to prove. .˙.

[Prop. 3, fig. 4 and 5] $^{9}_{B}$ Let us assume a circle through A, B, G, with tangents $^{9}_{A}$ EG and ED. $^{10}_{B}$ Let us draw from point E an arbitrary line cutting $^{11}_{B}$ the circle, namely line $^{10}_{A}$ EHB, and draw from point

D ${}^{12}_{B}$ a line parallel to line EB, namely line DA. Let us join AG, which cuts ${}^{11}_{A}$ ${}^{13}_{B}$ BH at Z;
Then I assert that BZ equals line ZH.
${}^{14}_{B}$ Proof: Let us specify the center of the circle, let it be ${}^{12}_{A}$ point T. Let us join ${}^{15}_{B}$ TD, TE, TG, TZ. Because line TD equals line TG, and line TE is ${}^{13}_{A}$ common [to both], lines ${}^{16}_{B}$ DT and TE equal lines GT and TE. Base ED is the same as base${}^{5}_{H}$EG, ${}^{14}_{A}$ thus angle DTE ${}^{17}_{B}$ equals angle GTE, and so angle DTG is twice the size of angle ETG. Angle DTG is twice the size of ${}^{15}_{A}$ ${}^{18}_{B}$ angle DAG, therefore angle DAG equals angle ETG. [B: But angle DAG equals ${}^{19}_{B}$ angle EZG], hence angle ETG equals angle EZG. ${}^{16}_{A}$ ${}^{20}_{B}$ Thus quadrilateral (14) ETGZ [lies] in a circle, ${}^{17}_{A}$ and so the angles ${}^{21}_{B}$ EGT and EZT are equal to each other. ${}^{18}_{A}$ Angle ${}^{22}_{B}$ EGT is right, thus angle EZT ${}^{19}_{A}$ is right ${}^{23}_{B}$ [B:, therefore line TZ is perpendicular to line HZ]. So a line has been drawn ${}^{24}_{B}$ from center T to ${}^{20}_{A}$ line BH, which cuts it at right angles ${}^{25}_{B}$, and consequently bisects it at Z. Hence line BZ ${}^{26}_{B}$ equals line ${}^{21}_{A}$ ZH. [B: And this is what we wanted to prove.] .˙.

[Prop. 4, fig. 6] ${}^{27}_{B}$ Let us assume an isosceles [B: equilateral] triangle ABG, ${}^{28}_{B}$ and draw line AD perpendicular to ${}^{22}_{A}$ line BG; ${}^{29}_{B}$ let us assume [B: take] [points E and Z on the extended line AB, such that] the square of BD [be] equal to the product ${}^{30}_{B}$ of BE and BZ, and join DZ, draw from ${}^{31}_{B}$ point ${}^{23}_{A}$ Z a line parallel to line BG, namely line ZH, and let us join ${}^{142r}_{B}$ EH;
Then I assert that angle EHG is twice the size of ${}^{90v}_{A}$ angle AZD.
Proof: Let us join DE and DH. Because ${}^{2}_{B}$ the product of EB and BZ equals the square ${}^{2}_{A}$ of DB, angle ZDB equals angle ${}^{3}_{B}$ BED. Angle ZDB equals angle ${}^{3}_{A}$ HZD. Thus angle HZD equals angle ZED. ${}^{4}_{B}$ But angle HZD equals ${}^{6}_{H}$ angle ${}^{4}_{A}$ ZHD, because triangle HDZ is isosceles. ${}^{5}_{B}$ Therefore angle ZED equals angle ZHD, and thus ${}^{5}_{A}$ quadrilateral EZDH (15) lies in a circle. ${}^{6}_{B}$ Let us extend line EH (rectilinearly) to point T, ${}^{6}_{A}$ so angle DHT equals angle ${}^{7}_{B}$ EZD, because it is an exterior [angle] of the ${}^{7}_{A}$ quadrilateral [B: EZDH]. Angle DZA equals ${}^{8}_{B}$ angle AHD, thus angle ${}^{8}_{A}$ AHT [B: AHD] is twice the size of angle AHD. But angle AHT equals ${}^{9}_{A}$ ${}^{9}_{B}$ angle EHG, and angle AHD equals angle ${}^{10}_{A}$ AZD [A: AZE], hence angle EHG ${}^{10}_{B}$ is twice the size of angle AZD. [B: And this is what we wanted to prove .˙. .˙.].˙.

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(14) "quadri" in A only in margin. (15) margin A: reductio ad absurdum.

[Prop. 5, fig. 7] Let us assume the semicircle through 11|A A, B, G. Let us bisect arc ABG at point B, extend line AG (rectilinearly) 12|A to D, join DB, [which cuts the semicircle at point Y,] bisect line BY at point E, specify the center of the semicircle, 13|A namely point Z, join ZE, and let us extend lines ZE and AB (rectilinearly) to point H; Then I assert 14|A that the ratio of AH to HB is the same as the ratio of DZ to ZB.

Proof: Let us draw from point B a line 15|A parallel to line ZH, namely line BT and let us join BZ, hence the ratio of 16|A AH to HB is the same as the ratio of AZ to ZT. The ratio of DZ to 17|A ZB is the same as the ratio of ZB to ZT, for triangle DZB 18|A is similar to triangle ZBT. AZ equals ZB. Angle TBD is right, because line ZE is drawn from center Z 19|A and bisects line BY at point E, so they comprise together a right angle. Angle ZED equals 20|A angle TBY, thus angle TBD is right. Hence the ratio of AH to HB is the same as the ratio of DZ to ZB. And this is 21|A what we wanted to prove.

[Prop. 6, fig. 8] 11|B Let us assume a semicircle through A, B, G, D and draw BD and GA, [A: which meet 22|A at point Z]; we also 12|B join BA and GD and extend both (rectilinearly)) until they meet at point 13|B E; Then I assert 23|A that the product of BD and DZ equals the product of GD and DE.

14|B Proof (16): When the product of BD and 9lr|A DZ is the same as the product of 15|B GD and DE, the ratio of BD to DG is the same as the ratio 2|A of ED to DZ. Thus when we join 16|B EZ, the triangles BDG and EDZ 3|A should be similar to each other, and angle ZBG should be equal 17|B to angle 4|A DEZ. When we join DA angle DBG is equal 5|A 18|B to angle DAG [A: because the base of both is the same arc]. Thus angle 6|A DAG should equal angle 19|B 7|B DEZ. And so quadrilateral 20|B EAZD must lie in a circle. That [B: it lies in a circle] is 7|A evident , because 21|B both angles EAZ and EDZ are right. Hence 22|B the product of BD 8|A and DZ must certainly equal the product of GD and DE. And this is what 23|B we wanted to prove. [A: And the composition of this proof is 9|A : Angle

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(16) margin A fol. 9lr : Proof of the proposition: Let us assert both angles EDZ and EAZ are right. Thus quadrilateral EAZD lies in a circle. Therefore angle DEZ is the same as angle DAZ, that is to say the same as angle DBG. And both angles D are right. Hence triangle EDZ is similar to triangle GEB. Thus was tested what was wanted. I had written down this remark before looking at the construction of the proof. So I apologize.

GDB is right and the remaining angle BDE is right, likewise angle GAB is right and the remaining angle 10/A ZAE is right, and so quadrilateral ZDAE lies in a circle. Angle DBG equals angle DAG 11/A and angle DAZ is the same as angle DEZ, because both their arcs are DZ. Thus angle DEZ is the same as angle DBG. The triangles 12/A BDG and DEZ are then similar to each other.Therefore the ratio, [i.e.] the quantity of BD over DG is the same as the quantity of DE over DZ, hence BD times 13/A DZ is the same as GD times DE. And this is as we wanted to prove it. .•.]

[Prop. 7, fig. 9] 23/B Let us assume a semicircle through A, B, G, D, 14/A and join 24/B AG and BD [A: which meet at point Y]; let the product of BD and DY be equal to the square of DZ and the product of 15/A GA and AY be equal 25/B to the square of AE, and let us join EB and ZG; Then I assert that line ZH is equal to line EH.

16/A Proof: 26/B Let us join BA and GB and extend both (rectilinearly) until they meet at point T. Then the product of BD 17/A and DY 27/B equals the product of GD and DT, [B: as has been proved in the preceding [proposition]], and the product of GA and AY equals the product of BA and 28/B AT. Thus the product of BA 18/A and AT equals the square of AE, [B: the product of] GD and DT equals the square of DZ, and angles 29/B TDZ and TAE are both 19/A right. Thus when we have joined ZT and TE, both angles 30/B TZH and TEH are 20/A right. On the ground that the product of BT and TA equals the product of GT and TD, that the product of 31/B BT 21/A and TA equals the product of BA and AT [plus the square of AT] (17), that the product of GT and TD equals the product 22/A 142v/B of GD and DT plus the square of TD, and that the two products 8/B of BA and AT and of GD and DT [B: the squares of BA, AT, GD, DT] 23/A 2/B equal the two squares of AE and DZ, the squares of TA and AE therefore equal 91v/A 3/B the squares of TD and DZ. [A: The sum of] the squares of TA and AE equals 4/B the square of TE, because angle TAE is right [A: and the squares 1/A of TD and DZ equal the square of TZ, because angle TDZ is right]. 5/B So the square of TZ equals the square of TE, 3/A and thus line TE 6/B equals line TZ. Therefore when we join EZ, angle TZE 7/B is equal to angle TEZ. But 4/A the right angle TZH and 8/B the right angle TEH are equal to each other. Hence the remaining angle EZH equals 5/A the remaining angle ZEH. 9/B And thus line ZH equals line EH. And this is what we wanted to prove.

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(17) in A left out, added in margin.

[Prop. 8, fig. 10] ${}^{6}_{A}$ ${}^{10}_{B}$ Let us assume an equilateral triangle ABG, and let us draw in it the perpendiculars AZ, BD, GE;
Then I assert ${}^{7}_{A}$ ${}^{11}_{B}$ that the perpendiculars AZ, BD, GE are of equal lenght.
Proof: As triangle ABG is isosceles ${}^{12}_{B}$ and ${}^{8}_{A}$ perpendicular AZ has been drawn in it, line BZ is equal to line ZG. Likewise, as triangle GBA ${}^{13}_{B}$ is ${}^{9}_{A}$ isosceles and perpendicular GE has been drawn in it, line AE is equal to line BE. Thus line ${}^{14}_{B}$ GZ equals line AE. Let us take ${}^{10}_{A}$ line AG common [to both], then lines AE and AG are equal to lines AG and GZ.${}^{15}_{B}$ Angle ${}^{11}_{A}$ GAE [A: angle A] equals angle AGZ [A: angle G] and thus base AZ [B: AB] equals base EG. ${}^{12}_{A}$ Likewise, ${}^{16}_{B}$ as triangle BGA is isosceles and ${}^{9}_{H}$ ${}^{13}_{A}$ perpendicular BD [A: BE] has been drawn in it, line ${}^{17}_{B}$ AD is equal to line DG [B: DE]. Thus line EB ${}^{14}_{A}$ equals line GD. Let us take BG common [to Both], then ${}^{18}_{B}$ lines EB and BG are equal to lines BG and GD. ${}^{15}_{A}$ Angle BGD [A: angle B] equals angle ${}^{19}_{B}$ GBE [A: angle G] and thus base BD equals base GE. It has already been ${}^{16}_{A}$ proved ${}^{20}_{B}$ that line GE equals line AZ, and so line BD equals line AZ. ${}^{21}_{B}$ Hence the [B: three] lines EG, BD, AZ ${}^{17}_{A}$ are of equal length. And this is what ${}^{22}_{B}$ we wanted to prove.

[Prop. 9, fig. 11] Let us assume an equilateral triangle ${}^{23}_{B}$ ABG and draw ${}^{18}_{A}$ in it perpendicular AD. ${}^{24}_{B}$ Let us study on line BD an arbitrary point, namely point E, and draw ${}^{19}_{A}$ from it [B: point E] ${}^{25}_{B}$ the two perpendiculars to lines GA and AB, namely lines ZE and EH;
Then I assert that line AD equals the two lines ${}^{20}_{A}$ ZE and EH.
${}^{26}_{B}$ Proof: Let us draw from point E a line parallel to line AG, namely line TE, ${}^{21}_{A}$ and let us draw from point ${}^{27}_{B}$ B a line which is perpendicular to line GA, namely line BY. Because triangle ABG ${}^{22}_{A}$ is equilateral ${}^{28}_{B}$ and line AG is parallel to line TE, triangle TBE is equilateral. ${}^{23}_{A}$ And ${}^{29}_{B}$ because line BY is perpendicular to line [B: AG, and line] AG parallel to line TE, ${}^{30}_{B}$ BK (18) is perpendicular to line TE. [A: Line ${}^{92r}_{A}$ EH is perpendicular to line TB [ms: TE], and so line BK equals line EH. Line KY ${}^{2}_{A}$ is parallel to line EZ, and is thus equal to it.] [B: Line KY equals ${}^{10}_{H}$ line EZ because figure ${}^{31}_{B}$ KEZY is a parallelogram.] Therefore the total line BY equals the two lines ZE and EH. ${}^{143r}_{B}$ ${}^{3}_{A}$ But line BY equals line AD. Hence line AD equals the two lines EH and EZ. ${}^{4}_{A}$ And this is what we wanted to prove.

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(18) K is not defined in the drawing of B.

[Prop. 10, fig. 12] 2/B Let us assume an equilateral triangle 5/A ABG, draw in it perpendicular AD and assume [B: study] in its interior 3/B an arbitrary point, 6/A namely point E. Let us draw [B: from it] [A: three] perpendiculars to the sides of the triangle, namely lines 4/B ZE, EH, 7/A ET;
Then I assert that AD equals [A: the sum of them] [B: the lines EZ, EH and ET].
Proof: Let us construct [B: draw] through 5/B point E a line parallel to line 8/A BG, namely line YEK. Since line YK is parallel to line 6/B BG and EZ is parallel to line DL, figure 9/A EZDL (19) is a parallelogram. Since triangle ABG 7/B is equilateral, [B: perpendicular AD has been drawn in it,] and line YK is parallel to the base, [B: namely line 8/B BG,] 10/A triangle AYK is equilateral. 9/B Since perpendicular AL has been drawn in it, and we study on line KLY [B: an arbitrary point, 10/B namely] 11/A point E, from which the perpendiculars are drawn to the lines AY and AK, 11/B namely lines EH 12/A and ET, line AL is then equal to [the sum of] these two [B: the lines EH and ET]. 12/B [B: It has already been proved that] line LD is the same length as line EZ. Consequently line 13/A 11/H AD 13/B equals the lines EZ, EH, ET. And this is what we wanted 14/B to prove.

[Prop. 11, fig. 13] 14/A 143v/B 12,6/H Let us assume a triangle ABG 2/B and draw from point A [A: on 15/A line AB] a line to line BG which encloses [B: with BA] an angle equal to angle G [B: AGB], namely 3/B line AD — thus angle BAD 16/A equals angle AGD —;
Then I assert that the product of GB and BD equals 17/A the square 4/B of AB.
Proof: Since angle AGB equals angle 18/A BAD and [B: we made] 5/B angle ABG [A: is] common to both triangles ABG and ABD, 19/A the remaining angle BDA is equal to angle 6/B BAG. Thus the triangles ABG and ABD have equal angles, 20/A consequently they are similar to each other. 7/B Thus the ratio of GB to BA is the same as the ratio of AB to BD, hence the product of GB 8/B and BD equals the square 21/A of AB. And this is what we wanted to prove. .·.

[Prop. 12, fig. 14] 143r,14/B 11,3/H Let us assume an isosceles triangle ABG, 22/A 15/B draw from point A a perpendicular on line AB, namely line AD and extend line BG 16/B (rectilinearly) 23/A until it meets [B: line AD at point D] [A: it, namely BD, and they meet at D]. Let us bisect

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(19) L is not defined in the drawing of A.

line AB at point $^{17}_{B}$ E, join EZD $^{92v}_{A}$ and draw from point Z a line parallel to line AB, namely line ZH;
Then I assert $^{18}_{B}$ that the product of DA and AH $^{2}_{A}$ equals the square of AG.
Proof: Let us extend ZH (rectilinearly) to point $^{19}_{B}$ T. Since triangle ABG $^{3}_{A}$ is isosceles and line ZT is parallel to line AB, line ZT $^{20}_{B}$ is equal to line ZG. Moreover, $^{4}_{A}$ since line AE equals line EB, and line EB is parallel to line HT, $^{21}_{B}$ HZ is equal to line ZT. $^{5}_{A}$ It has already been proved that line ZT equals line ZG [B: , and so line $^{22}_{B}$ ZH equals line ZG]. Thus the [B: three] lines ZH, ZT, ZG $^{23}_{B}$ are of equal length. And when we join $^{6}_{A}$ GH, angle HGT $^{24}_{B}$ is right, hence the two remaining angles ZHG and ZTG $^{7}_{A}$ $^{25}_{B}$ equal a right [angle]. $^{26}_{B}$ Angle ZTG $^{27}_{B}$ equals angle $^{8}_{A}$ B [B: ABG], and thus angle B [B: ABG] plus angle $^{12}_{H}$ ZHG equal $^{9}_{A}$ a right [angle]. Angle B [B: ABG] $^{28}_{B}$ plus angle $^{10}_{A}$ D [B: ADB] equal one right [angle], and thus angle D [B: ADB] equals angle ZHG. $^{29}_{B}$ Angle ZHG equals $^{11}_{A}$ angle ZGH, and thus angle D [B: ADB] equals angle ZGH. [A: Therefore the triangles [margin: i.e. triangles DAG and GAH] are similar.] Hence the product $^{143v}_{B}$ of DA and AH $^{12}_{A}$ equals the square of AG. [B: And this is what we wanted to prove. .·. .·.] .·.

[Prop. 13, fig. 15] $^{143v,8}_{B}$ $^{12,16}_{H}$ Let us assume an isosceles triangle $^{9}_{B}$ ABG, with the equal sides $^{13}_{A}$ lines AB and BG; let us draw from point A $^{10}_{B}$ a line, which is perpendicular to line BG, namely $^{14}_{A}$ line AD;
Then I assert that $^{13}_{H}$ twice the amount of the product of DG and GB $^{11}_{B}$ equals the square of AG.
Proof: Let us draw $^{15}_{A}$ from point A a line which is perpendicular to line AG, namely line AE, $^{12}_{B}$ and extend $^{16}_{A}$ line BG (rectilinearly) until it meets line AE [B: and let their meeting be] at point $^{13}_{B}$ E.
Since $^{17}_{A}$ angle EAG is right and line GB is the same as line BA, $^{18}_{A}$ $^{14}_{B}$ the [B: three] lines EB, BA, BG are of equal length, and EG $^{19}_{A}$ $^{15}_{B}$ is twice the length of GB. [A: The product of GD and GE is then twice the amount of the product of DG and GB.] And the product of EG and GD $^{20}_{A}$ equals the square of $^{16}_{B}$ GA, because angle GAE is right and DA is perpendicular [B: to line BG]. [Thus angle GAD equals angle AEG.] (20)
Hence twice the amount of the product of DG and GB $^{21}_{A}$ equals $^{17}_{B}$ the square of AG. And this is what we wanted to prove.

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(20) see remark on this proposition in Chapter III.

[Prop. 14, fig. 16] (21) Let us assume a triangle ABGD and draw 22/A from point 18/B A to line BG the perpendicular, namely AD;
Then I assert that the difference (22) of the square of BD over the square of DG 23/A is like the difference of the square 19/B of BA over the square of AG.
Proof: Since the difference of the square of BD over the square 93r/A 20/B of DG is the same as the difference of the squares of BD and DA over the squares of AD and DG, the squares 21/B of BD and DA equal 2/A the square of AB, the squares of AD and DG equal the square of AG, 22/B hence the difference of the square 3/A of BD over the square of DG is like the difference of the square of BA 14/H over the square 23/B of AG. And this is 4/A what we wanted to prove.

[Prop. 15, fig. 17] 144v,6/B 17,17/H Let us assume a line AB equal 5/A 7/B to a line AG and a line BD equal to a line DG, and let each of the angles BAG and BDG 18/H be right;
6/A 8/B Then I assert that angle ABD equals angle AGD.
Proof: Let 7/A us join BG. Since 9/B angle A is right, angles z and e 8/A are equal to a right [angle]. Also, since 10/B angle D is right, 9/A angles h and t are equal to a right [angle]. 11/B [B: Angles e and z are already equal to a right [angle].] Thus angles 12/B e and z equal angles 10/A h and t. [A: Angle e equals angle z, and angle h equals angle t. Consequently angle e 11/A equals angle h and angle z equals angle t] Hence the sum of the angles h and e equals 13/B the sum 12/A of the angles z and t. And this is what we wanted to prove.

[Prop. 16, fig. 18] 143v,23/B 14,2/H Let us assume a rectangular triangle ABG 13/A with angle A right, 24/B let us bisect BG at D and join AD;
Then I assert that the lines 14/A 25/B BD, DG, DA are of equal length.
Proof: Let us draw from point D a line parallel to line AB, 15/A 26/B namely line DE. Since line BD equals line DG and line DE is parallel to line AB, 16/A 27/B line AE is equal to line EG. Angle BAG is [B: assumed to be] right, thus angle h, which adjoins (23) it, is right 28/B

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(21) The marginal note in mirror-script does not belong to this proposition but to prop. 17 on the opposite page. (22) زيادة ziyāda literally: excess, surplus. (23) ولي waliya: to be near, lie next, adjoin, be adjacent. The technical term "waliya" is not often found in geometrical texts. Schramm has drawn my attention to the fact that Ibn al-Haytham in his " Sharḥ muṣādarāt Uqlīdis "[Ms. Istanbul Feyzullah 1359, 2,foll. 150v - 237r] also uses "waliya", although in a

and likewise angle $^{17}_{A}$ z. Since line AE equals line EG, line ED (24) $^{29}_{B}$ is common [to both], and angle $^{18}_{A}$ h equals angle z, base AD is equal $^{30}_{B}$ to base DG. $^{19}_{A}$ But line DG equals line DB. [A: Thus line AD equals line $^{20}_{A}$ DB.] Hence the three lines $^{31}_{B}$ AD, BD, DG are of equal length. And this is what we wanted to prove.

[Prop. 17, fig. 19] $^{21}_{A}$ Let us assume a rectangular triangle ABG with angle A right; let us extend line [AG]* $^{22}_{A}$ (rectilinearly) to point D, draw from point D perpendicular DE (25) [which meets AB at Z,] join GZ, and let the square of [BH]* be $^{23}_{A}$ equal to the product of AB and BZ;
Then I assert that when we join DH, [the square of DH is equal to the product]* $^{93v}_{A}$ of DE and ZD.
Proof: Let us join DB and extend GZ (rectilinearly) to point T. $^{2}_{A}$ Since in triangle DGB the perpendiculars DE and BA have been drawn and $^{3}_{A}$ line GZ has been extended to point T, line GT is perpendicular (26) to DB. Thus $^{4}_{A}$ the difference (27) of the square of DH over the square of HB [ms: GB] is equal to the difference of the square of DT $^{5}_{A}$ over the square of TB. The difference of the square of DT over the square of TB is like the difference $^{6}_{A}$ of the square of DZ over the square of ZB. Thus the squares of DH and BZ together (28) are like the two squares of HB [ms: GB] and DZ. Let us make the product $^{7}_{A}$ of

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slightly different context: "the angle which adjoins the inclination is smaller than the other angle" [fol. 158r, 10]. (24) not well-readable in B, because of a wurm-hole. (25) A: marginal note in mirror-script on opposite page [fol. 92v]: From the assumption of the proposition perpendicular DE meets side AB between the two points A and B. (26) A margin fol. 93v: The proof of this is: When we join line AE, quadrilateral AZEG will fall in a circle and quadrilateral ADBE will also fall in a circle. Thus angle EAZ is the same as angle EGZ and it is also the same as angle EDB. Thus angle EGZ is the same as angle TDZ and angle GZE is the same as angle DZT. Hence angle GEZ, which is right, remains the same as angle DTZ, and so it is right. Or we say angle AED is the same as angle AGZ and the same as angle ABD. Thus angle AGZ is the same as angle ABD and the two opposite angles Z are equal to each other. Thus angle GAB, which is right, remains the same as angle ZTB, and so it is right. Hence line GZT is perpendicular to DB. (27) compare with prop. 14. (28) "That is because the sum of the two extremities in this numerical relation is the same as the sum of the two mediums." on fol. 93v left margin.

DZ and ZE common [to both], which is like AZ times ZB. Because angle A is right and angle E is right, thus the points E, B, $^{8}_{A}$ A, D coil a circle. Hence [the two lines]* have to meet within a circle. Therefore the product of AB and BZ $^{9}_{A}$ plus the square of DH [ms: ZH] is equal to the product of DE and DZ plus the square of BH. And the product of AB and BZ equals the square $^{10}_{A}$ of BH. Consequently, the product of DE and DZ equals the square of DH. And this is what we wanted to prove. .˙.

[Prop. 18, fig. 20] $^{11}_{A}$ Let us assume the lines (29) AB and BG and assume on line AB an arbitrary (30) point, namely $^{12}_{A}$ point D, and let the square of AB be equal to the square of AD plus the square of BG. Let us join DG and bisect it $^{13}_{A}$ at point E, and join AE;
Then I assert that angle DAE equals angle DGB.
$^{14}_{A}$ Proof: Let us extend BA to point Z, and let ZA be equal to line AD. Since line ZD $^{15}_{A}$ has been bisected at point A, and [AB] is greater in length than AD [ms: DB], the product of ZB and BD plus the square of AD $^{16}_{A}$ is equal to the square of AB. But the square of AB is assumed to be equal to the two squares of DA and BG. Thus, when we have discarded the common square (31) of AD, $^{17}_{A}$ the product of ZB and BD remains equal to the square of BG. Thus, when we join ZG, angle $^{18}_{A}$ BZG is equal to angle BGD. And angle BZG equals angle DAE, because $^{19}_{A}$ AE is parallel to GZ. Hence angle DAE equals angle DGB. $^{20}_{A}$ And this is what we wanted to prove. .˙.

[Prop. 19, fig. 21] $^{14,15}_{H}$ $^{21}_{A}$ Let us assume $^{144r}_{B}$ an isosceles triangle ABG [A: with equal sides AB and AG,] $^{22}_{A}$ and let us draw from point A to line BG an arbitrary line, namely line $^{2}_{B}$ AD;
Then I assert that $^{23}_{A}$ the product of BD and DG plus the square of DA equals the square of AG.
$^{15}_{H}$ Proof: Let us draw from $^{3}_{B}$ point A $^{94r}_{A}$ to line BG perpendicular AE. Since line BG has been divided in two halves at point E and in two $^{4}_{B}$ different parts $^{2}_{A}$ at point D, the product of BD and DG plus the square of ED is equal to the square of EG. $^{5}_{B}$ Let us add $^{3}_{A}$ the square of AE common [to both]. Thus the product of BD and DG plus the squares of AE and DE $^{4}_{A}$ is equal $^{6}_{B}$ to the squares of AE and EG. But the squares of AE and EG equal $^{5}_{A}$ the square of AG, because angle $^{7}_{B}$ AEG is right. And the squares of DE and EA equal the square of AD [B: because an-

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(29) slight change in the ending of the word made by redactor.
(30) not really arbitrary. (31) "square" added by redactor.

gle 8/B AEG is right]. Hence the product of BD and DG 6/A plus the square of DA equals the square of AG. 9/B And this is what we wanted to prove.

[Prop. 20, fig. 22] 7/A Let us assume an isosceles triangle ABG [A: with equal sides AB and AG], 8/A and let us draw 10/B from point A two lines, namely lines AD and AE; let the ratio of the product of BD and DG to 9/A the square of DA be like the ratio 11/B of the product of GE and EB to the square of EA;
Then I assert that line DA equals line 10/A AE.
Proof: Because 12/B the ratio of the product of BD and DG to the square of AD is the same as the ratio of the product of GE and EB 11/A to the square 13/B of AE, therefore [A: I assert], when we compose, then the ratio of the product of BD and DG plus the square of DA 12/A to the square of AD is like the ratio of the product 14/B of GE and 16/H EB plus the square of AE to the square of EA. But the product 13/A of BD and DG plus the square of DA equals the square 15/B of AB, and the product of GE and EB plus the square of EA 14/A equals the square of AG. Hence the ratio of the square of BA to the square 16/B of AD is like the ratio of the square of GA 15/A to the square of EA. The front parts are equal to each other, thus the succeeding [parts] 17/B [A: likewise] [B: are consequently equal to each other]. Hence line DA equals 16/A line EA. And this is what we wanted to 18/B prove.

[Prop. 21, fig. 23] Let us assume a triangle ABG 17/A and bisect angle A 19/B with line AD;
Then I assert that the ratio 18/A of the two lines BA and AG together to line GB is the same as the ratio of AB to BD.
Proof: 19/A 20/B Since angle A from triangle ABG has been bisected by line AD, the ratio of BA to AG is 20/A 21/B the same as the ratio of BD to DG (32). When we alternate, the ratio of AB to BD is the same as the ratio 21/A of AG 22/B to GD. The ratio of the whole to the whole is [A: the same as] [B: like the ratio of] the single to the single. Hence the ratio 23/B of the two lines BA and AG to 22/A line BG is the same as the ratio of AB to BD. And this is what we wanted 24/B to prove.

[Prop. 22, fig. 24] Let us assume a triangle ABG, 23/A extend lines GA and BA (rectilinearly) to 25/B points D and E, join DG and BE, draw 17/H

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(32) A margin fol. 94r: This because of the third proposition in the second chapter from the third classification from the first part from mathematical sciences ... cf. Chapter III, prop. 21 and prop. 43.

from $^{94v}_{A}$ point E a line parallel to line DG, namely $^{26}_{B}$ line EZ, draw from point D a line parallel to line $^{2}_{A}$ EB, namely line DH, and join ZH [A: GZ] (33);
Then I assert $^{27}_{B}$ that it [B: line ZH] is parallel to line BG.
Proof: $^{3}_{A}$ Let us join ZG, HB [A: GB], DE. Thus triangle DEG [B: ZEG] is equal $^{28}_{B}$ to triangle DZG, because both cover the same base, [B: namely line DG [ms. ZG],] and both [lie] between $^{4}_{A}$ two lines parallel to each other, [B: namely the lines] $^{29}_{B}$ DG and EZ. Triangle DAG common [to both] is discarded, thus the remaining triangle DAE is clearly equal $^{5}_{A}$ to the remaining triangle GAZ. $^{30}_{B}$ Triangle DEB equals triangle EHB, because both cover the same base, [B: namely line EB,] $^{31}_{B}$ and [lie] between two lines $^{6}_{A}$ parallel to each other, [B: namely] EB and DH. Triangle EAB common [to both] is discarded, thus the [B: remaining] triangle DAE $^{144v}_{B}$ is equal to the [B: remaining] triangle ABH. $^{7}_{A}$ [B: But it has already been proved $^{2}_{B}$ that] triangle DAE equals triangle AGZ [B: GAB], thus triangle ABH equals $^{3}_{B}$ triangle AGZ. $^{8}_{A}$ Triangle AZH common [to both] is discarded, thus $^{4}_{B}$ the remaining triangle BZH [A: GZB] is equal to triangle HZG. $^{9}_{A}$ Both cover the same base $^{5}_{B}$, namely ZH, thus both [lie] between two lines parallel to each other. $^{10}_{A}$ $^{6}_{B}$ Hence line ZH [A: GZ] is parallel to line BG. And this is what we wanted to prove.

[B: $^{144v,14}_{B}$ Finished is the book of Archimedes on the Elements of Geometry $^{15}_{B}$ which consists of twenty (34) propositions. Praise be to Allah, $^{16}_{B}$ and his benedictions on his prophet Mohammed and his family.]

<u>From here on the propositions only appear in treatise A.</u>

[Prop. 23, fig. 25] ^{11} Let us assume a triangle ABG with lines AD and EB equal to each other ^{12} and AZ and GH also equal to each other, and join GE, GD, BZ, BH. Let BZ and GE meet at point W ^{13} and BH and GD meet at point S, let us join AS and extend it (rectilinearly) to Y and likewise ^{14} AWT;
Then I assert that line TB equals line GY.
Proof: Let us construct through point ^{15} A a line parallel to line BG, namely line KL, and extend lines BZ, BH, GD, GE to L, M, N, K.(35)

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(33) Several times in this proposition it is ambiguous whether in A "G" or "H" is written. (34) cf. Chapter I,3. (35) ms. LMNK. The points are also not in the correct order, it should be M, L, N, K.

[16] Line AZ equals line HG and HZ is common [to both], so line AH equals line GZ, and the ratio of AH to HG is the same as the ratio [17] of GZ to ZA. But the ratio of AH to HG is the same as the ratio of AL to GB, and the ratio of GZ to ZA is the same as the ratio of BG [18] to AM. Therefore the ratio of AL to [19] BG is the same as the ratio of BG to AM. Thus the product of AL [20] and [AM]* equals the square of BG. And in the same manner (36) we prove that the product [21] of [KA]* and AN equals the square of BG. Thus the product of LA and AM equals [22] the product of KA and AN, and so the ratio of LA to AN is the same as the ratio of KA to AM. [23] [The ratio]* of KA to AM is the same as the ratio of GT to TB and the ratio of AL to AN [96r] (37) is the same as the ratio of BY to YG [ms: BG]. Thus the ratio of GT to TB [ms: TE] is the same as the ratio of BY to YG [ms: BG]. Therefore when we compose, the ratio of GB [2] to BT is the same as the ratio of BG to GY. Thus the ratio of BG to both of the lines BT and GY is the same. Line [3] BT therefore equals line YG. And this is what we wanted to prove.

[Prop. 24, fig. 26] Let us assume a rectangular triangle [4] ABG with right angle B, let angle AGB be bisected by line GD, [5] and we assume angle DAE equal to one of the two angles AGB;
Then I assert that line GB is greater than [6] line GE.
Proof: Because angle DGA equals angle DAE, the product of GD and ED is [7] equal to the square of DA. The square of DA is greater than the square of DB because the ratio of AD to DB is the same as the ratio of AG [8] to GB, and line AG is greater than GB, hence line AD is greater than DB. We assume the product of GD and DZ [9] to be equal to the square of DB (38), and we join ZB. Angle ABZ equals angle ZGB and angle [10] ZDB is common to the triangles BDZ and BGD. Thus angle DBG is the same as angle DZB, and so both are right. [11] Thus line BG is greater than GZ, consequently it is much greater than GE. And this is what we wanted [12] to prove.

[Prop. 25, fig. 27] Let us assume a rectangular triangle with right angle [13] B. We divide angle G in two parts by line GD, with angle [14] BGD twice the size of angle DGA;
Then I assert that the product of BG and GA is greater [15] than the square of GD.

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(36) margin: "what way?" (37) fol. 95 does not belong to this treatise, cf. Chapter I,1. (38) The proof would be easier to understand with an insertion. cf. the proof written in symbols in chapter III.

Proof: When we assume [E, such that] the product of GA and GE [16] is equal to the square of GD, and join ED, angle DAG equals angle EDG. [17] Angle GDB is then equal to the two angles EDG and AGD, and so it is greater than angle EDG. Let angle [18] GDZ be equal to angle EDG, and let us bisect angle DGB with line GZ, [19] then line EG equals line GZ. Because angle BDG equals angles EGD and EDG, that which [20] is like angle GDZ, the common angle GDZ is discarded, and the remaining angle BDZ is equal [21] to the remaining angle EGD. Angle EGD equals angle DGZ, thus angle BDZ equals [22] angle ZGD, and thus line BG is greater than line ZG, because we have proved it previously. Line GZ equals [23] line GE, thus line BG is greater than line GE, and the product of BG and AG is greater than [96v] the product of AG and GE. The product of AG and GE equals the square of GD, hence the product of BG and GA is greater than the square of GD. [2] And this is what we wanted to prove. .˙.

[Prop. 26, fig. 28] Let us assume a semicircle through A, B, G, D, assume arc [3] AB to be like arc BG, join GD, and draw from B a line which is perpendicular [margin: to AD], namely line BE;
Then I assert that [4] line GD is shorter than line ED.
Proof: We join BD and draw from G a perpendicular to [5] AD, namely perpendicular GZ, which cuts through line BD at H. Thus AD times DZ is the same as BD times DH. [margin: For the triangles ADB and ZDH are similar to each other.] [6] And AD times DZ is the same as DG times itself. Hence DG times itself is the same as BD times DH. And also, [7] since angle ADB is like angle BDG and triangle DGZ is rectangular, [8] DG times DZ must be more than DH times itself. Thus the ratio of the square of DG to DG times DZ is smaller [9] than the ratio of the square of DG to the square of DH. The square of DG (,this) is the same as BD times DH. Thus the ratio of the square [10] of DG to DG times DZ is smaller than the ratio of BD [11] times DH to the square of DH. Consequently it is proved that the ratio of DG to [12] DZ is smaller (39) than the ratio of BD to DH. And the ratio of BD to [13] DH (,this) is the ratio of DE to DZ. Thus the ratio of DG tó DZ [14] is smaller than

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(39) margin fol. 96v: That because the "heights" (ارتفاع irtifā^{c}: elevation, rise) of the square of DG and the product of DG and DZ are equal. Thus the ratio of the square to the product is the same as the ratio of base DG to base DZ. And likewise because the heights of the product of BD and DH and the square of DH are equal. Hence the ratio of the product to the square is the same as the ratio of BD to DH.

the ratio of DE to DZ. Hence DG is smaller than DE. And this is what we wanted to prove. .˙.

[Prop. 27, fig. 30] [15] Let us assume a triangle ABG, bisect line BG at point D, and join AD; draw from [16] point B an arbitrary line to line AG, namely line BZE, join GZ and extend it (rectilinearly) [17] to point H, and join HE (40);
Then I assert that line HE (40) is parallel to line BG.
Proof: Let us construct through point [18] A a line parallel to line BG, namely line TAK, and extend [19] lines BE and GH (rectilinearly) to points T and K. Since line [20] GD equals line DB, line TA is equal to line AK. Thus the ratio [21] of GB to TA is the same as the ratio of GB to AK. But the ratio of BG to TA [22] is the same as the ratio of BH to HA (41), because lines TA and BG are parallel to each other and [23] lines AB and GT lie in between. The ratio of GB to AK is the same as the ratio of GE to EA. Thus the ratio of BH to HA (41) is the same as the ratio [97r] of GE to EA. Hence line HE (40) is parallel to line BG. And this is what we wanted to prove. .˙.

[Prop. 28, fig. 31] Let us assume a semicircle [2] through A, B, G, extend BG (42) [margin: to E], bisect arc BG at point A, and join GA [ms: GE] and AE [, which meets the semicircle at D];
Then I assert that [3] the product of EA and AD equals the square of AG.
Proof: Let us join AB, DG, DB. Since arc AB [4] is like arc AG, angle ABG is equal to angle AGB. And since angle ADB is the same as angles DBE and [5] DEB, and angle ADB equals angle AGB because both of them span the one arc AB, [6] angle ABG [ms: AGB] is equal to angles DBE and BED. And when [7] we discard the common angle DBE, the remaining angle ABD is [8] the same as the remaining angle DEB. Thus the product of AE and [9] AD equals the square of AB. And AB equals AG, hence the product of EA and AD equals the square of AG. And this is [10] what we wanted to prove. .˙.

[Prop. 29,1, fig. 32] Let us assume two intersecting circles through A, E, D and Z, H, G (43), draw [11] line Z[B]EH, which cuts through the circles arbitrarily, and join ZA, AB, ED, DH;
Then I assert that angle [12] ZAB equals angle EDH.
Proof: Let us join AD. Since figure ADBE is a quadri-[13]-lateral in a

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(40) could also read "GE". (41) could also read "GA". (42) "BG" difficult to read, probably changed by redactor. (43) D = G .

circle, angles BAD and BED are equal to two right [angles], and [since figure AZHD is a quadrilateral in a circle, angles ZAD and ZHD are also equal to two right angles. Thus] angles ZAD [14] and ZHD are equal to angles BAD and BED. And when we discard the common angle BAD, angles [15] ZAB and EHD remain equal to angle BED. Angle BED equals angles EHD and EDH. [16] Thus angles ZAB and EHD are equal to angles EHD and EDH. And when we discard the common angle EHD, [17] the remaining angle ZAB is the same as the remaining angle EDH. And this is what we wanted to prove. .˙. .˙.

[Prop. 29,2, fig. 32] [18] Let us keep the picture, extend lines HD and ZA to points T and K, and join BK and ET [meeting at point L];
Then I assert [19] that line BL equals line LE.
Proof [1]: Since [20] angle ZAB equals angle EDH, angle [21] ZAB, which is exterior to quadrilateral [22] ABET, equals angle TEB, and angle [23] EDH, exterior to quadrilateral BKED, [97v] equals angle KBH, angle TEB is equal to angle EBL. Consequently line BL equals [2] line LE. .˙.
[Proof 2:] Also, since angle ZAB equals angle EDH, angle BAT remains equal [3] to angle EDK. And since quadrilateral ABTE lies in a circle, angles BAT [4] and TEB are two right [angles]. And since quadrilateral EDBK lies in a circle, angles EDK and EBK are [5] two right [angles]. Thus angles BAT and TEB equal angles EDK and EBK. Angle BAT [6] equals angle EDK, thus angle EBL remains the same as angle BEL. Hence line BL is the same as line LE. And this is [7] what we wanted to prove.

[Prop. 30,1, fig. 33] Let us assume a rectangular triangle ABG, with its angle A right; [8] let us extend line BA at side A, be its extension line AD, and draw (44) from point D line [9] DE perpendicular to line BG [,meeting AG at Z];
Then I assert that the product of BD and DA equals the product of GZ and ZA plus the square [10] of ZD.
Proof: Since angle BAG is right and likewise angle ZEB, points B, A, E, Z lie [11] on the circumference of a circle. Thus the product of BD and DA equals the product of ED and DZ. But the product of ED and DZ [12] equals the product of EZ and DZ plus the square of ZD. And the product of EZ and DZ equals [margin:, for triangles GEZ and DAZ are similar to each other,] the product of GZ and ZA. .˙.
[Prop. 30,2, fig. 33] [13] Conversely: let the product of BD and DA be equal to the product of GZ and ZA plus the square of ZD;
Then I assert that angle [14] DEB is right.

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(44) ms. اخرج instead of لنخرج.

Proof: Since the product of BD and DA equals the product of GZ and ZA plus the square [15] of ZD, and the square of ZD equals the squares of DA and AZ, the product of BD and DA is equal to the product of GA and AZ [16] plus the square of AD. When we discard the common square of DA, the product of BA and AD is equal to the product [17] of GA and AZ, and the ratio of BA to AG is the same as the ratio of ZA to AD. Thus triangles BAG and DAZ [18] are similar to each other, and angle BDE is the same as angle AGB. Angle EZG is the same as angle [19] AZD, and so the remaining angle DAZ is the same as the remaining angle ZEG. Angle [20] DAZ is right, thus angle DEG is right. Hence angle DEB, which adjoins (45) it, is right. And this is what we wanted to prove.

[Prop. 31, fig. 34] [21] Let us assume a circle through A, B, T, D, draw the tangents EA and DE to it, join AD, draw line [22] EZ arbitrarily, draw [margin: on] line EZ perpendicular ZT which meets AD at point Y, [margin: and join] EY and ZD;
Then I assert [23] that angle DEY is twice the size of angle DZT.
Proof: Let us extend EZ (rectilinearly) to [98r] point B, join BT, draw line DL perpendicular to line DE, and extend line EY to point [2] K. Since angle BZT is right, BT is a diameter. Line DL is also a diameter since it has been drawn [3] on line ED in a right angle, point L is consequently the center of the circle. Hence angle DLT is twice the size of [4] angle DZT. Thus angle DLT has to be equal to angle DEY. When points [5] K, D, E, L lie on the circumference of a circle, this is true. And they are on the circumference of a circle when [6] both angles EDL and EKL are right. This is the case, since triangle AED is isos-[7]-celes and in it has been drawn line EY. Thus the product of AY and YD plus the square of EY equals the square [8] of AE. And the square of AE equals the product of BE and EZ. Hence the product of AY and YD plus the square of EY is [9] equal to the product of BE and EZ. The product of AY and YD equals the product of TY and YZ. Consequently the product [10] of TY and YZ plus the square of YE equals the product of BE and EZ, thus angle TKY is right, since [11] this has been proved in the preceding proposition (46). .˙.
We object, however, that line EY might be extended [12] to point L, or [to a point] in between L and B. .˙.
The answer is that when it is extended to point L, [13] angle ELT is right, according to the preceding proof, and equal to angles ELD and DEL. Thus we drop [14] the common angle ELD, and angle DLT remains the

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(45) cf. prop. 16. (46) namely prop. 30,2.

same as angle DEL. .˙.
Likewise, when it [= the extension of EY] falls between [15] L and B, it is also in a right angle [for instance when] (47) it falls in M. Then we join AM. Points A, E, L, M lie on [16] the circumference of a circle, since angle LME is the same as angle LAE. Thus the circle passing through points [17] L, E, M passes through point A. Angle ALB is the same as angle [18] AEM, since they stand on the same arc. [19] Thus angle MED (48) remains [20] the same as angle TLD, because the whole angle [21] AED is the same as angles ALB and DLT, [22] since angle AED plus angle ALD equal two right [angles], [23] and likewise angles ALB and DLT plus angle ALD equal two right angles. [And likewise angles [98v] ALB and DLT plus angle ALD equal two right [angles].] (49)
And this is what we wanted to prove. .˙.

[Prop. 32, fig. 35] [2] Let us assume a semicircle through A, B, G, and draw from point E a tangent, namely line AE, [3] and another tangent, namely line EB. Let us extend lines EB and AG until they meet at point D, draw [4] from point B a line parallel to line GA, namely line BZ, join DZ and AB, [margin: thus they meet at point H,] and draw from point [5] H a perpendicular to line AG, namely line HT;
Then I assert that point T is the center of the circle.
Proof: [margin: Let us join GH and YH] [, Y being the intersection point of line BZ and the circle]. [6] Since both lines AE and EB touch the circle, they are equal to each other. Hence the ratio of DE to EB is the same as the ratio [7] of DE to EA. But the ratio of DE to EA is the same as the ratio of DB to ZA, and the ratio of DE to EB is the same as the ratio of DA (50) [8] to ZB. Thus the ratio of DA to BZ is the same as the ratio of DB to ZA. When we alternate, the ratio of AD to DB is [9] the same as the ratio of BZ to ZA. Thus the ratio of the square of AD to the square of DB is the same as the ratio of the square of BZ to [the square of] (47) ZA. The square of BD [10] equals the product of AD and DG and the square of ZA equals the product of BZ and ZY. Thus the ratio of the square of AD to [11] the product of AD and DG is the same as the ratio of the square of BZ to the product of BZ and ZY. Consequently the ratio of line AD to DG is the same as the ratio [12] of line BZ to ZY.

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(47) inserted by redactor. (48) changed by redactor. (49) repetition, crossed out by redactor. (50) changed by redactor into "DB", with addition in margin: "to ZA and the ratio of DE to EB is also the same as the ratio of DA to".

The following text is rightly crossed out by the redactor:
"Thus when we separate, the ratio of AG to GD is the same as the ratio of BY to YZ. And when we alternate, [13] the ratio of AG to BY is the same as the ratio of GD to YZ. The ratio of GD to YZ is the same as the ratio of DH to HZ, since [14] the ratio of AD to DG is the same as the ratio of BZ to ZY and when we alternate, the ratio of AD to BZ is the same as the ratio of GD to [15] [YZ]*, but the ratio of AD to BZ [ms: YZ] is the same as the ratio of DH to HZ, since BZ is parallel to DA. Thus (triangle [..H]* is similar [16] to triangle [...]* and) the ratio of GD to YZ is the same as the ratio of DH to HZ, or the ratio of GD to DH is the same as the ratio [17] of YZ to ZH. These lines enclose equal angles, namely angles GDH and YZH. [18] Hence the triangles are similar to each other, and angle DHG equals angle YHZ."
This is replaced in margin by:
"Through alternation the ratio of AD to BZ is the same as the ratio of DG to YZ. The ratio of AD to BZ is the same as the ratio of DH to HZ. Thus the ratio of DG to YZ is the same as the ratio of DH to HZ. And angles GDH and YZH are equal to each other. Therefore triangles GDH and YZH are similar to each other, and angle GHD equals angle YHZ."
Now we continue with the main text:
Let angle YHD be [19] common, thus angle ZHD is equal to angle YHG. Angle [20] ZHD equals two right ⌊angles⌋, therefore so does angle YHG. [21] Thus line YHG is rectilinear. Since angle [22] BYH equals angle HGA, and angle BYH equals [23] angle BAG because they stand on the same arc, namely arc BG (51), angle BAG is equal to angle [99r] HGA. Hence line AH equals line HG. Line HT is assumed perpendicular to line AG, and so line AT equals line [2] TG. Line AG is a diameter of the circle, consequently point T is its center. And this is what we wanted to prove.

[Prop. 33, fig. 36] [3] Let us assume a triangle ABG, with line AG bisected at point D. Let us extend line BA [4] (rectilinearly) to point E and join EDZ;
Then I assert that the ratio of BE to AE is the same as the ratio of BZ [5] to ZG.
Proof: Let us draw from point A a line parallel to line BG, namely line AT. [6] Since it is parallel to line BG and line AD equals line

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(51) margin: Had he said, since line BY is parallel to line AG, arc AY is equal to arc BG, hence angle A equals angle G, it would have been more proper.

DG, line AT is equal to line ZG. [7] Because in triangle EBZ line AT has been drawn parallel to the base which is BZ, the ratio [8] of BE to EA is the same as the ratio of BZ to AT. It has already been proved that line AT equals line ZG, [9] thus it is obvious that the ratio of BE to EA is the same as the ratio of BZ to ZG. And this is what we wanted [10] to prove.

[Conversely:] Let us assume that the ratio of BE to EA is the same as the ratio of BZ [11] to ZG;

Then I assert that line AD equals line DG.

Proof: [12] The ratio of BE to EA is the same as the ratio of BZ to AT and line AT is parallel to line ZG. Therefore line AD is equal [13] to line DG. And this is what we wanted to prove. .˙.

[Prop. 34, fig. 37] Let us assume a triangle ABG, extend line BA [14] at side A, and let its extension be AE. Let us divide line BG in two halves at point Z and line AG [ms: AB] in two sections [15] at point D so that the ratio of BE to EA is the same as the ratio of GD to DA;

Then I assert that the line which [16] connects the points E, D, Z is rectilinear.

Proof: Let us construct through point A a line parallel [17] to line BG, namely line AT, and join line ETD. Since the ratio of BE to EA is the same as the ratio of GD to DA, [18] the ratio of GD to DA the same as the ratio of GZ to AT and line GZ equal to line BZ, the ratio of BE to EA is [19] the same as the ratio of BZ to AT. Let us draw from point B a line parallel to line ET, [20] namely line BY, and extend TA to Y, then the ratio of BE to EA is [21] the same as the ratio of YT to TA. The ratio of BE to EA is the same as the ratio of BZ to AT, thus line YT [22] equals line BZ. It is parallel to it, and line DE has been assumed parallel [23] to line BY, consequently line EZ is rectilinear (52). And this is what we wanted to prove.

[Prop. 35, fig. 38] (53) Let us assume a line AB [99v] with a point D above it, and draw [from point D] two [half-]lines which are both parallel to line AB, namely lines [2] ED and DZ;

Then I assert that lines ED and DZ have been joined in a straight line.

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(52) margin fol. 99r: If ZT were not to join ED in a straight line, they would have to be parallel, because both of them are parallel to BY and they meet in point T. (53) margin fol. 99v top: This proposition should precede the preceding proposition.

Proof: Let us assume on line BA an arbitrary point, namely point G and join GD. Since line ED is parallel to line AB, angles EDG and DGA are equal to two right [angles]. Likewise, since AB is parallel to line DZ, angles DGB and ZDG are together equal to two right [angles], and so the four angles are the same as four right angles. Two of these [margin: namely the angles G] [are the same as] two right [angles], [therefore] two [margin: the others, namely the angles D] remain [the same as] two right [angles]. Hence line EDZ is rectilinear. And this is what we wanted to prove. .˙.

[Prop. 36, fig. 39] Let us assume three lines AB, AG, AD, draw to these from point E the lines EZ [,meeting AD, AG, AB at L, T, Z] and EB [,meeting AD, AG at D, G], and let the ratio of EZ to ZT be the same as the ratio of EL to LT;
Then I assert that the ratio of EB to BG is the same as the ratio of ED to DG.
Proof: Let us construct through point T a line parallel to line BE, namely line HTY. Since in triangle EZB line HT has been drawn parallel to its base, the ratio of EZ to ZT is the same as the ratio of EB to HT. The ratio of EZ to ZT is the same as the ratio of EL to LT, and the ratio of EL to LT is the same as the ratio of ED to TY, because line TY is parallel to line DE. (54) Thus the ratio of EB to HT is the same as the ratio of ED to TY. And when we alternate, the ratio of EB to ED is the same as the ratio of HT to TY. Since in triangle ABD line AG has been drawn arbitrarily, and line BD parallel to line HY, the ratio of HT to TY is the same as the ratio of BG to GD. It has already been proved that the ratio of HT to TY is the same as the ratio of BE to ED, thus the ratio of BE to ED is the same as the ratio of BG to GD. And when we alternate, then the ratio of EB to BG is the same as the ratio of ED to DG. And this is what we wanted to prove.

[Prop. 37,1, fig. 40] Let us assume a rectangular triangle ABG, with angle A right. [100r] In it lines AD and AE have been drawn, with angle DAE equal to angle ABG, and there has been drawn from point B to line AE perpendicular BZ, which [meets AD at T, and] is extended to point H [on AG];

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(54) l. 14 (last two words), l. 15, l.16, l. 17 (first four words) have been crossed out, probably by the redactor. They contain the same argument as has already been written in the previous lines.

Then I assert that the product of DA and AT plus the product of GB (55) |3 and BD equals the product of HB (55) and BT plus the product of GA and AH.
Proof: Since angle DAE equals |4 angle ABG and we add the common angle BAD [to both], angle BAE is the same as angle ADG. Because angle AZH |5 is right, the remaining angles ZAH and AHZ are equal to one right [angle]. Angle BAH is assumed |6 right, and so angles ZAH and ZHA equal angle BAG. We discard the common angle ZAH, thus angle |7 BAZ remains the same as angle AHZ. It has been proved that angle ADG equals angle BAZ, therefore angle |8 AHZ is equal to angle ADG. [margin: Hence angle GHT plus angle GDT is the same as two right [angles].] Thus the points H, T, D, G lie on the circumference of a circle, and the product of DA and AT is |9 equal to the product of GA and AH, and the product of HB and BT equals the product of GB and BD. It follows that the product of DA and AT |10 plus the product of GB and BD equals the product of HB and BT plus the product of GA and AH. .˙. .˙.
[Prop. 37,2, fig. 40] Let us keep the picture in the same position; |11 Then I assert that the product of GB and BD plus the product of GA and AH equals the square of AB.
Proof: [Let |12 us assume the product of GB and BD to be equal to the product of AB and BY and let us join TY. The product of GB and BD is certainly |13 equal to the product of HB and BT, hence the product of AB and BY is equal to the product of HB and BT.] (56) Thus points |14 A, H, Y, T lie on the circumference of a circle. [Therefore angle BYT equals angle AHT. It has already been proved that angle AHT |15 equals angle ADG, and so angle BYT equals angle ADG,] (56) thus (56) points

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(55) The difference between HB and GB in this proposition often has to be deduced from the context. (56) crossed out in ms., with "thus" being changed into "and". The substitute text in the margin reads [fol. 100r, 2. marginal note]: Let us draw from point T perpendicular TY on line BH. Since angle ATB is obtuse, because angle ATH is acute, point Y falls between points A and B. Hence rectangular triangle ABH is similar to rectangular triangle YB[T]*. Line TY is drawn in it parallel to ZA. Because the ratio of HB to BA is the same as the ratio of Y[B]* to BT, the product of HB and BT is the same as the product of AB and BY. [Insertion: The product of HB and [B]*T is the same as the product of GB and BD, thus the product of AB and BY is the same as the product of GB and B[D]*] And angle TYB is equal(?) to angle(?) AH[B]* ... angle [.]DB is the same as two right[angles]. Likewise angle AYT plus angle AHT is the same as two right [angles].

B, Y, D, T lie on [16] the circumference of a circle. So the product of BA and AY equals the product of DA and AT. And the product of DA and AT equals the product of GA [17] and AH. It has already been proved (57) that the product of GB and BD equals the product of AB [18] and BY, hence the product of GA and AH plus the product of GB and BD is equal [19] to the product of AB and BY plus the product of AB and AY. The product of BA and AY [20] plus the product of AB and BY is like the square of AB, the square of AB therefore is equal to the product of GB and BD [21] plus the product of GA and AH. And this is what we wanted to prove.

[Prop. 38,1, fig. 41] Let us assume three lines of equal length [22] AB, AG, AD, and join DG, GB, BD;
Then I assert that angle GBA plus angle GDB is the same as a right [angle].
[23] Proof: Let us draw line AZ [in the extension of BA,] equal to one of them, and join GZ. Since lines ZA, GA, BA [100v] are of equal length, angle ZGB is right, and angles [G]Z[B] and [G]B[Z] are the same as a right [angle] [2] (, because lines ZA, GA, BA are of equal length). Thus points Z, G, B lie [3] on the circumference of a circle. It has already been assumed that lines DA, GA, [4] BA are of equal length, thus points D, G, B, Z lie on the circumference of a circle. [5] Thus angle BZG equals angle GDB. It has already been proved that angle GZB plus angle GBZ equals one right [angle], [6] and so angle GDB plus angle GBZ is right.
[Prop. 38,2, fig. 41] Also, let lines BA and AG be of equal length, [7] and angle GBA plus angle GDB be equal to one right [angle];
Then I assert that lines AB, AG, [8] AD are of equal length.
Proof: Let us extend line BA to point Z, let line AZ be equal to one [9] of the lines AB or AG and let us join GZ. Thus angle ZGB is right, and the angles GZB and GBA together are right. [10] Angles GBZ and GDB are assumed as one right [angle], angle GDB therefore equals angle GZB. [11] And so points G, D, Z, B lie on the circumference of a circle. Lines ZA, BA, GA are of equal length, hence point A is the center of the circle. [12] AD then equals each of AG and AB. And this is what we wanted to prove.
[Prop. 38,3, fig. 41] Moreover, let DA and BA be [13] of equal length, and the angles GBA and GDB be equal to one right [angle];

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(57) "proved" should be "assumed", cf. l. 12. Only in the case of the marginal text "proved" is correct.

Then I assert that lines DA,[14] GA, BA are of equal length.
Proof: Let line AZ be equal to each of the lines BA and AD, and let us join [15] DZ. Thus angle B[DZ]* is right, thus the two angles [D]*ZB [and DBZ] are the same as a right [angle]. Angles GBA and GDB are assumed [16] equal to a right [angle]. We discard the common angle DBZ, and angles GBD and GDB remain equal [17] to angle D[Z]*B. We add the common angle BGD, then angles BGD and DZB equal [18] the angles of triangle BGD. Thus angles DZB and BGD equal two right [angles], and points B, Z, D, G [19] lie on the circumference of a circle. Lines ZA, DA, BA are of equal length, so point A is the center of the circle. Hence line GA [20] equals DA. And this is what we wanted to prove. .˙.
[Prop. 38,4, fig. 42] Also, let lines DA and GA be of equal length, [21] and the angles GBA and GDB be equal to one right [angle];
Then I assert that lines DA, GA, [22] BA are of equal length.
Proof: Let us draw from point A to GD perpendicular AE, [meeting DB at Z,] and let us join ZG. [23] Thus line DE is equal to line EG, and angle DZE is equal to angle GZE. [101r] Since angle DEZ is right, angles EDZ and EZD are equal to a right [angle]. It has already been assumed (58) [2] that angles GBA and GDB equal a right [angle], therefore angle EZD equals angle [3] GBA. But angle EZD equals angle EZG, and so angle EZG [4] equals angle GBA. Thus points Z, G, B, A lie on the circumference of a circle, and angle [5] AGB equals angle AZB. But angle AZB equals [6] angle DZE, and angle DZE equals angle GBA, hence angle AGB equals angle [7] GBA. [margin: Hence line AG is the same as BA]. And this is what we wanted to prove. .˙.

[Prop. 39, fig. 43] Let us assume triangle ABG and draw in it line [8] AD, such that angle DAG is equal to angle ABG;
Then I assert that the ratio of BG to GD is the same as the ratio [9] of the square of BG to the square of GA and the same as the ratio of the square of BA to the square of DA.
Proof: Since angle ABG is assumed [10] equal to angle GAD, and angle AGD is common to triangle ABG and triangle AGD, the remaining angle [11] BAG is the same as the remaining angle ADǴ [ms: ADB]. Thus the ratio of BG to AG is the same as the ratio of AG to GD and [the same as] the ratio [12] of AB to AD. Hence the ratio of BG to GD is the same as the ratio of the square of BG to the square of GA and the same as the ratio of the square of BA [13] to the square of AD. And this is what we wanted to prove. .˙.

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(58) ms.: "proved". This is the same mistake as in prop. 37,2.

[Prop. 40, fig. 44] [14] Let us assume a rectangular triangle ABG, with angle [15] B right; we bisect angle A with line AD, draw line AE arbi-tra-[16]-rily, construct from point E a line parallel to line AD, name-ly line EZ, and join ZB [meeting AD at T,] and ET;
Then I assert [17] that the ratio of AZ to ZT, this is [the same as] the ratio of AE to ET. (59)
Proof: Let us make the product of AT and TY [18] like the product of ZT and TB (60), and join ZY and EY [added: and BY]. Since the product of ZT and TB equals the product of AT [19] and TY, angle ZAT is equal to angle ZBY. But angle ZAT equals [20] angle TAB, thus angle TAB equals angle ZBY.
The following text is crossed out by the redactor:
"Because the product of ZT and TB [equals]* [21] the product of AT and TY, angle BZY is also equal to angle BAT. Consequently it equals an-gle [22] [YB]*Z. Thus the product of AY and YT [ms: BT] equals the square of BY [ms: ZY]. Also, since the product of ZT and TB equals the product [23] of AT and TY, angle BAY is equal to angle [BZ]*Y. But angle BAY is the same as angle GAY, thus angle [101v] GAY is the same as angle BZY. And so the product of AY and YT (61) equals the square of ZY. It has been proved that it equals [2] the square of BY,"
This is replaced in margin by:
"Since angle TBY is the same as angle TAB and angle BYT is common [to both], the ratio of AY to YB is the same as the ratio of BY to YT. Thus the product of AY and YT equals the square of BY. Since the ra-tio of AT to TB is the same as the ratio of ZT to TY and the two an-gles T are opposite each other, hence angle TZY is the same as angle BAT or as angle TAZ. Angle TYZ is common [to both]. Therefore the ra-tio of AY to YZ is the same as the ratio of YZ to YT, thus the pro-duct of AY and YT is the same as the square of YZ,"
Now we continue with the main text:
hence line BY equals line YZ. Since angle ABD is right, [3] angles ADB and DAB are the same as a right [angle]. But angle DAB equals angle ZBY and [angle]* [4] ADB is, due to parallelism, the same as angle ZED, so angles ZBY and DE[Z]* [5] are the same as a right [angle]. Line BY equals line ZY, lines BY, [6] ZY, YE are therefore of equal length (62).

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(59) "T" is difficult to decipher. It is again written in margin where it is also difficult to read. (60) margin: Thus point Y, either it falls below point D or on it or beyond it. (61) could also read "BT". (62) margin: through the proposition conveyed on this, with part four [= prop. 38,4].

Then the product of AY and TY equals |7 the square of YE (63) [and angle YET equals angle YAE]. Thus the ratio of AY to YT (63) is the same as the ratio of the square of AE to the square of ET (64). [But]* |8 the ratio of AY to YT is the same as the ratio of the square of AZ to the square of ZT (65). Hence the ratio of AE to ET [is like]* |9 the ratio of AZ to ZT. And this is what we wanted to prove.

[Prop. 41,1, fig. 45] Let us assume a rectangular triangle A[BG]*, |10 with angle A right; we draw from point A to line BG lines AD and AE, such that angle |11 DAG is equal to angle GAE;
Then I assert that the ratio of BE to EG is the same as the ratio of BD to D[G]*.
|12 Proof: Let us construct through point G a line parallel to line AB, namely line HGZ (66) [and join DZ] (67). Since |13 angle BAG is right, angle AGH is right, thus angle AGZ, which adjoins (68) it, is right, and angle DAG [equals]* |14 angle GAE. The two triangles are therefore similar to one another, and so line GZ is the same as line HG. Thus the ratio of AB to HG is the same as the ratio |15 of BE to EG and the same as the ratio of AB to GZ. The ratio of AB to GZ is the same as the ratio of BD to DG, hence the ratio of BE |16 to EG is the same as the ratio of BD to DG. And this is what we wanted to prove. .˙.
[Prop. 41,2, fig. 45] [Conversely:] Also, let the ratio of BE to E[G]* be |17 the same as the ratio of BD to DG [,and let angle BAG be right];
Then I assert that angle DAG [ms: HAG] equals angle GAE.
Proof: |18 Let us construct through point G a line parallel to line AB, namely line HGZ. Since the ratio of BE to |19 EG is the same as the ratio of BA to HG and the ratio of BD to DG is the same as the ratio of AB to GZ, the ratio of AB to [HG]* is |20 the same as the ratio of AB to GZ, hence line GH equals line GZ. .˙. Since angle BAG|21 is right, angle AGH is right, and so angle AGZ is also |22 right. Thus lines AG and GH equal lines AG |23 and GZ, and angle AGH equals angle AGZ. [Base AH |102r equals base AZ, thus triangle GAH is the same as triangle AGZ.] (69) Therefore angle ZAG is the same as angle GAH. And this

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(63) "Y" could also read "B". (64) margin: through the preceding proposition on this [= prop. 39]. (65) margin: also through the preceding proposition [= prop. 39]. (66) could also read "GHZ", clear from drawing. (67) crossed out and replaced in margin by: "and extend AD until it meets HGZ in Z". (68) cf. prop. 16. (69) crossed out by redactor who, although this sentence is not necessary, could have kept the second part of it.

is [2] what we wanted to prove. .'.
[[Prop. 41,3, fig. 45] (70) Furthermore, let angle EAG be the same as angle GAD, and the ratio of BD to DG the same as the ratio of BE to EG;
Then I assert that angle GAB is a right angle.
Proof: Let us make in point A of line AG a right [angle] and let its new side be line AT. Either AT falls on AB, or on the side of G, or towards the opposite of it. Since the ratio of TD to DG is the same as the ratio of TE to EG, and the ratio of BD to DG is the same as the ratio of BE to EG by assumption, and by alternation of the rear parts, the ratio of EG [ms:ED] to DG is the same as the ratio of ET to TD and the same as the ratio of EB to BD. Hence the ratio of ET to TD is the same as the ratio of EB to BD. And so by comparison the ratio of ED to DT is the same as the ratio of ED to DB. Thus DT is the same as DB. Thus line AT falls exactly on line AB. It has been assumed that they do not fall together. Therefore the assumption is impossible because of the congruence of ... Thus angle GAB is consequently(?) right. And that is what we wanted.]

[Prop. 42, fig. 46] Let us assume a triangle ABG, bisect angle B with line BD [3] and angle G with line GD, and join AD;
Then I assert that angle A has also been bisected.
[4] Proof: Let us assume [on line BG] a line BE equal to line BA (71), and a line GZ equal to line GA, and join DZ and DE. [5] Since line AB equals line BE and line BD is common [to both], angle BAD is equal to angle [6] BED. Also, since line AG equals line GZ and line GD is common [to both], [7] therefore angle DAG is the same as angle DZG. Also, since line ZD equals [8] line DA, and it has already been proved that line DA equals line DE, [9] line DZ is the same as line DE, thus angle EZD equals angle ZED. But angle ZED equals [10] angle DAB, and angle EZD equals angle DAG, hence angle BAD equals angle [11] GAD. And this is what we wanted to prove.

[Prop. 43, fig. 47] (72) Let us assume a rectangular triangle ABG with [12] angle B right; let us bisect angle A with line AD, draw from point

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(70) This addition has been written in the margin of fol. 102r, probably after an improvement was made to prop. 43. The proof stands on fol. 101v, margin bottom. (71) margin: Thus point E [ms: D], either it falls between B and G, or on G, or away from it towards the opposite of B; and in this manner is the remark on point Z in relation with B.

A an [13] arbitrary line, namely line AE, draw from point E a line parallel to line AD, namely line EZ, [14] and join ZB [meeting AD at T,] and ET;
Then I assert that the ratio of AZ to ZT is the same as the ratio of AE to ET.
Proof: [15] Let us bisect angle ABZ with line BH and join ZH, then angle AZB has been bisected [16] with line ZH. We draw from point B a line which is perpendicular to BH, namely line BK, and join ZK [ms: DK], [17] EH (73), EK (74). Since angle HBK is right and angle ABH is equal to angle HBT, the ratio of KA [18] to AH is the same as the ratio of KT to TH. And since angle AZH equals angle HZT, angle HZK is [19] right; likewise angle HBK [is right], thus points B, H, Z, K lie on the circumference of a circle. Because angle [20] ABG is right and equal to angle HBK which is also right, we drop the common angle HBE, and [21] angle DBK remains equal to angle ABH which is the same as angle HBT. We add angle TBD common [to both], [22] then angle HBD becomes equal to angle KBZ which is the same as angle KHZ. We extend furthermore line HZ (rec-[23]-tilinearly) to point Y, thus angle KHZ is equal to angle EZY because of parallelism. (Angle [102v] ... (73) is certainly equal to angle DHZ.) Therefore angle [2] YZE (73) is equal to angle DBH. So points E, Z, H, [3] B lie on the circumference of a circle. [When we join EH] (75), then [4] angle HEK (73) is right, and the ratio of KA to AH [5] is the same as the ratio of KT to TH, therefore angle AET has [6] been bisected with line EH, and the ratio of AH to HT [7] is the same as the ratio of AE to ET (76). But the ratio of AH to HT is the same as the ratio of AZ to [8] ZT, hence the ratio of AE to ET is the same as the

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(72) Prop. 43 has the same assumption and assertion as prop. 40. cf. chapter III. (73) changed by redactor. (74) margin fol 102v, left bottom: We extend AD until it meets BK at K, because angle BHT is acute. Since the two angles ABZ and BAZ together are smaller than two right [angles], the two angles ABH and BAH together, which are half of the previous [ones], are smaller than one right [angle], hence angle AHB is obtuse. (75) crossed out by redactor and replaced by [fol. 102v, right margin middle]: Points B, H, Z, K lie on the circumference of a circle. Hence all the five points B, H, Z, E, K lie on the circumference of one circle. There does not exist another when the intersection of two circles is in more than two points, and points B, H [ms: changed into K(!)], Z belong to both quadrilaterals. (76) margin fol. 102v (beside the end): This because of the third proposition in the second chapter in the third classification in a book by al-Kamāl. cf.

ratio of AZ to ZT. And this is what we wanted to prove. .˙.
9 Finished are the propositions.
10 Praise be to Allah.

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*

The two remaining marginal remarks on the last page [fol. 102v] read, right upper margin: "And the amount of propositions by Aqāṭun is 43, counting improvements and additions(?)".
and on the top of the page: "By Aqāṭun. He has explained this book, he has ascertained it, and he has corrected it, may Allah, who is sublime, compensate him, the slave, the fakīr, who needs Allah, Muḥammad ibn Sartāq from Marāgha, in the madrasah al-Niẓāmiyyah (77) ...
... the meeting in the year 628. And he has solved it all in Marāgha in one night before this approximately(?). And a year follows. [Thus let his intelligence be admired and the independence of time in it be praised.](?) The End.(78)"

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prop. 21. (77) the most famous madrasah (~ religious boarding school) in Baghdād. cf Chapter II,1. (78) والسلم instead of والسلام.

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Max Krause [II], Die Sphärik von Menelaos aus Alexandrien in der Verbesserung von Abū Naṣr Manṣūr b. ʿAlī b. ʿIrāq. Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Phil.-Hist. Klasse, 3. Folge, 17 (Berlin, 1936)

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Samenvatting.

Dit proefschrift gaat over het traktaat Aya Sofya 4830,5 "Kitāb al-mafrūḍāt lī Aqāṭun" (afgekort A), bestaande uit drieenveertig stellingen over vlakke meetkunde. Negentien stellingen uit de eerste helft hiervan zijn ook zelfstandig overgeleverd in het traktaat Bankipore 2468,29 "Kitāb Arshimīdis fī'l-Uṣūl al-handasīya"(afgekort B). Hoofdstuk I beschrijft en vergelijkt de beide handschriften en concludeert, dat het niveau van traktaat A in het algemeen hoger ligt dan dat van traktaat B. Bovendien geven de kanttekeningen in A ook inzicht in de belangstelling van de ongetwijfeld intelligente bewerker. Hoofdstuk II bespreekt de twee mogelijkheden voor de titel en de naam van de auteur. Het blijkt dat "Kitāb al-mafrūḍāt" plausibel is en dat er geen reden is om een andere auteur dan Aqāṭun aan te nemen. In hoofdstuk III wordt de inhoud van traktaat A op de voor ons vertrouwde schrijfwijze weergegeven. Hierbij wordt bovendien ingegaan op de betrekkingen, die tussen A en voorafgaande of latere wiskundige werken schijnen te bestaan. Deze invloeden zijn samengevat in hoofdstuk IV, dat tot slot de conclusies over dit traktaat geeft. Als hoofdstuk V volgt dan de vertaling van de arabische tekst, die achter in dit proefschrift bijgevoegd is.

STELLINGEN

I

Van de tijdelijk onvindbare manuskripten van het Wiskundig Genootschap — daterend van 1592 tot 1896 — zijn vooral de oudere exemplaren interessant. De zeventiende-eeuwse manuskripten laten zien, hoe gretig internationale resultaten in die tijd in Nederland ingang vonden en hoeveel bekwame wiskundigen hier toen gewerkt hebben, die voor een deel weer volkomen in vergetelheid geraakt zijn.

Yvonne Dold-Samplonius, "Die Handschriften der Amsterdamer mathematischen Gesellschaft", in Janus LV (1968) 241 - 303.

II

Het valt niet te rechtvaardigen uit de inhoud, dat de verhandeling van Archimedes over rakende cirkels zoveel later uit het arabisch vertaald is dan zijn "Lemmata".

Archimedes Opera Omnia, J. L. Heiberg et al. ed., 3 vols.(Stuttgart, 1972)
Idem, vol. IV, "Über einander berührende Kreise", Yvonne Dold-Samplonius, Heinrich Hermelink, Matthias Schramm ed. (Stuttgart, 1975)

III

Of het anonieme traktaat zonder titel AS 4830,4 , dat over het verdelen van gegeven lijnen in gegeven verhoudingen gaat, werkelijk van Abū Sahl Waijan ibn Rustam al-Qūhī is, gelijk een latere kanttekening beweert, is moeilijk vast te stellen. Een zekere overeenkomst van stijl met al-Qūhī's verhandeling over de zevenhoek zou er voor kunnen pleiten.

Max Krause, "Stambuler Handschriften islamischer Mathematiker", in Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung B, vol. 3 (Berlin, 1934) 522.
Yvonne [Dold-]Samplonius, Die Konstruktion des regelmäßigen Siebenecks nach Abū Sahl al-Qūhī Waiǧan ibn Rustam", in Janus L (1963) 227 - 249.

IV

De Banū Mūsā schrijven in de "Verba Filiorum" : "This [= Archimedes'] method of finding the ratio of the diameter to the circumference, although it does not reveal a true ratio that can be reckoned with, still does yield a ratio of the one to the other which is an approxi-

mation to any limit the investigator of this subject desires".
Clagett's gevolgtrekking, dat de Banū Mūsā hiermee op de irrationaliteit van π zouden wijzen, is niet steekhoudend.

Marshall Clagett, "Archimedes in the Middle Ages", vol. I, The Arabo-Latin Tradition (Madison, 1964) 265/266, 358.

V

Het beeld dat Brecht van Galilei geeft is eenzijdig en strijdig met de feiten.

Le Opere di Galileo Galilei, Antonio Favaro ed., Vols. X - XVIII, Lettere (Firenze, 1900 - 1906)
Bertolt Brecht, "Leben des Galilei" (Frankfurt, 1962)

VI

De perfekte of konische passer (al-birkār al-tāmm), een instrument om mechanisch kegelsneden te tekenen, heeft, hoewel in enige traktaten beschreven, nooit veel toepassing gevonden.

Franz Woepcke, "Trois traités arabes sur le compas parfait", in Notices et extraits de la Bibliothèque nationale, 22, part 1 (1874)

VII

De aantekening van Schipperges "Profandarstellungen waren im Islam unnötig angesichts der Unfaßlichkeit Allahs" is een omdraaiing van de historische, werkelijke ontwikkelingsgang.

Heinrich Schipperges, "Arabische Medizin im Mittelalter", in Sitzungsberichte der Heidelberger Akademie der Wissenschaften Math.-Naturw. Klasse (1976) 2. Abhandlung, 39.

VIII

Het verdient aanbeveling het arabisch als fakultatief vak in het hoger middelbaar onderwijs op te nemen.

IX

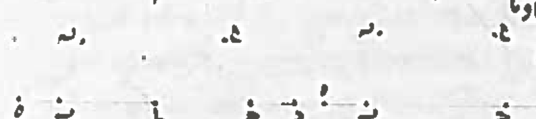
Meer begrip bij zowel de wiskundige voor de geschiedenis als bij de historicus van de exakte wetenschappen voor de wiskunde zou nuttig zijn voor beiden.

Yvonne Dold-Samplonius　　　　Amsterdam, 16 februari 1977

ARABIC TEXT

بسم الله الرحمن الرحيم ... الهندسة ... مسائل هندسية

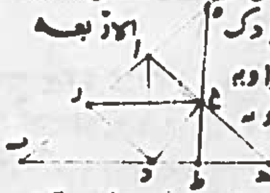
لنخرج من ... علیه اکح ولنخرج خط دح علی استقامته فی کلتی الجهتین الی نقطتی ه ز ونعرض خطی ه ح ح مساویین ولنخرج من نقطتی ه ز خطین مماسان نصف دایرة اکح وهما خطا ه ت ز ك ونصل ح ك فاقول ان خط ح ك مواز لخط ه ز : برهانه انا نستخرج مرکز دایرة اکح وهو نقطة ط ونصل به خطین فلان خط ه ح مساو لخط ح ز نجعل ط ح مشترکا فیکون خطا ه ط مساو لخط ط ز وخط ه ط مساو لخط ح ه فسطح ح ه فی ه ز مساو لسطح ز ه فی ه ح ولکن سطح ه ح فی ه ز مساو لمربع ه ت وسطح ز ه فی ه ح مساو لمربع ز ك فمربع ه ت مساو لمربع ز ك ولان خطی ه ط ط ز مساویان لخطی ز ط ط ه وقاعدة ه ت مساویة لقاعدة ز ك تکون زاویة ه ط ت مساویة لزاویة ز ط ك فقوس ت ح مساویة لقوس ح ك فخط ت ك مواز لخط ه ز وکذلك اذا کانت القطعة اقل من نصف دایرة وان کانت اکبر من نصف دایرة فقد یمکن ان یلتقی خط ح ك مع خط ه ز الی الجهة ... الاخری والبرهان واحد وقد یمکن ان یکونا متوازیین فلو کانا متوازیین وهما متساویان وعن ... الخطوط التی ... بین اطرافها متوازیة متساویة : ومن اجل ان سطح ح ه فی ه ز مساو لسطح ز ح فی ح ه وسطح ح ه فی ه ز مساو لمربع ه ت وسطح ز ح فی ح ه مساو لمربع ز ك فیکون مربع ز ك مساویا لمربع ه ت لخط ه ت مساو لخط ز ك فیخرج



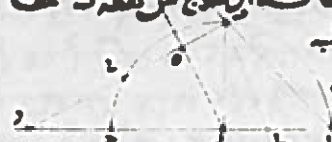
خطی ه ت ز ك حتی یلتقیا علی نقطة ط لخط ت ط مساو لخط ط ك لانهما خرجا من نقطة واحدة وهی نقطة ط مماسان الدایرة وقد کان تبین ان خط ه ت مساو لخط ز ك فخط ح ك مواز لخط ه ز وذلك ما اردنا ان نبین ... علیها اکح وعلی ه ز خطی ... ه ح ح ز مماسا ونصل ك ... ونخرج علی استقامته الی نقطة ه ولنخرج من نقطة ه خطا ... الدایرة ویلقی خطا ه ت علی نقطة ت وهو خط ه ت ك فاقول ان نسبة

ط ه الی ه ز کنسبة ط ك الی ك ز برهانه انا نخرج من نقطة ز خطا موازیا لخط ه ت وهو خط ز ح فنسبة ط ه الی ه ز کنسبة ط ح الی ز ح ولکن خط ز ح مساو لخط ح ك لخط ح ز مساو لخط ز ح ومن اجل ان نسبة ط ه الی ه ز مثل نسبة ط ح الی ز ح وخط ز ح مساو لخط ز ك تکون نسبة ط ه الی ه ز کنسبة ط ك الی ز ك ولکن خط ز ك مساو لخط ط ك لانهما مماسان للدایرة وخط ح ز مساو لخط ا ز فنسبة ط ه الی ه ز کنسبة ط ك الی ا ز وذلك ما اردنا بیانه ❀ لنفرض دایرة اب ج وخطی ه ك ه ز مماسانها ولنخرج من نقطة ه خطا یقطع الدایرة کیف ما وقع وهو خط ه ح ت ولنخرج من نقطة ز خطا موازیا لخط ه ت وهو خط د ا ونصل ا ك ویقطع ت ح علی ا فاقول ان ت ك مساو لخط ا ح برهانه انا نستخرج مرکز الدایرة وهو نقطة ط ونصل ط ك ط ه ط ز فلان خط ط ك مساو لخط ط ز وخط ط ه مشترکا یکون خطا ه ك ط ه مساویین لخطی ه ز ط ه وقاعدة ه ك کقاعدة ه ز فزاویة ه ط ك کزاویة ه ط ز فزاویة ز ط ك ضعف زاویة ه ط ك وزاویة ز ط ك ضعف زاویة ز ا ك فزاویة ز ا ك مساویة لزاویة ه ط ك فزاویة ه ط ك مساویة لزاویة ه ز ... فهذه اضلاع ه ط ك ز تقع فی دایرة ... الخلف فزاویتا ه ط ك ه ز ك متساویتان وزاویة ه ط ك قایمة فزاویة ه ز ك قایمة فقد اخرج من مرکز ط الی خط ت ح خط یقطعه علی زوایا قایمة فهو اذا یقطعه بنصفین فخط ا ت مساو لخط ا ح ... فلنفرض مثلثا ... علیه اکا ولنخرج خط اد عمودا الی خط ك ا ونعرض مربع ت د مساویا لسطح د ه فی ت ز ونصل د ز ولنخرج من نقطة ز خطا موازیا لخط ت ك وهو خط ز ح ونصل ه ح فاقول ان زاویة ه ح ...

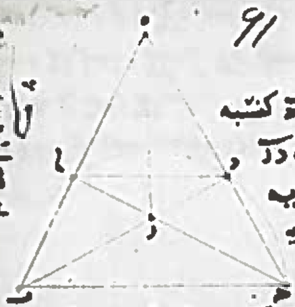
زاويه ا ر ك برهانه ان نصل ده دح فلان سطح ه ب يا ... مساو لمثلث ح
د ب فكون زاويه ر د ب مساويه لزاويه ب ه د وزاويه ر د ب مساويه لزاويه
ه ر د فزاويه ه ر ك مساويه لزاويه ده ك ولكن زاويه ه ر ك مساويه لزاويه
ر ح ك لان مثلث ح د ا متساوي الساقين فزاويه ده ك مساويه لزاويه ر ح ك فذو
اربعه اضلاع ه ر د ح تقع في دايره ولنخرج خط ه ح على استقامه الى نقطه ط



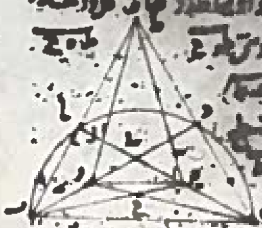
فزاويه ر ح ك مساويه لزاويه ه ر ك لان خارج ذي الاربعه
الاضلاع وزاويه د ر ا مساويه لزاويه ا ح ك فزاويه
ا ح ك ضعف زاويه ا ح ك ولكن زاويه ا ح ك مساويه
لزاويه ه ح ط وزاويه ا ح ك مساويه لزاويه
ا ر ك فزاويه ه ح ط ضعف زاويه ا ر ك ه لنفرض نصف دايره ه
على ا ك ولنقسم قوس ا ك بنصفين على نقطه ب ولنخرج خط ا د على استقامه
الى د ونصل د ب ونقسم خط ب ... بنصفين على نقطه ه ونخرج مركز نصف الدايره
وهو نقطه ر ونصل ر ه ونخرج خط ر ه ا ب على استقامه الى نقطه ع فاقول
ان نسبه ا ع الى ح ب كنسبه د ا الى د ب برهانه انا نخرج من نقطه ب خطا



موازيا لخط د ع وهو خط ب ك ونصل ر ب فنسبه
ا ع الى ح ب كنسبه ا د الى د ك ونسبه د ا الى
د ب كنسبه د ب الى د ك لان مثلث د ب ك
... مثلث ... واد مشترك ... وزاويه ... لان خط ر ه خرج من مركز ا
... خط ... بنصفين على نقطه ه فهو يحيط معه بزاويه قايمه وزاويه ده ك مساويه
... زاويه ... قايمه فنسبه ا ع الى ح ب كنسبه د ا الى د ب وذلك
... لنفرض نصف دايره عليه ا ك ولنخرج ... ونصل
... ونصل ... وخرجها على استقامه حتى يلتقيا على نقطه ك فاقول
... مساو لسطح ... برهانه انه اذا كان سطح ...



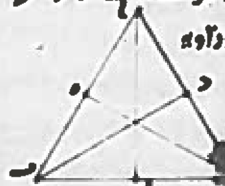
د ا مثل سطح د ه في د ه يكون نسبه د ه الى د ه كنسبه
ه ه الى د ا فاذا وصلنا ه ... يكون مثلثا ... ه د ا
متشابهان ويكون زاويه ... مساويه لزاويه
ده ك واذا وصلنا د ا كانت زاويه د ... مساويه
لزاويه د ا ه لان قاعدتهما قوس واحده فزاويه
د ا ه مساويه لزاويه ده ك فذوا اربعه اضلاع ه ا ر ... تقع في دايره وكما تبين
انه كذلك ان كل واحده من زاويتي ... ه د ... قايمه فقد وجب ان يكون سطح ...
في د ... مساويا لسطح ... في ... وذلك ما اردنا ان نبين وتاليف هذا البرهان
ان زاويه ... قايمه ونبين زاويه ... قايمه وكذلك زاويه ... [illegible]
... قايمه فذوا اربعه اضلاع ... تقع في دايره وزاويه ... مساويه لزاويه ...
وزاويه د ا ... مثل زاويه ده ... لان قوسهما ... فزاويه ... [illegible]
متشابهان ... قدر ... عند ... كقدر ... [illegible]
في ... كضرب ... [illegible] ... اردنا بيانه ... نفرض نصف دايره عليه ...
ونصل ... [illegible] ... نقطه ... ونبين سطح ... مساويا ...
... مساويا ... ونصل ... فاقول ان خط ... مساو ...
برهانه انا نصل ... ونخرجها على استقامه حتى يلتقيا على نقطه ... فسطح ...
في ... مساو ... وسطح ... في ... مساو لسطح ... في ...
... مساو ... [illegible] ... كل واحده



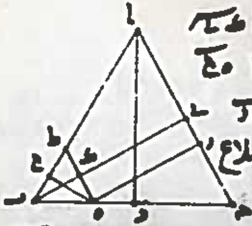
منها قايمه ... [illegible] ... كل واحد ...
قايمه ... [illegible] ... وسطح ...
في ... مساو ... وسطح ... [illegible]
... [illegible] ... وسطح ...
مساويا ... [illegible]

لمربع ط د د ز ومربعا ط ا ا ه مجموعين مساو لمربع ط ه كان زاوية ط ا ه قايمة ومربعا
ط د د ز مساويان لمربع ط ز كان زاوية ط د ز قايمة فمربع ط ز مساو لمربع ط ه
فخط ط ه مساو لخط ط ز فان وصلنا ه ز كانت زاوية ط ز ه مساوية لزاوية ط ه ز لكن
زاوية ط ز ح القايمة وزاوية ط ه ح القايمة متساويتان فتبقى زاوية ه ز ح الباقية مساوية
لزاوية ز ه ح الباقية فخط ز ح مساو لخط ه ح وذلك ما اردنا ان نبين

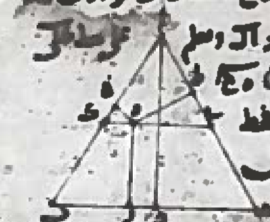
لنفرض مثلثا متساوي الاضلاع عليه ا ب ج ولنخرج فيه اعمدة ا د ب ه ج ز فاقول
ان اعمدة ا د ب ه ج ز متساوية برهانه انه لان مثلث ا ب ج متساوي الساقين وقد
اخرج فيه عمود ا د يكون خط ب د مساويا لخط د ج وايضا لان مثلث ج ب ا متساوي
الساقين وقد اخرج فيه عمود ج ز يكون ا ز مساويا لخط ز ب [illegible] ونجعل



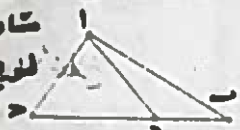
خط ا ج مشتركا فيكون خطا [illegible] مساويين لخطي [illegible] وزاوية
[illegible] مساوية لزاوية [illegible] فقاعدة [illegible] مساوية لقاعدة [illegible]
وايضا من اجل ان مثلث [illegible] متساوي الساقين وقد اخرج
فيه عمود ب ه يكون خط [illegible] مساويا لخط [illegible]
مساو لخط [illegible] ونجعل [illegible] مشتركا فيكون خطا [illegible]
وزاوية [illegible] مساوية لزاوية [illegible] فقاعدة [illegible] مساوية لقاعدة [illegible]
تبين ان خط [illegible] مساو لخط [illegible] فالخطوط [illegible]
متساوية وذلك ما اردنا ان نبين [illegible] متساوي الاضلاع عليه ا ب ج ولنخرج
عمود ا د ونعلم على خط ب ج نقطة كيف ما وقعت وهي نقطة ه ولنخرج
منها على خطي ب ا ا ج عمودين وهما خطا ه ز ه ح فاقول ان خط ا د مساو لخطي
ه ز ه ح برهانه انا نخرج من نقطة ه خطا موازيا لخط [illegible] وهو خط [illegible]
ولنخرج من نقطة [illegible] موازيا لخط [illegible] وهو خط [illegible] فلان مثلث ا ب ج
متساوي الاضلاع وخط [illegible] موازي لخط [illegible] يكون مثلث [illegible] متساوي الاضلاع
[illegible] عمودا على خط [illegible] وخط



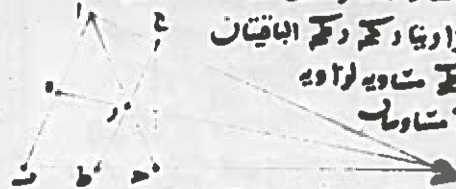
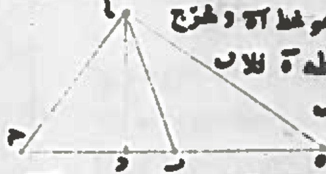
ه ح عمود على خط [illegible] فخط [illegible] مساو لخط ه ح وخط [illegible]
موازي لخط [illegible] مساو [illegible] ولذلك يكون جميع خط [illegible] مساو لخطي [illegible] ه ح
ولكن خط [illegible] مساو لخط ا د فخط ا د مساو لخطي ه ح [illegible]
وذلك ما اردنا ان نبين لنفرض مثلثا متساوي الاضلاع
عليه ا ب ج ولنخرج فيه عمود ا د ولنفرض داخله نقطة
كيفما وقعت وهي نقطة ه ولنخرج الى اضلاع المثلث ثلاثة اعمدة وهي خطوط ه ز ه ح
ه ط فاقول ان ا د مساو لجميعها برهانه انا نخرج من نقطة ه خطا موازيا لخط
[illegible] وهو خط [illegible] فلان خط [illegible] موازي لخط [illegible] يكون سطح
[illegible] متوازي الاضلاع ولان مثلث ا ب ج متساوي الاضلاع وخط [illegible] موازي لقاعدة
يكون مثلث [illegible] متساوي الاضلاع وقد اخرج فيه عمود [illegible]

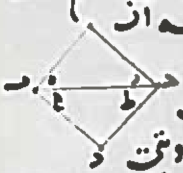


نقطة [illegible] واخرج منها عمودان على خطي [illegible]
[illegible] يكون خط [illegible] مساويا [illegible]
[illegible] مساو لخطوط [illegible] وذلك ما اردنا ان نبين
لنفرض مثلثا عليه ا ب ج ولنخرج من نقطة ا من
خط ا ب [illegible] زاوية [illegible] مساوية لزاوية [illegible]
مساوية لزاوية [illegible] فاقول [illegible]



[illegible]
تكون زاوية [illegible] الباقية مساوية لزاوية [illegible]
[illegible]
[illegible] وذلك ما اردنا ان نبين لنفرض مثلثا [illegible]
نخرج من نقطة ا عمودا على خط [illegible] وهو خط [illegible] ونخرج [illegible]
حتى تلقاه [illegible]

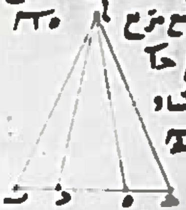
ولنخرج من نقطة ز خطا موازيا لخط ا ب وهو خط ز ح فاقول ان سطح د ا في ا ح
مساو لمربع ا د برهانه انا نخرج ز ح على استقامة الى نقطة ك فلان مثلث ا ك د
متساوي الساقين وخط ز ك موازٍ لخط ا ب يكون خط د ك مساويا لخط ز ك وانما كان
اجل ان خط ا ه مساو لخط ه ب وخط ه ب موازٍ لخط ح ك تكون ز ح مساويا لخط ز ك
وقد بين ان خط ز ك مساويا لخط ز د فخطوط ز ح ز ك ز د متساوية واذا وصلنا

ح د تكون زاوية ح د ك قائمة فزاويتا د ح ز د ك ح الباقيتان
مساويتان لقائمة وزاوية د ح ز مساوية لزاوية
ح د ز وزاوية ك مع زاوية د ح ز مساويتان
لقائمة وزاوية ك مع زاوية
ك مساوية لقائمة واحدة فزاوية ك مساوية لزاوية د ح ز وزاوية د ح ز مساوية
لزاوية د ح ز فزاوية ك مساوية لزاوية د ح ج فالمثلثان متشابهان فسطح د ا في ا ح
مساو لمربع ا د ه لنفرض مثلث متساوي الساقين عليه ا ب ج ولكن ساقاه
المتساويان خطا ا ب ا ج ولنخرج من نقطة ا خطا يكون عمودا على خط ب ج وهو
خط ا د فاقول ان سطح ب د في د ج مع مربع ا د مساو لمربع ا ب برهانه انا نخرج

من نقطة ا خطا يكون عمودا على خط ا ب وهو خط ا ه ونخرج
خط ب ج على استقامة حتى يلقى خط ا ه على نقطة ه فلان
زاوية ب ا ه قائمة وخط ا د عمود يكون
خطوط ه د ب د ا د متناسبة
ضعف د ب في سطح ه د في د ب ضعف سطح ه د في د ب وسطح ه د في د ب
مساو لمربع ا د لان زاوية ه ا ب قائمة و د ا عمود فسطح ه د في د ب مع ب د
مساو لمربع ا د وذلك ما اردنا ان نبين لنفرض مثلث عليه ا ب ج ولنخرج
من نقطة ا الى خط ب ج عمودا وهو ا د فاقول ان زيادة مربع ب د على مربع د ج
مثل زيادة مربع ا ب على مربع ا ج برهانه لان زيادة مربع ب د على مربع


د ج كزيادة مربع ب د على مربع ا د ج ومربعا ب د د ا مساويان
لمربع ا ب ومربعا ا د د ج مساويان لمربع ا ج فزيادة مربع
ب د على مربع د ج مثل زيادة مربع ا ب على مربع ا ج وذلك
ما اردنا ان نبين لنفرض خط عليه ا ب مساويا
لخط ا ج وخط ب ج مساويا لخط د ج ولكن كل واحدة من زاويتي ب ج د قائمة
فاقول ان زاوية ا ب د مساوية لزاوية ا ج د برهانه انا

نصل ب ج فمن اجل ان زاوية ا قائمة تكون زاويتا ب ج
متساويتين لقائمة وايضا من اجل ان زاوية د قائمة تكون
زاويتا ح ج متساويتين لقائمة فزاويتا ب ج متساويتان لزاويتي
ح ج وزاوية ب مساوية لزاوية ب وزاوية ج مساوية لزاوية ج فزاوية ب
مساوية لزاوية ج وزاوية ب مساوية لزاوية ج فجميع زاويتي ب ج مساوية لجميع
زاويتي د ج وذلك ما اردنا ان نبين لنفرض مثلث قائم الزاوية عليه ا ب ج
ولكن الزاوية القائمة ا ولنقسم خط ب ج بنصفين على د ولنصل ا د فاقول ان خطوط
ب د د ج د ا متساوية برهانه انا نخرج من نقطة د خطا موازيا لخط ا ب
وهو خط د ه فمن اجل ان خط ب د مساو لخط د ج وخط د ه موازٍ لخط ا ب يكون
خط ا ه مساويا لخط ه ج وزاوية ه ا د قائمة فزاوية ج ا د قائمة وكذلك زاوية
د من اجل ان خط ا ه مساو لخط ه ج وخط ه د مشترك وزاوية

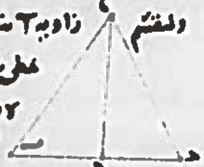
ج مساوية لزاوية د تكون قاعدة ا د مساوية لقاعدة د ج
ولكن خط د ج مساو لخط ب د فخط ا د مساو ...
د ب فخطوط ا د ب د د ج الثلاثة متساوية وذلك ما اردنا ان نبين
لنفرض مثلث قائم الزاوية ... ولنخرج خط
على استقامة الى نقطة ك ولنخرج من نقطة ... وصل ... ه
مساويا لسطح ا ب في ب ج فاقول ان ا د ...

الى خط دا عمود اه فلان خط دا قد قسم بنصفين على نقطة ه ونقسم بمختلفين
على د فيكون سطح به في دح مع مربع هد مساويا لمربع هح ونجعل
مربع اه مشتركا فيكون سطح بد في دح مع مربع اد مساويا
لمربعي اه هد ولكن مربعا اه هد مساو
لمربع اد لان زاوية اهد قائمة ومربعا ده اد مساويان لمربع اد فسطح بد في دح
مع مربع دا مساو لمربع اح وذلك ما اردنا ان نبين

لنفرض مثلثا متساوي الساقين عليه ابح وليكن ساقاه المتساويان اب اح
ونخرج من نقطة ا خطين الى خط بح وهما اد اه وليكن نسبة سطح بد في دح الى
مربع دا مثل نسبة سطح به في هح الى المربع ها فاقول ان خط دا مساو لخط
اه برهانه لان نسبة سطح بد في دح الى المربع اد كنسبة سطح به في هح
الى المربع اه فاقول انا اذا ركبنا كانت نسبة سطح بد في دح مع مربع دا
الى المربع اد مثل نسبة سطح به في هح مع مربع اه الى المربع اه ولكن سطح
بد في دح مع مربع دا مساو للمربع اح وسطح به في هح مع مربع
اه مساو للمربع اح فنسبة مربع اح الى المربع اد مثل نسبة مربع اح
الى المربع اه والمقدمان متساويان فالتاليان كذلك خط دا مساو
لخط اه وذلك ما اردنا ان نبين لنفرض مثلثا عليه ابح

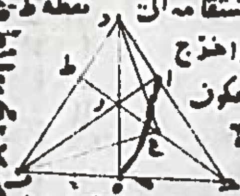


والمثلث زاوية ا بنصفين بخط اد فاقول ان نسبة
خطي با اح جميعا الى الخط بح كنسبة اد الى دح برهانه
لان زاوية ا قسمت بنصفين بخط اد تكون نسبة با الى اح
كنسبة بد الى دح واذا ركبنا كانت نسبة اب اح الى اح كنسبة



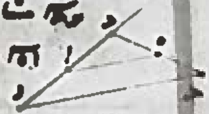
بح الى دح ونسبة الجميع الى الجميع كالواحد الى الواحد فنسبة خطي با اح الى
خط بح كنسبة اد الى دح وذلك ما اردنا ان نبين لنفرض مثلثا عليه ابح
ونخرج خطي دا دب على استقامة الى نقطتي ده ونصل ده ونخرج من

ده الى ز برهانه ان نصل دب ونخرج حد على استقامة الى نقطة ك



من اجل ان خط دب قد اخرج فيه عمود اد وهو اخرج
خط دح الى نقطة ك فيكون خط دك عمودا على دب فيكون
زيادة مربع دح على مربع حب مساويا لزيادة مربع دك
على مربع كب وزيادة مربع دك على مربع كب مثل ذلك
على مربع دب فيكون مربعا دح حب مجموعين مثل مربعي حد دز ونجعل سطح
دز في زح مشتركا وهو مثل اح في حب لان زاوية اح قائمة وزاوية ا قائمة فنقطة ب
ا د نرسم عليها دايرة فيكون خطان قد تقاطعا في دايرة فيكون سطح اح في حب
مع مربع دح مساويا لسطح دز في زح مع مربع حز وسطح اح في حب مساو لمربع
حز فسطح دز في زح مساو لمربع دح وذلك ما اردنا ان نبين

لنفرض خطوطا عليها اب بح ونفرض على خط اب نقطة كيفما وقعت وهي
نقطة د وليكن مربع اب مساويا لمربع اد مع مربع دب فنصل دح ونقسم
بنصفين على نقطة ه ونصل اه فاقول ان زاوية داه مساوية لزاوية دحب
برهانه انا نخرج بح الى نقطة ح وليكن دا مساويا لخط اح فلان خط دح
قد قسم بنصفين على نقطة ه وزيد في طوله دب فيكون سطح دب في بح مع مربع اد
مساويا للمربع اب ولكن مربع اب فرض مساويا لمربعي دا دب فاذا القينا اد المشترك
بقي سطح دب في بح مساويا للمربع دب فاذا وصلنا دب تكون زاوية
دحب مساوية لزاوية دحب وزاوية دحب مساوية لزاوية داه لان
اه موازي دح فزاوية داه مساوية لزاوية دحب

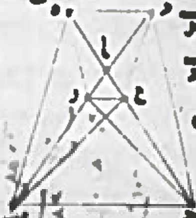


وذلك ما اردنا ان نبين

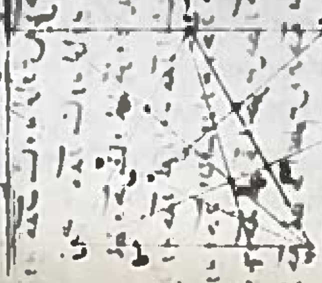
لنفرض مثلثا متساوي الساقين عليه ابح وليكن ساقاه المتساويان اب اح
ونخرج من نقطة ا الى خط بح خطا كيفما وقع وهو خط اد فاقول ان
سطح بد في دح مع مربع دا مساو لمربع اح برهانه ان نخرج من نقطة ا

بيان ذلك ان نصل
خط اه فيكون ذو
اربعة اضلاع ارده
وافعاني دايرة وكذلك
اربعة اضلاع اكده
ايضا وافعا في دايرة
فزاوية ه اد كزاوية
ه حد وهي ايضا
كزاوية ه كد
كزاوية
وزاوية
كزاوية
في زاوية
القايمة
وتعلل زاوية
كزاوية
احد وكذلك
زاوية
المتقابلان
زاوية

نقطة ه خطا موازيا لخط د ح و هو خط ه ز و نخرج من نقطة ک خطا موازيا لخط
ه ب و هو خط ه ح و نصل ز ح فاقول انه موازی لخط ب ج برهانه
ان نصل د ز ح ب د ه فمثلث د ه ح مساو لمثلث د ه ز لانهما على قاعدة واحدة وفيما
بين خطين متوازيين د ه و ز و نلقى [illegible] د ه ه المشترک فيبقى مثلث ح ه ه الباقي مساو
لمثلث ح ا ز الباقي و [illegible] مساو لمثلث ه ح ب لانهما على قاعدة واحدة وبين خطين
متوازيين ه ب د ح ونلقى [illegible] المشترک فمثلث د ا ه مساو لمثلث ا ح ح

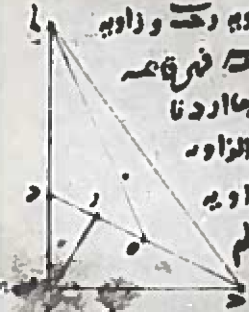


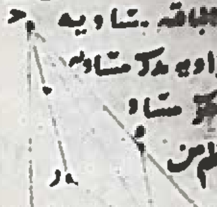
ومثلث د ا ه مساو لمثلث ا [illegible] مثلث ا ح ب مساو لمثلث ا ح ز
ونلقى مثلث ا ح ح المشترک [illegible] الباقي کمثلث ح ز د
وهما على قاعدة واحدة و [illegible] فهما في خطين متوازيين
فخط ح ز مواز لخط ب ج و ذلک [illegible] اردنا ان نبين
لنفرض مثلثا عليه ا ب ج [illegible] ب ج مساو من
و ا د ح مساو من ايضا و [illegible] ب ج و نقاطع [illegible] لا نتقاطع و
ونقاطع ح على نقطة [illegible] ونصل [illegible] ونخرج [illegible] على استقامة [illegible]
او [illegible] فاقول ان خط [illegible] مساو لخط [illegible] برهانه [illegible] على نقطة
ه خطا موازيا لخط [illegible] و [illegible] ونخرج خطوط [illegible]
لخط ا ز مساو لخط [illegible] و [illegible] مشترک فخط ا ح مساو [illegible]
ح الى د ا و لکن نسبة ا ح [illegible] کنسبة ا ک الى [illegible] و [illegible]
الى [illegible] نسبة [illegible] الى [illegible]



[illegible] لنسبة [illegible] مسطح [illegible]
[illegible] مسطح
[illegible] مساو [illegible]
لنسبة [illegible] نسبة [illegible] الى [illegible] کنسبة [illegible] الى [illegible]
[illegible] کنسبة [illegible] ونسبة [illegible] الى [illegible]

كنسبة سطح ا د الى سطح د ب ونسبة خط ا ل الى ل ط كنسبة سطح ا ل الى [illegible] فاذا ركبنا يكون نسبة ج ب
الى [illegible] كنسبة [illegible] الى [illegible] ونسبة [illegible] الى كل واحد من خطي [illegible] الى واحده فخط
[illegible] مساو لخط [illegible] وذلك ما اردنا ان نبين لنفرض مثلثا قايم الزاويه
عليه ا ك [illegible] ولتكن زاويته القايمه زاويه [illegible] ولنقسم زاويه ا [illegible] بنصفين بخط د [illegible]
ونفرز من زاويه [illegible] مساويه لاحدى زاويتي [illegible] فاقول ان خط [illegible] اعظم
من خط [illegible] برهانه لان زاويه [illegible] مساويه لزاويه [illegible] يكون سطح [illegible] في [illegible]
مساو لمربع [illegible] ومربع [illegible] اعظم من مربع [illegible] لان نسبة [illegible] الى [illegible] كنسبة [illegible]
الى [illegible] وخط [illegible] اعظم من [illegible] فخط [illegible] اعظم من [illegible] فالضرب من سطح [illegible] في [illegible]
مساو بالمربع [illegible] ونصل [illegible] فزاويه [illegible] مساويه لزاويه [illegible] وزاويه
[illegible] مشترك لمثلثي [illegible] فزاويه [illegible] كزاويه [illegible] فهي قايمه
فخط [illegible] اعظم من [illegible] فهو اعظم كثيرا من [illegible] وذلك ما اردنا
ان نبين لنفرض مثلثا قايم الزاويه عليه [illegible] والزاويه
القايمه [illegible] ونقسم زاويه [illegible] بنصفين بخط [illegible] ولتكن زاويه
[illegible] ضعف زاويه [illegible] فاقول ان سطح [illegible] في [illegible] اعظم
من مربع [illegible] برهانه ان الضرب من سطح [illegible] في [illegible]
مساو بالمربع [illegible] ونصل [illegible] فزاويه [illegible] مساويه لزاويه [illegible] ويكون
زاويه [illegible] مساويه لزاويتي [illegible] فهي اعظم من زاويه [illegible] ولكن زاويه
[illegible] مساويه لزاويه [illegible] ولنقسم زاويه [illegible] بنصفين بخط [illegible]
[illegible] مساو لخط [illegible] ولان [illegible] مساويه لزاويه [illegible] التي مثل
زاويه [illegible] زاويه [illegible] المشترك يكون [illegible] الباقيه مساويه
لزاويه [illegible] الباقيه وزاويه [illegible] مساويه لزاويه [illegible] فزاويه [illegible] مساويه
لزاويه [illegible] فخط [illegible] اعظم من خط [illegible] مساو
[illegible] فخط [illegible] اعظم من خط [illegible] يكون سطح [illegible] اعظم من

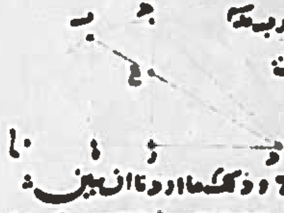


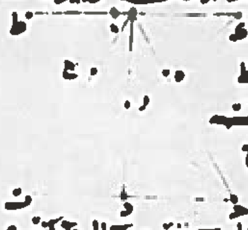


هذا هو الشكل الذي بينه محمد بن موسى مما كان في الكرة الذي احال عليه
كل مخروط [illegible] قاعدته دايره فان [illegible] المواذي قاعدته يصله مشتركه
ايضا دايره والخط الخارج من [illegible] الى [illegible] قاعدته [illegible] مشترك [illegible]
المخروط [illegible] وقاعدته دايره [illegible] ونصل سطح مواذي سطح [illegible]
فاقول ان [illegible] ايضا دايره [illegible] ونخرج من نقطه ا
الى [illegible] المخروط الى مركز قاعدته [illegible] نقطه [illegible] خط [illegible]
نقطه [illegible] فاقول ان نقطه [illegible] ايضا مركز دايره [illegible] برهانه [illegible]
[illegible] نصف محيط دايره [illegible] ونصل خط [illegible]

[illegible]

سطح اه في هة وسطح اه في هة مثل مربع هة فسطح كه في جا اعظم من مربع هة
وذلك ما اردنا ان نبين. د لنفرض نصف دايرة على اكد ونفرض قوس
اب مثل قوس بكد ونصل هة ونخرج من به خطا يكون عمودا على اد وهو خط بة فاقول ان
خط هة اقصر من خط دك برهانه انا نصل بد بك ونخرج من كد عمودا على
اد وهو عمود حز ونقطع خط بة على ح فضرب اد في دز كضرب بد في دح
وضرب اد في دز كضرب دك في مثله فضرب دك في مثله كضرب بد في دح وايضا
لما كانت زاوية ادب مثل زاوية بكد ومثلث دحز قايم الزاوية وجب ان يكون ضرب
دح في دا اكثر من ضرب دح في مثله فنسبة مربع دك الى ضرب دك في دا اصغر
من نسبة مربع دح الى مربع دح ومربع دة هو كضرب بد في دح فنسبة مربع
دك الى ضرب دة في دا اصغر من نسبة ضربه
في دح الى مربع دح مثلا فان نسبة دة الى
دا اصغر من نسبة بة الى دح ونسبة بة الى
اكة من نسبة دة الى دا فنسبة دك الى الكل
اصغر من نسبة دة الى دا فدك اصغر من دة وذلك ما اردنا ان نبين.
لنفرض مثلثا على اكد ونقسم خط كد بنصفين على نقطة ز ونصل از ونخرج من
نقطة ب خطا كيف ما وقع الى خط اكد وهو خط بح ونصل كح ونخرجه الى
ان نقطة ع ونصل حة فاقول ان خط حة موازي لخط كد برهانه ان الخط على نقطة
ح خطا موازيا لخط كد وهو خط ط ونخرج
خط بة على الاستقامة الى ان يلقاه على خط
حة مساو لخط دك فيكون خط كا مساويا لخط اكة فنسبة
حة الى كا كنسبة اد الى اكة وكذلك نسبة الى
نسبة ج الى اكة لان خط كا مساويا لان قع
عليها حطا ان حا ونسبة اح الى اكة كنسبة اد الى دة فنسبة ج الى كنسبة

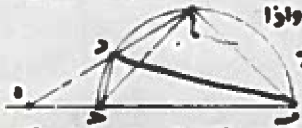




على اكم

لتشابه
مثلثي آكب
دكح

هة الى اة خطا هة موازي لخط كد وذلك ما اردنا ان نبين. لنفرض نصف دايرة د
على اكد ونخرج ... ونقسم قوس كد بنصفين على نقطة ز ونصل ...اة فاقول ان
سطح هز في زا مساو لمربع ... برهانه انا نصل اك دك دك فلان قوس اك
مثل قوس اد يكون زاوية اكد مساوية لزاوية اكد ... وزاوية ادك ... دة
دك وزاوية ادك مساوية لزاوية اكد لان كل واحدة منهما يوتر ها قوس اك
يكون زاوية اكد مساوية لزاويتي دهة ... ولذا

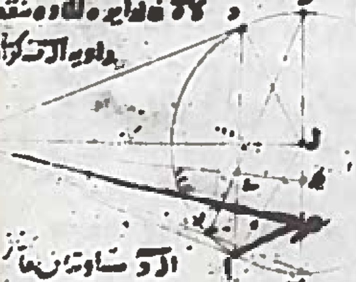


التي زاوية دهة المشتركة يكون زاوية اكد
الباقية كزاوية دكة الباقية فسطح اه في
اد مساو لمربع اد ... فسطح هز في اك مساو لمربع اد وذلك
ما اردنا بيانه. ه لنفرض دايرتين يتقاطعان عليها اهد دح وبخرج
خط دج يقطع الدايرتين كيف ما وقع ونصل ها اد هد دج فاقول ان زاوية
دال مساوية لزاوية هدج برهانه انا نصل اد لان شكل اد ذو الاربعة
اضلاع في دايرة يكون زاويتا ... مساوية لزاويتين قايمتين فيكون زاويتا ... دا
... مساويتين لزاويتي ... فاذا القينا زاوية ... المشتركة بقي زاوية
... مساوية لزاوية ... وزاوية ... مساوية لزاوية
يكون زاويتا ... مساويتين لزاوية ... فاذا القينا زاوية ... المشتركة
يكون زاوية ... الباقية كزاوية ... الباقية وذلك ما اردنا ان نبين
ونثبت الصورة ونخرج خط ... الى التقاطع ... ونصل ... فاقول
ان ... مساو لخط ... برهانه لان
زاوية ... مساوية لزاوية ... وزاوية
... التي هي خارج ذي الاربعة اضلاع
... مساوية لزاوية ... وزاوية
... الداخلة عن ذي الاربعة اضلاع ...

الى

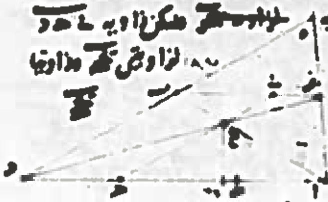
مساويه لزاويه ه ح ط تكون زاويه ط ه د مساويه لزاويه ه ح ك فاذا خط ح ك مساو
لخط ك ه وايضا لان زاويه ك ه د مساويه لزاويه د ح ك تبقى زاويه ه ا ط مساويه
لزاويه ه د ك ولان ذا الاربعه اضلاع ا ح ط ه في دايره تكون زاويتا ا ط
ه ح ط قائمتين وذو الاربعه اضلاع د ح ط ه في دايره تكون زاويتا ه د ك ه ح ط
قائمتين فزاويتا ه ا ط ه د ك متساويتان لزاويتي ه ح ا ه ح ك وزاويه ه ا ط
مساويه لزاويه ه ح ك تبقى زاويه ه ح ك كزاويه ه ح ا فخط ح ك كخط ح ا
ما اردنا بيانه لنفرض مثلثا قايم الزاويه عليه ا ب ح وزاويه ا منه قايمه
ولنخرج خط ح ا على استقامه ولتكن المخرج منه خط ا د واخرج من نقطه د خط
د ه عمودا على خط ا د فاقول ان سطح ح د في د ا مساو لسطح ح ا في ا ح مع مربع
د ه برهانه لان زاويه ا د ه قايمه وكذلك زاويه ه ا ح تكون نقطه ا ه د
على محيط دايره فسطح ح د في د ا مساو لسطح ه د في د ح ولكن سطح ه د في د ح
مساو لسطح ه د في د ح مع مربع د ه وسطح ه ا في ا ح مساو لسطح ح ا في ا ح
وايضا فلكن سطح ح د في د ا مساو لسطح ح ا في ا ح مع مربع د ه فاقول ان زاويه
ه د ح قايمه برهانه لان سطح ح د في د ا مساو لسطح ح ا في ا ح مع مربع
د ه ومربع د ه مساو لمربعي د ا ا ه يكون سطح ح د في د ا مساويا لسطح ح ا في ا ح
مع مربع ا د فاذا القينا مربع د ا المشترك يكون سطح ح ا في ا د مساو لسطح
ح ا في ا د ويكون نسبه ح ا الى ا ه كنسبه ه ا الى ا د فمثلثا ح ا ه د ا ه
متشابهان فزاويه ح ه ا كزاويه ا د ه وزاويه ه ح ا كزاويه
ا ه د فزاويه د ه ح الباقيه كزاويه ح ه ا الباقيه فزاويه
د ه ح قايمه فزاويه د ه ح قايمه وذلك ما اردنا ان نبين
نفرض دايره عليها ا ب ح د ولنخرج خطي ا ح د ح متساويين ولنصل ا د ولنخرج خط
... ونخرج عمود ح ك مقاطع ا د على نقطه ل ... فاقول
ان زاويه د ه ح ضعف زاويه د ح ا ... على استقامه الى ...

نقطه ح ونصل ه ك ونخرج خط د ك عمودا على خط د ه ونخرج خط ه ح الى نقطه
ك فلان زاويه ه د ك قايمه يكون ه ك قطرا وخط د ك ايضا قطرا لانه قد اخرج
من خط ه ك على زاويه قايمه فنقطه ك اذا مركز الدايره فزاويه د ك ه ضعف
زاويه د ح ه فبطل ان تكون زاويه د ك ه مساويه لزاويه د ه ح فان كانت نقطه
ح و ك على محيط دايره كان ذلك كذلك وانما تكون على محيطها اذا كانت
كل واحده من زاويتي د ك ه ك ه د قايمه وبه كذلك لان ... مساويين ك
الساقين وهما خرج بين خط ه ح ا فسطح ا ح في ا د مع مربع ه ح مساو لمربع
ا ه ومربع ا ه مساو لسطح ب ح في ح ا فيكون سطح ا ح في ح د مع مربع ه ح
مساويا لسطح ب ح في ح ا وسطح ا ح في ح د مساو لسطح ح ا في ح ب فاذا سطح
ح ا في ح د مع مربع ه ح مساو لسطح ب ح في ح ا فيكون زاويه ح ه د قايمه لما
قد بيناه في الشكل قبل هذا فان اعترض بان خط ه ح اذا خرج على استقامه
فقد يخرج الى نقطه ك ويكون ... فالجواب انه اذا خرج الى نقطه ك تكون
زاويه ك ه ح قايمه على ما تقدم في المقدمه وتكون زاويتا ك ه د د ه ح متساويتين
زاويه ك ه د المستقيمه بقي زاويه ك ه ح كزاويه د ه ح وكذلك كان ذلك بيانه
... يكون ايضا على زاويه قايمه ... ثم نصل ... فيكون نقطه ... على
محيط دايره لان زاويه ... كزاويه ... فزاويه ... ونقط
... وزاويه ... كزاويه
... على نقطه ا ويكون
... لان قاعدتها قوس
واحده ... زاويه ...
كزاويه ... اذ كل زاويه
... كزاويه ... وكذلك
لان زاويه ... مع زاويه
... متساويتان ...
وكذلك زاويتا ... مساويتان ... متبين ...



... ذلك ما اردنا ان نبين

لنعرض نصف دايرة عليه اكد ونخرج من نقطة ه خطا ماسا وهو خط اه
واخرج ماسا وهو خط ه ب ونخرج خط ... اكد حتى يلتقيا على نقطة ب ونخرج
من نقطة ب خطا موازيا لخط ... وهو خط ... ونصل ... ونخرج من نقطة
ج عمودا على خط اكد وهو خط حك فاقول ان نقطة ك مركز الدايرة برهان ذلك
فلان كل واحد من خطي ... ماس الدايرة يكونان متساويين فنسبة ده الى ... كنسبة
ده الى ... وكذلك نسبة ده الى ... كنسبة ... الى ... فنسبة ده الى ... كنسبة
... فتكون نسبة ... الى ... كنسبة ... الى ... واذا ... تكون نسبة ... الى ...
كنسبة ... الى ... فنسبة مربع اكد الى المربع ... كنسبة مربع ... الى ... ومربع ...
مساو لسطح ... فتكون نسبة مربع ... الى
سطح ... كنسبة مربع ... الى السطح ... فاذا نسبة خط ... الى ... كنسبة
[illegible]



... مثلث ... [illegible]
متشابهه فيكون زاويتا ... متساويتين
... متساويتين ...
لذلك خط ... مستقيم ولان زاويه
... مساويه لزاويه ... وزاويه ... مشتركة
... قوس واحد ... يكون زاويه ... مساويه لزاويه

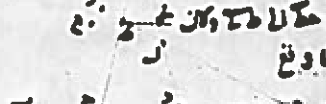
... خط اكد مساو لخط ... وخط ... فرض عمودا على خط اكد فخط اكد ... ولخط
... وخط ... قطر الدايرة فنقطة ك اذا مركزها وذلك ما اردنا ان نبين
لنعرض مثلثا عليه اكد ... خط اكد نصفين على نقطة ... ونخرج خط ...
على استقامة الى نقطة ... ونصل ... فاقول ان نسبة ... الى ... كنسبة ...
الى ... برهان ذلك انا نخرج من نقطة ... خطا موازيا لخط ... وهو خط ...
فلانه مواز لخط ... وخط ... مساو لخط ... يكون خط ... مساويا لخط ...
ولان خط ... قد اخرج فيه خط ... موازيا لقاعدته التي هي ... تكون نسبة
... الى ... كنسبة ... الى ... وقد كان تبين ان خط ... مساو لخط ...
فاذا نسبة ... الى ... كنسبة ... الى ... وذلك ما اردنا
ان نبين ولنفرض ان نسبة ... الى ... كنسبة ...
الى ... فاقول ان خط ... مساو لخط ... برهانه
ان نسبة ... الى ... كنسبة ... الى ... وخط ... مواز لخط ... فخط ... مساو
لخط ... وذلك ما اردنا ان نبين لنعرض مثلثا عليه اكد ونخرج خط ...
على ... ولكن المخرج ... ونقسم خط ... نصفين على نقطة ... وخط ...
على نقطة ... فتكون نسبة ... الى ... كنسبة ... الى ... فاقول ان ...
... برهان ذلك انا نخرج من نقطة ... خطا موازيا
لخط ... وهو خط ... ونصل ... فلان نسبة ... الى ... كنسبة ... الى ...
ونسبة ... الى ... كنسبة ... الى ... وخط ... مساو لخط ... تكون نسبة ... الى ...
كنسبة ... الى ... ونخرج من نقطة ... خطا موازيا لخط ...
وهو خط ... ونخرج ... فتكون نسبة ... الى ...
كنسبة ... الى ... ونسبة ... الى ... كنسبة ... الى ...
... مساو لخط ... وهو موازله وقد فرض خط ... موازيا
لخط ... اذا مستقيم وذلك ما اردنا ان نبين لنعرض مثلثا عليه ...

ونقطة ز فوقه ونخرج خطين يكون كل واحد منها موازيا لخط اب وهما خطا
ه ز و ز ح فاقول ان خطي ه ز و ز ح قد اتصلا على استقامة برهانه انا نفرض
على خط ب ا نقطة كيف ما وقعت وهي نقطة ط ونصل ط ز فلان خط ه ز
موازٍ لخط اب تكون زاويتا ه ز ط و ز ط ب متساويتين لقائمتين وايضا لان
اب موازٍ لخط ز ح تكون زاويتا ح ز ط و ز ط ا

متساويتين لقائمتين فتكون الاربع الزوايا
كاربع زوايا قائمة الاثنتان منها كـ[illegible]
الاثنتان كـ[illegible] فخط ه ز ح مستقيم وذلك ما اردنا ان نبين

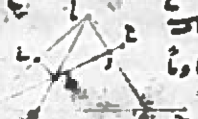
لنفرض ثلثة خطوط عليها اب ا ح ا د ونخرج اليها من نقطة ه خطي
ه ز ه ح ولتكن نسبة ه ز الى ز ح كنسبة ه ط الى ط ك فاقول ان نسبة ه ح الى
ح ك كنسبة ه ز الى ز ط برهانه انا نخرج على نقطة ك خطا موازيا لخط ه ح
وهو خط ك ل فلان مثلث ه ح ك قد خرج فيه خط ك ل موازيا لقاعدته
تكون نسبة ه ز الى ز ك كنسبة ه ح الى ح ل ونسبة ه ز الى ز ك كنسبة ه ل
الى ل ك ونسبة ه ك الى ز ك كنسبة ه ل الى ل ط لان خط ط ل موازٍ لخط ه ح
[illegible]
[illegible]
[illegible]
[illegible] نسبة ه ح الى ح ك كنسبة ه ز الى ز ط

واذا [illegible] الى ه ز كنسبة ح ك الى ك ط ولان [illegible]



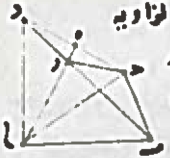
مثلث اب ح قد خرج فيه خط ا ه كيف ما وقع
وخط د ه موازٍ لخط ب ح يكون نسبة ح ه الى
ه ك كنسبة ا ك الى ك د [illegible] نسبة ب ا الى ا ح كنسبة ا ه الى ه د يكون
نسبة ب ا الى ا ك كنسبة ا ح الى ا د [illegible] نسبة ه ح الى ح د كنسبة ه د
الى د ح وذلك ما اردنا ان نبين لنفرض [illegible] الزاوية عليها اب ح وزاوية ا ح ك

فقد اخرج فيه خط ا ز ا ه فكانت زاوية د ا ه مساوية لزاوية ا ك ح وقد اخرج من نقطة
ح الى خط ا ه عمود ح ط وانفذ الى نقطة ع فاقول ان سطح د ه في ا ه مع سطح ح ه
في ه ك مساوٍ لسطح ح ب في ب ك مع سطح ح ا في ا ع برهانه انا زاوية د ا ه مساوية
لزاوية ا ك ح لنجعل زاوية ب ا ه مشتركة تكون زاوية ب ا ه كزاويتي ا د ح و د ا ز زاويتي ا ز ح
قائمة تكون زاويتا د ا ع ا ح الباقيتان متساويتين لقائمة واحدة وزاوية ا ع ح
قائمة فزاويتا د ا ع و ع ا ح متساويتان لزاوية ه ا د ولنقل زاوية د ا ع المشتركة فتبقى زاوية
ه ا ح كزاوية ا ح ك وقد كان تبين ان زاوية ا د ح مساوية لزاوية ا ك ح تكون زاوية
ا ح ك مساوية لزاوية ا د ح فنقط ا ع ح ك على محيط دائرة ويكون سطح د ه في ه ا
مساويا لسطح ع ه في ا ه وسطح ح ه في ه ك مساوٍ لسطح ح ع في ع ك فسطح د ه في ا ه
مع سطح ح ه في ه ك مساوٍ لسطح ع ه في ا ه مع سطح ح ع في ع ك ولنثبت الصورة على حالها
فاقول ان سطح ح ب في ب ك مع سطح ح ا في ا ع مساوٍ لمربع ا ح برهانه انا نخرج
[illegible]
[illegible]
[illegible]
[illegible] ونقط ا ع ح ك على
محيط دائرة ومسطح ح ا في ا ع مساوٍ لسطح د ه في ا ه وسطح د ح في ح ك مساوٍ لسطح ح ب

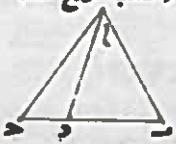


في ب ك وقد كان تبين ان سطح ح ب في ب ك مساوٍ لسطح ا ح
في ح د فيكون سطح ح ا في ا ع مع سطح ح ب في ب ك مساويا
لسطح ا ح في ح د وا ح في ا د وسطح ح ا في ا د
مع سطح ا ح في ح د هو مثل مربع ا ح فيكون مربع ا ح مساويا لسطح ح ب في ب ك
مع سطح ح ا في ا ع وذلك ما اردنا ان نبين لنفرض ثلثة خطوط متساوية عليها
ا ب ا ح ا د ونصل ب ح ح د فاقول ان زاوية ب ح د مع زاوية د ب ح تقابله
برهانه انا نخرج خط ا ح [illegible] متساويا لا [illegible] فلان خطوط ب ا ح ا د

ان خطوط دآ حا با متساويه فتكون نقط ز د ح
على محيط دايره وقد كانت فرضت خطوط دآ حا
با متساويه فنقط ز د ح آ على محيط دايره
فزاويه دزح مساويه لزاويه حدح وقد كان تبين ان زاويه حزح مع زاويه حدح قايمتين
واحده فزاويه حدح مع زاويه حبز قايمه وايضا فلنكن خطا با اح متساويين
وزاويه حبا مع زاويه حدح متساويتين لقايمه واحده فاقول ان خطوط احد ادآ
آد متساويه برهانه انا نخرج خط دآ الى نقطه ز ونكن خط اآ مساويا لخط
على اح ونصل حز فتكون زاويه زحد قايمه فزاويتا حزح حدح مساويتان
وزاويتا حبز حدح فرضتا كقايمه واحده فزاويه حدح مساويه لزاويه حزح
فنقط ح د ز ح على محيط دايره وخطوط دآ با حا متساويه فنقطه آ مركز الدايره
فاد مساو لكل واحد من احد اح وذلك ما اردنا بيانه وايضا لنكن دآ حا
متساويان وزاويتا حبا حدح متساويتان لقايمه واحده فاقول ان خطوط دا
حا با متساويه برهانه لنكن خط اآ مساويا لكل واحد من خطي با اح ونصل
دح فزاويه بحد قايمه فزاويتا حدح كقايمه وفرضت زاويتا حبا حدح
مساويتان لقايمه فتلقى زاويه دحا المشتركه فزاويتا حدح حدح مساويتان
لزاويه حدح ونجعل زاويه حدح مشتركه فتكون زاويتا حدح ودحا مساويتين
لزوايا مثلث حدح فزاويتا دحا حدح مساويتان لقايمتين فنقط د ح ز ح
على محيط دايره وخطوط دآ دآ با متساويه فنقطه آ مركز الدايره فخط حا
مساو لخط دآ وذلك ما اردنا بيانه وايضا لنكن خطا دآ حا متساويين
وزاويتا حبا حدح متساويتان لقايمه واحده فاقول ان خطوط دآ حا
با متساويه برهانه انا نخرج من نقطه آ الى خط حد عمود داه ونصل زح
فيكون خط ده مساويا لخط ... فتكون زاويه دزه مساويه لزاويه حزح



... زاويتي حا حدح متساويتان لقايمه فزاويه دزح مساويه لزاويه
با ولكن زاويه دزح مساويه لزاويه دزح فزاويه دزح
مساويه لزاويه حبا فنقط ز د ح آ على محيط دايره فزاويه
احد مساويه لزاويه ادح ولكن زاويه ادح مساويه
لزاويه دزه وزاويه دزه مساويه لزاويه حا فزاويه احد مساويه لزاويه
حبا وذلك ما اردنا ان نبين . لنفرض مثلث عليه احد ونخرج فيه خط
اد ويكون زاويه دآ مساويه لزاويه احد فاقول ان نسبة حد الى حد كنسبة
مربع حا الى مربع حا وكنسبة مربع دآ الى مربع دآ برهانه ان زاويه احد
مساويه لزاويه حدآ وزاويه احد مشتركه فمثلث احد ومثلث احد يكون زاويه
احد الباقيه كزاويه ادح الباقيه فنسبة حد الى دآ كنسبة اد الى حد ونسبة
اد الى اد فنسبة حد الى حد كنسبة مربع حا الى مربع حا وكنسبة مربع دآ
الى مربع اد وذلك ما اردنا ان نبين



لنفرض مثلث قايم الزاويه عليه احد وزاويته القايمه زاويه
حا ونقسم زاويه آ بنصفين بخط اد ونخرج خط اد كيف ما
وقع ونخرج من نقطه د خطا موازيا لخط اد وهو دز ونصل دح فاقول
ان نسبة اد الى دح مثل نسبة دا الى [illegible] ان مثلث احد ذا
من مثلث دح [illegible] ونصل دح لان مثلث دح [illegible] ومثلث دح
[illegible] فتكون زاويه داح مساويه لزاويه دح وتكون زاويه داح مساويه
لزاويه حاد فزاويه داح مساويه لزاويه دح [illegible]

~~[illegible]~~

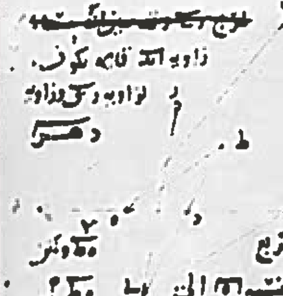
~~[illegible]~~

~~[illegible]~~

ولان زاوية طب ح كزاوية طاب وزاويه ب ح ط مشتركه
من نسبة اح الى ح د كنسبة ب ح الى ح ط فنصف [illegible] في
طاب في [illegible] د ولان نسبة [illegible] الى ط ب كنسبة د ح الى ط ح وزاويتا
[illegible] ط ا ا لزاويه ط [illegible] د زاويه ط ح د
[illegible]

فنقطع خط ... اما ... [illegible]

خط احد كنا اعوج



~~[illegible]~~

زاويه اكه قايمه يكون لوضع ح ت خطا ت مساو لخط بز ولان

لزاويه وحق وزاويه زاويتا ادب دات كقايمه لكن زاويه دات مساويه

دهـ ادت للموازاة كزاويه زهك فزاويتا دك

كقايمه وخط بح مساو لخط دب فخطوط حك

دب بك متساويه فسطح ابزهك مت و

ه ك و كز لوضع ح فنسبه اب الى بك كنسبه مربع اب الى المربع

نسبه اب الى بك كنسبه مربع اب الى المربع وكذ فنسبه اب الى ... ه ك ح

نسبه اب الى بك وذلك ما اردنا ان نبين لنفرض مثلث قايم الزاويه عليه ابح

ولتكن زاويه القايمه زاويه ا ولنخرج من نقطه ا الى خط ب ح خط ا د ولتكن زاويه

داح مساويه لزاويه حاب فاقول ان نسبه بح الى ح د كنسبه ... الى ...

برهانه انا نخرج من نقطه د خطا موازيا لخط اب وهو خط ده ~~[illegible]~~ فلان

زاويه باد قايمه تكون زاويه ادح قايمه فزاويه ابح التي عليها قايمه وزاويه داب مساويه

لزاويه حاد فيكون المثلثان متشابهين ... فنسبه اب الى ... كنسبه

... الى ... وكنسبه ... الى ... ونسبه ... الى ... كنسبه ... الى ... فنسبه

الى ... نسبه ... الى ... وذلك ما اردنا ان نبين وايضا فلتكن نسبه ... الى ...

كنسبه ... الى دك فاقول ان زاويه حاد مساويه لزاويه حاب برهانه

انا نخرج من نقطه د خطا موازيا لخط اب وهو خط ده فلان نسبه ... الى

... كنسبه ... الى ... ونسبه ... الى ... كنسبه ... الى ... يكون نسبه ... الى ...

كنسبه ... الى ... فخط ... مساو لخط ... ولان زاويه ...

قايمه تكون زاويه ... فزاويه ... ايضا

قايمه فخط ... مساويا لخط ...

... وزاويه ... مساويه لزاويه ~~[illegible]~~

~~[illegible]~~ فزاويه دكه كزاويه حاح وذلك

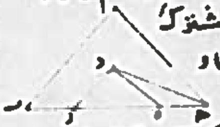
ما اردنا ان نبين لنفرض مثلثا عليه ابح ولنقسم زاويه ب بنصفين بخط بد

وزاويه ح بنصفين بخط حد ونصل اد فاقول ان زاويه ا قد انقسمت بنصفين

برهانه انا نخرج من خط ب ... مساويا لخط ب ا وخط ... مساويا لخط ... ونصل د ز د ه

فلان خط ... مساو لخط ب ... وخط ... مشترك تكون زاويه باد مساويه لزاويه

... وايضا لان خط ... مساو لخط ... وخط ... مشترك



فزاويه داح كزاويه ... وايضا لان خط ... مساو

لخط ... وقد كان تبين ان خط ... مساو لخط ... يكون

خط ... كخط ... فزاويه ... مساويه لزاويه ... لكن زاويه ... مساويه

لزاويه ... وزاويه ... مساويه لزاويه ... فزاويه ... مساويه لزاويه

حاد وذلك ما اردنا ان نبين لنفرض مثلثا قايم الزاويه عليه ابح ولتكن

زاويه القايمه ... ولنقسم زاويه ا بنصفين بخط اد ولنخرج من نقطه ا خطا

كيف ما وقع وهو خط ... ولنخرج من نقطه ... خطا موازيا لخط اد وهو خط ...

ونصل ... فاقول ان نسبه ... الى ... كنسبه ... الى ... برهانه

انا نقسم زاويه ... بنصفين بخط ... ونصل ... فزاويه ... قد انقسمت بنصفين

بخط ... ولنخرج من نقطه ... خطا يكون عمودا على ... وهو خط ... ونصل ...

... فلان زاويه ... قايمه وزاويه ... مساويه لزاويه ... يكون نسبه ...

الى ... كنسبه ... الى ... وزاويه ... مساويه لزاويه ... تكون زاويه ...

قايمه وكذلك زاويه ... فيكون نقط ... على محيط دايره ولذلك تكون زاويه

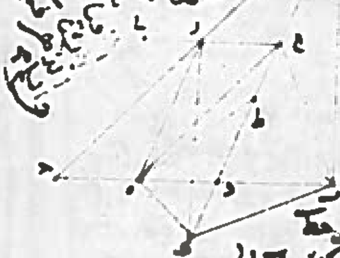
... قايمه ومساويه لزاويه ... القايمه ايضا فيسقط زاويه ... المشترك فيبقى

زاويه ... مساويه لزاويه ... التي هي كزاويه ... ونجعل زاويه ... مشتركه

فيصير زاويه ... مساويه لزاويه ... التي هي كزاويه ... لم ... على

استقامه ... فتكون زاويه ... مساويه لزاويه ... لموازاة ... وقد كان ...

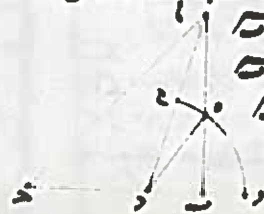
مساوية لزاوية د ه ز فتكون زاوية
ز ح ه مساوية لزاوية د ح ه فنقطة ز ح
ح على محيط دائرة ... فتكون
زاوية ح ه ك قائمة ونسبة ك ا الى ا ح
كنسبة ح ك الى ك ح وزاوية ه ك قد
انقسمت بنصفين بخط ه ح فنسبة ا ح الى ح ك
كنسبة ا ه الى ه ك ولكن نسبة ا ح الى ح ك كنسبة ا ا الى
ر ك فنسبة ا ه الى ه ك كنسبة ا ا الى ر ك وذلك ما اردنا ان نبين



تمت الاشكال ... والحمد لله

مقدمة يصرف في كتاب ثاوذوسيوس في الاكر

ليكن مثلث ا ب ج قائم الزاوية وقد اخرج من ا خط ا د الى ضلع ب ج فاقول ان
نسبة ج ب الى ب د اعظم من نسبة زاوية ا د ب الى زاوية ج برهانه انا نخرج
نقطة د من خط ب ج كزاوية ... زاوية ... ونخرج ... الى ... ونجعل مثل د ه
ونخط على مركز د وببعد د ه دائرة تقطع ضلع ... على ... فنسبة زاوية ح د ه
الى الزاوية ه د ز كنسبة قطاع ح د ه الى القطاع ه د ز ونسبة مثلث ا د ه الى المثلث
ه د ب اعظم من نسبة قطاع ح د ه الى القطاع ه د ز فنسبة مثلث ا د ه الى المثلث
ه د ب اعظم من نسبة زاوية ا د ه الى الزاوية ه د ب
واذا ركبنا تكون نسبة مثلث ا د ب الى المثلث ه د ب
اعظم من نسبة زاوية ا د ب الى الزاوية ه د ب
ونسبة مثلث ا د ب الى المثلث ه د ب كنسبة خط
ا ب الى الخط ب ه ... الى ... وزاوية
ه د ب كزاوية ج وذلك ما اردنا ان نبينه



بسم الله الرحمن الرحيم

لنفرض ثلثة اعداد متناسبة فنسبة ا الاصغر الى ب ا الاوسط كنسبة ب ا الى ج د الاعظم
وليكن ا مجهولا فاذا ضربنا ب ا في نفسه وقسمناه على ج د خرج ا وان كان ب ا المجهول
ضربنا ا في ج د واخذنا جذر المجتمع فيكون ب ا وان كان ج د المجهول ضربنا ب ا في نفسه
وقسمناه على ا فخرج ج د ولنفرض كل واحد من ا ب ا مجهولا والمعلوم جميعهما وهو
... فلان نسبة ا الى ب ا كنسبة ب ا الى ج د تكون اذا ركبنا نسبة ا و ب ا المعلوم الى ب ا
المجهول كنسبة ب ا و ج د المجهول الى ج د المعلوم والمجتمع من ضرب ا ب ا الاول في
ج د الرابع معلوم فالمجتمع من ضرب ب ا في جميع ج د ب ا معلوم فنخرج ج د الى ...
ولكن ... مساوي ب ا فيكون ... معلوما وخط ج د معلوم قد زيدت فيه زيادة
... وضرب ... في ... معلوم فاذا قسمنا ج د المعلوم بنصفين وضربنا نصفه في
نفسه وزيادة ما اجتمع من ... المعلوم واخذنا جذره كان معلوما فاذا القينا
منه نصف خط ج د المعلوم كان ... معلوما وهو مساوي ب ا فهو معلوم فاذا القيناه
من جميع ا ب ا المعلوم كان الباقي معلوما وهو ا فهو معلوم ولكن كل واحد من ا ...
مجهولا والمعلوم جميعهما ... فالمجتمع من ا في ج د مساوي ب ا المعلوم في نفسه فيكون
المجتمع من ا في ج د معلوما وجميعهما معلوم فاذا ضربنا نصف جميعهما في نفسه والقينا
منه المجتمع من ضرب ا في ج د واخذنا جذر الباقي كان معلوما وهو فضل نصف جميعهما
المعلوم على ا فاذا القيناه من نصف الجميع كان معلوما وهو ا فهو معلوم واذا القيناه من
جميعهما المعلوم كان ج د معلوما ولكن كل واحد من ج د ب ا مجهولا والمعلوم جميعهما
وعدد ا فيكون اذا ركبنا نسبة جميع ج د ب ا الاول الى ب ا الثاني كنسبة ... الثالث الى ا
الرابع فالمجتمع من ضرب جميع ج د ب ا المعلوم في ا المعلوم معلوم وهو ...
ضرب جميع ... في ... فتعدد ا زيدت فيه زيادة ... معلوم و ا
معلوم فاذا قسمنا ا بنصفين وضربناه في نفسه وزدنا عليه
المجتمع من ... واخذنا جذره كان معلوما