AL-KĀSHĪ'S ITERATION METHOD FOR THE DETERMINATION OF SIN 1°

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1. The subject of the present paper, an algorithmic method of solving a special cubic equation, stems from the construction of trigonometric tables to be applied in astronomical computations. A central problem in this construction is that of determining sin 1°. The standard Islamic method[1] for approximating sin 1°, which in its essential part goes back to Ptolemy,[2] proceeded from certain inequalities which, however, themselves set a limit to the accuracy of the result. In order to obtain better approximations by these procedures one would have to perform unreasonably long computations. Ptolemy was well aware of the shortcoming of his method and deplores that the chord of the third of an arc with known chord cannot be found by means of geometrical construction (διὰ τῶν γραμμῶν), stating that otherwise one could find *crd* $^1/_2$° from the known *crd* $1^1/_2$°.[3]

However, the Iranian astronomer and mathematician Jamshīd Ghīāth ud-Dīn al-Kāshī (d. 1429[4]) set up the trisection equation and succeeded in devising an ingenious method to solve it which yields sin 1° with any desired accuracy. Al-Kāshī was an eminent computer as is witnessed by his works, in particular his computation of 2π to 10

[1] See f. ex. L. Sédillot, *Prolégomènes des tables astr. d'Oloug-Beg* (Paris, 1853), p. 75 ff.

[2] *Almagest*, I, 10, IV, see f. ex. K. Manitius: *Des Cl. Ptolemäus Handbuch der Astronomie*, I (Leipzig, 1912), p. 32 ff.

[3] *Ibid.*, p. 32–33.

[4] According to a marginal note on the last folio of the India Office copy (MS 430, Ethé 2232) of al-Kāshī's astronomical tables, the *zīj-i Khāqānī*. Dr. E. S. Kennedy, who is preparing an edition of this work, has kindly given me this new date, for which I wish to thank him as well as for his explanation of Sédillot's misreading and his translation of the marginal glosses from the *zīj-i Khāqānī* which appear in sec. 5 of this paper. The text is available through the courtesy of India Office, London.

[5] (*a*) P. Luckey, "Der Lehrbrief über den Kreisumfang ... von ... al-Kāšī," herausgeg. v. A. Siggel, *Abh. d. deutschen Akad. d. Wiss. zu Berlin, Kl. f. Math. u. allg. Nat.*, Jg. 1950 Nr. 6 (Berlin, 1953).

For other works of al-Kāshī see:

(*b*) P. Luckey, *Math. Annalen*, 120. Bd., 1948, p. 217–274.

(*c*) P. Luckey, "Die Rechenkunst bei ... al-Kāšī...," *Abh. f. die Kunde des Morgenlandes*, v. XXXI, 1 (Wiesbaden, 1950).

correct sexagesimal, or 17 correct decimal, digits[5]—he gives both representations, the consistent use of decimal fractions being an invention of his.[6]

The method is described by Miriam Chelebi[7] (d. 1524/25), a grandson of ar-Rūmī,[8] al-Kāshī's successor as director of the observatory at Samarkand, in his commentary to the tables of Ulugh Beg.[9] It has been treated by Woepcke,[10] who rather obscures its structure by employing infinite series in his discussion, by Hankel,[11] who undoubtedly understood the method, but fails to justify it, and who does not compute sexagesimally, and by Braunmühl,[12] who follows Hankel.

2. In the following all numbers will be written in the sexagesimal system, unless otherwise stated, the digits being separated by commas and the integral and fractional parts by semicolons, e. g., 1,58; 0,32 means $1\cdot 60 + 58 + 0\cdot 60^{-1} + 32\cdot 60^{-2}$.

Furthermore, the sine function of an arc A used in Islamic astronomy—we shall denote it by Sin A—is $1{,}0\cdot\sin A$, thus exhibiting the same sequence of sexagesimal digits as sin A but with the semicolon moved one place to the right, or Sin A is sin A "once elevated" as it is called. Sin A is therefore the length of half the chord of the arc 2A measured with a unit which is 0;1 times the radius of the circle.

Our method proceeds from the identity:

$$\sin 3A = 3\cdot\sin A - 4\cdot\sin^3 A,$$

or rather:

$$\mathrm{Sin}\ 3A = 3\cdot\mathrm{Sin}\ A - 0;\,0{,}4\cdot\mathrm{Sin}^3 A,$$ [13]

or specially:

$$\mathrm{Sin}\ 3^\circ = 3\cdot\mathrm{Sin}\ 1^\circ - 0;0{,}4\cdot\mathrm{Sin}^3 1^\circ.$$

Sin 3° = Sin (18° − 15°) can be determined with any desired accuracy by operations the most complicated of which is extraction of

(d) E. S. Kennedy, "A Fifteenth Cent. Lunar Eclipse Computer," SCRIPTA MATH., v. XVII, 1951, p. 91–97.
(e) E. S. Kennedy, "An Islamic Computer for Planetary Latitudes," *Journ. Am. Or. Soc.*, v. 71, 1951, p. 13–21.

[6] See f. ex. Luckey, *loc. cit.* (in note 5(*a*)), Vorwort des Verfassers.

[7] H. Suter, *Die Mathematiker und Astronomen der Araber und ihre Werke* (Leipzig, 1900), p. 188.

[8] *Ibid.*, p. 174.

[9] Sédillot, *loc. cit.* (in note 2), p. 77–83.

[10] F. Woepcke, *Journ. de Math. pure et appl.*, Tome XIX, 1854, p. 153–176 and p. 301–303.

[11] H. Hankel, *Zur Geschichte d. Math. in Alterthum und Mittelalter* (Leipzig, 1874), p. 289–293.

[12] A. v. Braunmühl, *Vorl. ü. Geschichte der Trigonometrie*, I (Leipzig, 1900), p. 72–75.

[13] For the derivation of this formula, or its equivalent, see Sédillot, *loc. cit.* (in note 2), p. 81 ff.

square roots. The problem of finding Sin 1° therefore reduces to that of solving the cubic equation:

$$\text{Sin } 3^\circ = 3\cdot x - 0;0,4\cdot x^3$$

or:

$$x = \frac{x^3 + 15,0\cdot \text{Sin } 3^\circ}{45,0}.$$

Substituting the previously found value for Sin 3°, namely 3;8,24,-33,59,34,28,15 (Chelebi has erroneously 3;8,24,33,59,*32*,28,15[14]), we get:

$$x = \frac{x^3 + 47,6;8,29,53,37,3,45}{45,0}. \qquad (1)$$

x must be less than 1,0—indeed we know from cruder methods[15] that Sin 1° $\sim$ 1;2,...—so let $x = a_1;\, a_2,\, a_3,\, \ldots.$ Using but the fact that x must satisfy (1) we now proceed to determine the a's successively. We have:

$$a_1;\ldots = \frac{a_1;\ldots^3 + 47,6;\ldots}{45,0}.$$

However, if we are interested only in the integral part a_1 of the solution it seems plausible that we can disregard $a_1;\ldots^3 \sim 1$ since we have to divide it by 45,0. Postponing for the moment the accurate justification of this—or rather the similar steps below—we obtain:

$$a_1;\ldots = \frac{47,6;\ldots}{45,0}.$$

This division gives the quotient 1 and the remainder $r_1 = 2,6;8,29,\ldots$ Thus $a_1 = 1$. We now have:

$$1;a_2,\ldots = \frac{1;a_2,\ldots^3 + 47,6;8,\ldots}{45,0}$$

or:

$$0;a_2,\ldots = \frac{1;a_2,\ldots^3 + 2,6;8,\ldots}{45,0} = \frac{1;a_2,\ldots^3 + r_1}{45,0}.$$

Since we now are concerned with the first place after the semicolon we can replace $1;a_2,\ldots^3$ by 1^3 and obtain:

[14] *Ibid.*, p. 81.

[15] E. g., Ptolemy gives in the table of chords, *Almagest*, I, 11 (Manitius, *loc. cit.* (in note 3) p. 37) *Crd* 2° $\sim$ 2;5,40, or see any of the Islamic astron. tables.

$$0;a_2,\ldots \sim \frac{1^3 + r_1}{45,0} = \frac{2,7;8,\ldots}{45,0}.$$

This division gives the quotient 0;2 and the remainder $r_2 = 37;8,29,\ldots$. Thus $a_2 = 2$.

Going one step further we find:

$$1;2,a_3,\ldots = \frac{1;2,a_3,\ldots{}^3 + 47,6;8,\ldots}{45,0}$$

or:

$$0;0,a_3,\ldots \sim \frac{1;2^3 + 47,6;8,\ldots - 45,0\cdot 1;2}{45,0} = \frac{r_2 + (1;2^3 - 1^3)}{45,0}.$$

This division gives the quotient 0;0,49 and the remainder $r_3 = 0;29,42,$-1,37,3,45. Thus $a_3 = 49$.

The above enables us to verify immediately Chelebi's rule: "Then one takes the cube of the n quotients (i. e., forms $a_1;\ldots,a_n{}^3$) then the difference between this cube and that of the first $n - 1$ quotients (i. e., $a_1;\ldots,a_{n-1}{}^3$). This is added to the remainder of the nth division. One has now the $n + 1$'st quotient (i. e., a_{n+1})." Chelebi makes this statement for the cases $n = 1, 2, 3, 4$ and then adds: "and one may continue the computation in the same manner as far as desired." Thus he obtains Sin 1° = 1;2,49,43,11.

3. That we are justified in replacing $a_1;a_2,\ldots,a_{n-1}{}^3$ for $a_1;\ldots,a_{n-1},a_n,\ldots{}^3$ when we are only concerned with the nth digit of the solution of (1) may be seen as follows: Let us take the worst possible case where we use $1,0^{-n+2}$ for $0;0,\ldots,0,a_n,\ldots$, and let us use the too large value 1;3 for $a_1;\ldots,a_{n-1}$ (this is justified by other ways of approximating Sin 1°, as stated above). We get then:

$$(1;3 + 1,0^{-n+2})^3 = 1;3^3 + 3\cdot 1;6,9\cdot 1,0^{-n+2} + 3\cdot 1;3\cdot 1,0^{-2n+4} + 1,0^{-3n+6} < 1;3^3 + 4\cdot 1,0^{-n+2} \text{ for } n \geqq 3.$$

Fortunately this has to be divided by $45,0 = \frac{3}{4}\cdot 1,0,0$ (see (1)) so that the final influence of disregarding everything beyond the first n-1 digits is less than $6\cdot 1,0^{-n}$, or 6 in the $n + 1$'st digit.

Yet, this may from time to time change a_n to a value which is 1 too low. But if we for a moment think of the method as an iteration process for approximating a solution of the equation $x = f(x)$, noticing that $|f'(x)|$ is much less than 1 in the neighborhood of the point of intersection between $y = x$ and $y = f(x)$, the convergence of the approximants becomes evident. Thus the error in a_n will eventually be compensated, and it will happen by one of the subsequent a's, usually a_{n+1}, assuming a value greater than 60.

4. That this elegant method is due to al-Kāshī seems quite certain. Chelebi mentions as its inventor a person whose name Sédillot transliterates as *Atab-Eddin-Djemshid*. This was doubtlessly a misreading of the Arabic characters of the MS for *Ghīāth ud-Dīn Jamshīd*, al-Kāshī's given name and honorific title. For a missing dot over the initial *ghain* makes it an *'ain*, hence Sédillot's *A*. The only way of distinguishing between a *ya* and a *ta* (Sédillot's *t*) is by the presence of two dots under or over the letter, and probably the dots were missing in the MS. In the same manner *tha* and *ba* are distinguishable only by dots. The other variants in the transliteration are vowels, not normally indicated in Arabic script.

Further, al-Kāshī describes in his astronomical tables, the *zīj-i Khāqānī*, the then standard method of approximating Sin 1°. On folio 32 recto of the India Office copy of this work, mentioned already in note (4), he arrives at his final result, Sin 1° $\sim$ 1;2,49,43,13,5. But opposite this, and running down the left hand margin of the folio, is a series of glosses. They are apparently all in the same hand, but it must be presumed that they were entered successively in some previous copy of the work by various individuals, beginning with al-Kāshī himself. In the following translation[16] of these glosses the italicized material is in Arabic in the MS, the remainder in Persian:

"The ancients found no method for obtaining accurately the sine of the third of an angle whose sine is known. We have developed a method and have written an essay explaining it to its limit. We have extracted the sine of one degree by that method; it is 1,2,49,43,11,14,-44,16,19,16 ninths. *This is an annotation of the author; may God forgive him.*"

"Mostly (?) the successful completion of this essay was not attained. After the death of the author, *upon him (be) forgiveness!*, this weak person (i. e., the present scribe) saw his book in which he had commanded the extraction (i. e., he had extracted) this sine of one degree by the method of algebra without question. . . (word illegible), and on its margin a writing that, '*This is the sine of one degree; we extracted it by inspired strength from the Eternal Presence.*'"

"*Our master*, Mu'īn ud-Dīn, *the (astronomical) observer.*"

"This indigent destitute person Muḥammad. . . (word illegible) after the martyrdom of the martyred Sultan Ulugh Beg (saw, or found?) in

[16] My thanks are also due to Dr. Ahmad Fakhri for help with the translation.

the portfolio of that presence (i. e., honorable person) a scrap (of paper?) in the handwriting of our forgiven (i. e., deceased) master."

The above-mentioned essay, whether completed or not, is mentioned in the preface to al-Kāshī's *Key to Arithmetic*,[17] and the value of Sin 1° determined by this method is used in his computation of π.

[17] See Luckey, *loc. cit.* (in note 5(*c*)), p. 6.

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