motion scales to lay out on the inner ring an arc equal to the mean travel of the planet in question in the time elapsed between the given date and one of the years permanently marked. Then rotate the inner ring so that the initial point of the marked arc lies along the projection of the fixed point just utilized. The desired mean position will now be at the projection on the outer ring of the arc's terminal point.

Kāšī remarks that if each of the circular scales is made so that it can be individually rotated, there will be no need for the inner ring.

For material on f. 15r:1 - 17v see Section 43.

20. Determination of the Solar True Longitude (f. 19v:4 - 20r:1)

The Ptolemaic model of the sun's motion being extremely simple, the determination of its true position with the instrument is equally simple. The plate circumference itself represents the solar deferent, the apogee (represented in Figure 1 and 8 by the plate tongue) having already been fixed at its proper longitude. The fictitious center (F in either of Figures 1 or 8) stands for the center of the universe. Thus CF, the distance from the plate center to the fictitious center, is the solar eccentricity. Having computed the sun's mean longitude, lay it off as the arc AM by using the divisions of the ring. M is Kāšī's mark of the mean. Then the angle at which M is observed from F, measured from the equinoctial direction, is the true longitude. To evaluate this angle, lay the edge of the ruler along FM and rotate the alidade until it is parallel to the ruler. Then the arc AT is the required true longitude.

The angular difference between the true and the mean longitudes, arc MT, is the solar equation. The length of MF, measured

Figure 8. The Instrument, Set Up for Finding the Sun's True Longitude

with the scale marked off on the ruler, is the earth-sun distance. In this case there is no need for a norming operation since the units are the standard ones in ancient astronomy,
sixtieths of the deferent radius.

21. Determination of the Lunar True Longitude (f. 20v:5 - 20v:13)

In using the instrument to find the true ecliptic position of the moon at a given time, first compute from the mean motion table: (1) the lunar mean longitude, (2) the lunar mean anomaly, and (3) the solar mean longitude. Then mark P (see Figure 1), the position of the mean moon, on the edge of the ring. Find e, the moon's mean elongation, being the difference between (1) and (3) above. Starting from P measure clockwise along the edge of the ring an arc PS equal to 2e, the double elongation. Rotate the plate until its tongue comes opposite S and fix it here with the peg. By means of the alidade mark E the intersection of CP with the moon's deferent. E is Kāshī's mark of the center. Put the ruler along E and N, the point opposite, and rotate the alidade until it is parallel to the ruler. Mark B, the intersection of the alidade edge with the edge of the ring, the directed segments CB and NE thus being parallel and in the same sense. As the author says, it is from this point that the mean anomalistic motion is to be measured. Now rotate the alidade clockwise the amount of the anomaly, (2) above, through the arc BK. This puts the alidade in the position shown in our figure.

In order to complete the Ptolemaic lunar configuration all that remains is to lay off the epicycle radius in proper length and direction (parallel to the alidade edge) from E. Its endpoint L will then correspond to the position of the moon in space. However, this construction presents some difficulty from the practical point of view, for, as is the case with our figure, L may run off the plate entirely. Moreover, we are not primarily interested in locating L, but rather in determining the direction of the vector CL. We recall that a lunar difference mark (Cf. f. 12r:9) has been put on the alidade edge at a distance from the center equal to the epicycle radius. The author specifically cautions us that the position of the alidade should now be such that the difference mark falls opposite the end of anomalistic motion, i.e. D being the difference mark, vector CD must have a sense opposite to vector EL. Mark the position of D on the plate and lay the ruler along DE. Since the latter equals the required vector sum of CE and EL, if the alidade is now rotated parallel to the ruler, its intersection, G, with the edge of the ring will give the required true longitude of the moon. The final passage in the chapter (f. 20v:12-13) applies only if the plate has not been set in the ring to show the proper lunar apsidal longitude at the start of the operation.

22. Determination of a Planetary True Longitude (f. 21r:1 - 21r:7)

The solution of this problem with the equatorium resembles that of the analogous lunar problem. First compute from the table of mean motions the mean longitude of the sun for the instant required, and either the planet's mean longitude (for a superior planet) or the compound anomaly (for an inferior planet). Presumably the plate has already been fixed inside the ring so that the solar and planetary apogees have their proper longitudes. Letter references in the sequel are to Figure 9, which shows the final positions of alidade and ruler in the solution of such a problem for the planet Mars. The drawing is to scale.

Rotate the alidade until its edge is at the planet's mean
and simultaneously parallel to the alidade edge. Where the ruler intersects the deferent, E, make the center-mark. This locates the planet’s epicycle center at the given time.

For a superior planet turn the head of the alidade, that end of it opposite the side on which the difference marks (cf. f. 12r:9) have been made, until it reaches the mean solar longitude, $H$, and mark on the plate the point where the difference mark, $D’$, then lies. Now put the ruler so that its edge lies along $D$ and $E$, and rotate the alidade until it is parallel to the ruler. Then the intersection $M$ of the head of the alidade with the divisions of the ring gives the required true longitude. For the vector $D’C$ has been constructed in magnitude and direction equal to the vector $EM$ from the epicycle center to the planet, and side $CM$ of parallelogram $CD’EM$ gives the direction of the vector sum of $CE$ and $D’C$. And since $EM$ is parallel to $CH$, $EM$ has the required direction of the mean sun.

For the inferior planets, Venus and Mercury, the construction is of the same sort, bearing in mind that their mean longitude is the mean longitude of the sun. The same figure may be used to illustrate the configuration, although of course it will no longer be to scale. Now $L$ is the sun’s mean longitude and arc $PA’H$ is the compound anomaly.

23. The Planetary Equations (f. 21v:8 - 22r:11)

The term equation (ta’dil) was in general used in ancient and medieval astronomy to denote a correction, in general small, applied to a function representing some phenomenon. The usage has survived in modern astronomy in such expressions as "the equation of time".

A planet’s equation in longitude was defined as the difference between its true and mean longitudes. The equation was
in turn made up of two components. The \textit{first equation}, or \textit{equation of the center}, is that caused by the eccentric \textit{quant}, the \textit{second equation} is the effect of the planet's motion on the epicycle. (Cf. [4], p. 96.) Clearly the algebraic sum of the first and second equations equals the equation of the planet.

All \textit{zijs} (astronomical handbooks) contained tables of both equations, essential for the computation of true longitudes. In addition, since the second equation is dependent on the first, a fairly involved interpolation scheme was necessary, to estimate the effect of the first equation on the second. It is the great advantage of an analog computer such as the equatorium that true longitudes are determined without the interposition of the equation or its components at all.

Nevertheless, should the user want to find the equations with the instrument he can do so by following the instructions in Chapter II, 6. The author may have had in mind an individual who is computing a position accurately with a \textit{zi}, but who wants a quick, crude check of his partial results.

For whatever reason, having made the prescribed marks, at \textit{L}, the mean position in Figure 9, and \textit{F}, the projection of the epicycle center on the ring, it is clear that the arcs \textit{LF} and \textit{FM} are the first and second equations respectively.

When a mean position was measured from the apsidal line of the particular planet it was called the \textit{center}, here arc \textit{A''HL}. Addition of the first equation to the center gave the adjusted center, here arc \textit{A''HF}.

The closing statement of the chapter, that if the adjusted mean longitude is subtracted from the sun's mean (for the superior planets), or from the compound anomaly (for the inferior planets), the adjusted anomaly remains, is equivalent to the valid expression

\[ PA'H = PA'F + FA'H. \]

24. \textbf{The Lunar Equations (f. 22r:2)}

In the ancient and medieval lunar theory the terms \textit{first equation} and \textit{second equation} did not denote angles analogous to those associated with the same terms in the planetary theory.

The lunar anomaly was not laid off from the epicyclic apogee of the model. Ptolemy found it necessary instead to count the anomaly from an epicyclic apogee as determined with respect to the "point opposite" (N on Figure 1), not with respect to the center of the universe, C. (Cf. [41], p. 195).

The \textit{first lunar equation} was defined as the angular displacement in the epicyclic apogee caused by this situation. (See [23], f. 78r). On Figure 1 it is angle \textit{CEN}. And since \textit{CH} has been constructed parallel to \textit{EN}, angle \textit{BCP} also equals the first equation. But this angle, being a central angle on the plate, is measured by arc \textit{PB}. \textit{P} is evidently the "second mark" of our passage in the text, and \textit{B} is the "mark of the beginning of the anomalistic motion", i.e. the statement in the manuscript is valid.

The \textit{second lunar equation} was defined as the effect of the anomaly on the mean longitude, provided that the epicycle center were at maximum distance from the center of the universe. Since, in general, the actual distance was less than the maximum, in using the lunar tables it was necessary to add to the second equation a suitable correction. On the configuration on the instrument, however, this second equation does not appear as such, and the author of the manuscript makes no mention of it.
25. The Latitude of the Moon (f. 22r:13 - 22v:7)

The moon's orbit can be thought of as lying in a plane which makes a fixed angle of about five degrees with the plane of the ecliptic. The line of nodes, the intersection between the two planes, is not fixed, but rotates slowly in a negative direction with respect to the vernal point. Let \( \beta \) be the moon's latitude, \( \lambda \) its longitude, and \( \lambda_n \) the longitude of the ascending node, the ecliptic point through which the moon passes in going from southern to northern latitudes. Then we have

\[
\sin \beta = \sin 5^\circ \sin (\lambda - \lambda_n) = \sin 5^\circ \sin \omega,
\]

an exact spherical-trigonometric relation involving \( \omega \), the argument of the latitude, the moon's ecliptic distance from the ascending node.

In computing lunar latitudes Ptolemy did not use (1), but the equivalent of the expression

\[
\beta = 5^\circ \sin \omega,
\]

a reasonably good approximation. (Cf. [43], ed. of Halma, vol. i, p. 316).

We have already described how to find \( \lambda \) and \( \lambda_n \). Once put on the ring of the instrument, their differences can be noted directly. All versions of the text (f. 22r:13, NS p. 277) say add, rather than subtract, \( \lambda \) and \( \lambda_n \). This usage is explained by recalling that the motion of the node is contrary to the order of the zodiacal signs. Hence the nodal positions are to be plotted, as we would say, negatively, and algebraic subtraction in this case becomes addition.

Having determined \( \omega \), to complete the operation rotate the alidade until its pointer crosses the graduations of the ring at the point corresponding to \( \omega \). Then note at which of the latitude lines the edge of the alidade crosses the moon's latitude circle described in Section 12 above. The result is the lunar latitude.

The operation is pictured in Figure 10, where we note that

\[
\sin \theta = \sin 5^\circ \sin \lambda
\]

by similar triangles and use of the properties of the latitude lines (cf. Section 12),

\[
\frac{\sin \theta}{\sin \omega} = \frac{\sin 5^\circ}{1.0},
\]

or

\[
\sin \theta = \sin 5^\circ \sin \omega.
\]

Now comparison of expressions (1) and (3) shows that \( \theta = \beta \).
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that is, Kāshi's process utilizes the exact expression for determining the lunar latitude. In theory it is better than the result of using expression (2).

In the original version of the Nuzhah (NS, p.277) the method of determining the lunar latitude is as follows. Elevate the alidade so that it makes an angle of five degrees with the equating diameter. Mark on the edge of the alidade the point corresponding to the lunar latitude on the scale. Project this point parallel to the equating diameter on to the divisions of the ring. The number on this scale corresponding to the projection is the lunar latitude. It is easy to show that this process also yields the result of applying (3).

The Chaucer equatorium manuscript [42] gives a method of determining the lunar latitude as resulting from (2). There is no provision for the computation of planetary latitudes.

26. Planetary Latitudes in the Almagest

Chapter II,7 is by far the largest section of our text, taking up about a quarter of the entire treatise. That this is the case is not surprising, for the topic of which it treats, planetary latitudes, is the complicated result of a complicated phenomenon. In determining planetary longitudes it is convenient to regard all motion as taking place in the plane of the ecliptic, which has the effect of making the problem a two-dimensional one. This cannot be done with the motion in latitude. The accompanying theory assumes that the planes of both deferent and epicycle make small but non-zero angles with the ecliptic and with each other, some of the angles being variable.

Ptolemy regarded the latitude of a planet at any instant as being the algebraic sum of two or three component parts, known as the first latitude \( \beta_1 \), second latitude \( \beta_2 \) and, in the case of the inferior planets, the third latitude \( \beta_3 \). In order to define these components it will be necessary for us first to introduce a number of other terms.

It was customary to designate as first diameter of the epicycle, the line of true epicyclic apsides, that is, the diameter joining the true epicyclic apogee and perigee. In Figure 11, BC, B'C', and B"C" are three positions of the first diameter. The second diameter of the epicycle is the diameter perpendicular to the first ([4], pp.61, 64). Examples from the same figure are DF, D'F' and D"F".

Figure 11. The Tilted Deferent and Epicycle

The first latitude is the angle made by the ecliptic plane
with the line joining the center of the universe and the epicycle center. In other words, $\beta_1$ is the portion of the total latitude due to the deviation (hereafter denoted by $i$) the angle at the line of nodes between the deferent and ecliptic planes.

The second latitude is the component due to the inclination (denoted by $j$), the tipping of the epicycle about its second diameter, the latter being maintained parallel to the ecliptic plane in the case of the superior planets.

With the two inferior planets, however, a third latitude is involved, caused by the obliquity, a tilting of the epicycle about its first diameter.

A useful concept is that of the center of latitude, denoted by $\tilde{\omega}$ and defined as being the distance on the ecliptic from the ascending node of a planet to the true longitude of its epicycle center. In Figure 12 the expression

$$\tilde{\omega} = \angle NEC = \angle NEA + \angle AEC$$

holds identically for all positions of C and A. Angle AEC is called the adjusted center and is denoted by $\tilde{\alpha}_a$. In the case of the inferior planets, $\tilde{\omega} = \tilde{\alpha}_a + 90^\circ$, the line of apsides and the line of nodes being for them perpendicular. For the other planets the angles between these two lines are given in the table on f. 22v.

Chapter II,2 can be regarded as comprising four sections. The first, f. 22r:13 - 22v:7, disposes of the lunar latitude. This we have already commented on, in Section 25. The second section, f. 22v:7 - 25r:11, the longest and most involved of the four, describes simultaneously the determination of the latitudes of the superior planets and $\beta_2$ for the inferior planets. We separate the commentary on this section into two parts, Sections 28 and 30 below. The third section, f. 25r:11 - 26r:7, explains how to determine $\beta_3$ for an inferior planet and is discussed by us in Section 29. Finally f. 26r:7 - 26v:6 has to do with $\beta_1$ for an inferior planet. As being the simplest operation of the planetary latitude group, we describe it in the section immediately following this.

27. The First Latitude of the Inferior Planets (f. 26r:7 - 26v:6)

The geometric model used in the Almagest for the inferior planets has the deferent plane seesawing through a small angle north and south of the ecliptic plane about an axis in the
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ecliptic plane perpendicular to the (deferent) line of apsides and passing through the center of the universe. The deviation of the deferent plane is given by the expression (cf. [8], p.199)

\[ i = i_m \sin \omega, \quad i_m = \begin{cases} 0°10' & \text{for Venus}, \\ -0°45' & \text{for Mercury.} \end{cases} \]

Then, if in Figure 11, \( H' \) is the location of the epicycle center at a given time, in the spherical triangle \( NH'K \), arc \( H'K \) is \( \beta_1 \), the desired first latitude, angle \( H'KN \) is a right angle, and since \( i \) (angle \( H'NK \)) is small the approximate equality

\[ \beta_1 \approx i \sin \omega = (i_m \sin \omega) \sin \omega = i_m \sin^2 \omega \]

subsists. This is the Ptolemaic theory.

The construction described in our text yields a result which is very slightly different. The alidade is elevated from the equating diameter by an angle equal to \( \omega \). We then note at which of the latitude lines the edge of the alidade intersects the proper latitude circle (cf. Section 12). Suppose the designation of this latitude line is \( x \). This number will then satisfy the expression

\[ \sin x = \sin i_m \sin \omega, \]

the transformation being of the same sort as that described for the lunar latitude in Section 25 above.

Now put the alidade perpendicular to the equating diameter and mark on its edge the point where the \( x \) latitude line crosses it. The distance from the mark to the plate center will then be \( \sin x \). Return the alidade to its original position so that it makes an angle \( \omega \) with the equating diameter, and note the latitude line on which the mark falls. If this is the \( y \) latitude line \( y \) is the desired first latitude.

COMMENTARY

The second part of the operation is an iteration of the first with \( i_m \) replaced by \( x \) and \( x \) replaced by \( y \). Hence we have

\[ \sin y = \sin x \cdot \sin \omega = (\sin i_m \cdot \sin \omega) \sin \omega = \sin i_m \cdot \sin^2 \omega. \]

Comparison of expression (5) with (7) and the small size of \( i_m \) verifies the approximate equality of the results obtained with the two expressions.

In practice the construction is completely out of the question because of the minute semicircle required to be drawn on the plate. Even from a strictly theoretical point of view, what was justified in the case of the lunar latitude cannot be defended here. In both cases a right spherical triangle was involved, the solution of which involved an expression analogous to (6). But (4) is a device for giving a simple harmonic motion, and is not an approximate equality obtained from an accurate expression such as (6).


In Figure 11 three positions of a planetary epicycle are shown, its center lying at \( N, H, \) and \( H' \). In the first position the inclination \( j \), the angle the first diameter makes with the radius vector from the center of the universe to the epicycle center, is zero. In the second case it has taken the maximum value, \( i_m \), and the third case illustrates a general position intermediate between the two. In all three situations the second diameter \( (D^*F^*, DF, \text{and} D'F') \) is shown parallel to the ecliptic plane, hence \( \beta_3 \) and the obliquity are zero. Values
of \( j_m \) for each planet are to be found in the table on f. 23r. They are identical with those of the Almagest ([43], ed. of Halma vol.ii, p.255). In the transcription we have put a minus sign in parentheses before the entry for Venus so that the ultimate result will have the proper sign, north being taken as positive. A precise and general statement of the value of \( j_m \), angle \( B^H E \), for all positions on the deferent, is

\[
(8) \quad j(\\lambda_a) = j_m \sin \lambda_a = j_m \sin (\omega - 90^\circ) = -j_m \cos \omega
\]

for the inferior planets, and

\[
(9) \quad j(\omega) = j_m \sin \omega
\]

for the superior planets. Note that for an inferior planet maximum inclination occurs when the epicycle center is on the nodal line, in contrast to the situation pictured in Figure 11. This figure, however, shows all the elements affecting the latitude \( \beta = \beta_1 + \beta_2 \) of a superior planet. For the intermediate position shown, \( \omega \) is the arc NK, the inclination, angle \( B^H E \), is

\[
(\angle BHE) \sin \angle NEK,
\]

and \( \beta \) is the angle \( P^HE \) makes with the ecliptic plane. Also, if the relative position of the ecliptic plane there shown is ignored, the same figure may be used to illustrate the determination of the second latitude of an inferior planet.

Ptolemy regarded as prohibitively complicated a direct, general computation and tabulation of \( \beta_2 \) as a function of two variables, \( \lambda_a \) and \( a \). He therefore adopted the simplification sketched below, the ensuing sacrifice of accuracy not being large. He computed \( \beta_2 \) for general values of \( a \), but for \( \lambda_a = 90^\circ \), i.e. for the position of the epicycle center at \( H \) which gives maximum inclination of the first diameter, hence maximum \( \beta_2 \). Here this max \( \beta_2(a) \) is the angle \( PE \) makes with the deferent plane. He then defines the second latitude in general as

\[
(10) \quad \beta_2(\lambda_a, a) = \max \beta_2(a) \cdot \sin \lambda_a,
\]

expressed in modern notation.

Kāshī finds max \( \beta_2 \) for a general \( a \) by means of a clever construction in the manner of descriptive geometry in which the single plane of the instrument's plate is regarded as containing the planes of the deferent, the plane denoted by \( v \) in Figure 11, and the plane of the epicycle. The equating diameter (UC in Figure 13) represents the intersection between \( v \) and the deferent plane, with \( H \) (the center of the instrument) representing the center of the epicycle. \( E \) is the latitude point (cf. Section 13) of the particular planet being dealt with, so placed that \( EH \) equals the distance (along the perpendicular to the line of nodes) from the center of the universe to the deferent of the planet. The plane \( v \) is folded down into the plane of the plate, about \( EH \), in such fashion that its trace with the epicycle plane takes the position \( H3 \). And the plane of the epicycle is rotated into the deferent plane through an angle of \( j_m \) about its second diameter \( DF \) so that its first diameter takes the position \( BC \).

If, now, the true length of \( PE \) (in Figure 11) can be constructed, as well as the perpendicular from \( P \) to the plane of the deferent, then the problem will be as well as solved. For if a right triangle is constructed with \( PE \) as hypotenuse and altitude equal to the perpendicular just referred to, the acute angle at its base will be the desired max \( \beta_2 \).
The alidade is put so that its pointer crosses the ring at the graduation corresponding to the amount of the anomaly (a). The "first mark" (1 in Figure 13) is then made on the plate at the position of the permanent difference mark on the alidade edge. This makes H1 equal in length to the epicycle radius of the planet. Now make the "second mark" (2 on Figure 13), being the projection of 1 on the equating diameter.

By use of the alidade make the "third mark", 3, on the plate so that angle 3H2 = j_m and H2 = H3. Through point 3 draw a line M3 parallel to the equating diameter. The distance between this pair of parallel lines is the altitude of the desired right triangle.

To find the hypotenuse of the triangle mark point S the "substitute for the latitude point" on the equating diameter so that S2 = E3. It is as though, when the epicycle is rotated into the deferent plane, the right triangle EP3 also is flattened down into the deferent plane without change of size, the vertex E being made to slide outward along EH, 3 taking the position 2, and P the position 1. Then point S1 is the true length of EP in Figure 13.

Now, with center S and radius S1 draw an arc cutting M3 in R. Then the angle RSH is the desired max β_2(a), and it can be measured in the usual manner by placing the alidade parallel to SR and noting the point on the divisions of the ring where the pointer of the alidade falls.

The final steps in the determination consist of performing an operation analogous to the lunar latitude determination, the only differences being that the lunar latitude semicircle of radius Sin 50 is replaced by a mark on the alidade edge whose distance from the plate center is Sin max β_2(a). Thus the resulting β_2 from the instrument is given by the expression

\[ \sin \beta_2(a) = \sin \max \beta_2(a) \cdot \sin \bar{x}_a, \]

which leads to a result differing slightly from that of (10). The criticism made of the β_1 determination in Section 27 above applies also to this section of the text.

The passage concludes (f. 25r:3-25r:11) with the usual complicated rules for determining the sign of the result. Negative numbers were not known to the medieval scientists. In cases where a result involved one of two directions, the result itself having issued from a combination of two or more elements, each with a direction of its own, there was nothing for it but to enumerate all possibilities in a set of rules.
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The term parecliptic, occurring in this passage, requires explanation. Following Nallino ([3], vol. ii, p. 352) we translate by it the word mumaththal, which is a contraction of al-falak al-mumaththal li-falak al-burūj. It is a sort of reference circle, one for each planet, lying in the plane of the ecliptic and having its center at the center of the universe. Apparently the Islamic astronomers made use of the concept because they thought of the universe in terms of a set of more or less concentric and overlapping shells, each shell containing the deferent and epicycle of a planet. On the parecliptic of each planet were projected the positions of that planet. If in any context the word ecliptic is substituted for parecliptic the essential meaning will be unchanged.

29. The Third Latitude of the Inferior Planets (f. 25r:11 - 26r:17)

In approximating the final component of the latitude Ptolemy neglects the simultaneous but small effects on it of the first two components. Thus in Figure 14, which shows the upper half (BPC) of the epicycle tilted about its first diameter (BC), the plane of EON' may be considered either as the plane of the deferent or as that of the ecliptic. E is the center of the universe, P a general position of the planet corresponding to an anomaly of α. EM is tangent to the epicycle and the right spherical triangle CM'N' represents a portion of the celestial sphere, N' being the projection of M' on the ecliptic arc ON'. OM, which is approximately equal to q, its projection on the ecliptic, is the "second equation" of the planet. Clearly q is a function of α.

The obliquity defined in Section 26 above is measured by the spherical angle M'ON'. This angle varies sinusoidally as

\[ \frac{MN}{q} \approx q\left(\frac{M'N'}{ON'}\right) = qk \]

for any fixed obliquity. The number k depends on the size of the epicycle, the deferent, the eccentricity and the maximum obliquity. Making the additional assumption that, for fixed q, β₃ varies directly as the obliquity, angle M'ON', his Almagest tables ([43], ed. of Halma, vol. ii, p. 414; [3], vol. ii, p. 250) are computed as though

Figure 14. The Epicycle Tilted for the Third Latitude Component

H travels around the deferent, being zero when H is at the nodes and a maximum or minimum when it is in the line of apsides.

Ptolemy treats the spherical triangles MON and M'ON' as though they were plane triangles, using what is tantamount to the crudely approximate relation

\[ \frac{MN}{q} \approx q\left(\frac{M'N'}{ON'}\right) = qk \]
were the general definition of the third latitude for Venus and Mercury. The opposite signs for \( k \) imply opposite tilting of the epicycle planes in the two cases. The two different values of \( k \) for Mercury are to compensate roughly for the eccentric deferent of this planet. As the epicycle leaves the ascending node it travels toward the apogee; hence, in general it is then farther from the center of the universe than after leaving the descending node. Thus the same obliquity produces less effect at the greater distance, so the value of \( k \) is smaller in absolute value on the apogee side of the line of nodes. The eccentricity of Venus is so small that the corresponding effect for it is neglected.

Except for differences in parameters, Kāshī's method is a graphical duplication of the whole scheme. The text prescribes taking a third of a sixth of Venus' second equation, i.e. \( 1/18 \). For Mercury one is to multiply the second equation by either 0° or 0° depending respectively on whether the epicycle center is or not on the same part of the deferent as is the apex of the line of nodes.

It is to be noticed that in two cases Kāshī's values for \( k \) differ from the Ptolemaic ones by as much as a digit in the first sexagesimal place.

It then remains only to impose on the extreme obliquity thus found the now familiar sinusoidal transformation which employs the latitude lines on the plate. The final result is thus implicit in the expression

\[
\sin \beta_3(a, \bar{w}) = \sin kq \cdot \sin \bar{w},
\]

which is very nearly equivalent to (13).

**Commentary**

**X. Latitude of the Superior Planets (f. 22v:7 - 25r:11)**

The Almagest arrangements for the superior planets are simpler on two counts than those for the inferior planets. For one thing, there is no obliquity, hence \( \beta_3 = 0 \). And secondly, both \( \beta_1 \) and \( \beta_2 \) vary (roughly) sinusoidally with \( \bar{w} \), not one with \( \bar{w} \) and the other with \( \bar{w} \) and \( \hat{\lambda} \) as above. In Figure 11, for instance, when the epicycle center is at \( N \) both \( \beta_1 \) and \( \beta_2 \) are zero; at \( H \) both are maximal. In the general position \( H' \), \( \beta_1 \) equals the great circle arc \( KH' \), which is approximately equal to \( i_m \sin \bar{w} \).

Ptolemy therefore computes as previously max \( \beta_2(a) \) by finding the angle \( EP \) makes with the deferent plane. Now he takes

\[
\max \beta(a) = \max \beta_1 + \max \beta_2(a) = i_m + \max \beta_2(a)
\]

for general values of \( a \). Finally he defines

\[
\beta(a, \bar{w}) = \max \beta(a) \cdot \sin \bar{w}.
\]

But, in contrast to the inferior planets, the deferents of the superior planets are not symmetrically placed on the line of nodes; in general, if the epicycle center is on the northern part of the deferent (i.e. \( 0^\circ < \bar{w} < 180^\circ \)), it will be farther from the center of the universe than when it occupies a corresponding place on the southern portion \( (180^\circ < \bar{w} < 360^\circ) \). Hence both Ptolemy and Kāshī use two different values for \( EH \) (Figures 11 and 13) for each superior planet, depending on whether it is in the first or last two quadrants. These pairs of distances are the distances of the latitude points given in the table on f. 9r and discussed by us in Section 13. Except for this, Kāshī's construction for max \( \beta_2(a) \) is as explained in Section 28 above. To this add the corresponding constant \( i_m \). Then perform the customary transformation on it involving \( \sin \bar{w} \) to obtain the \( \beta \) implicit in the expression.
\[
(16) \quad \sin \beta (\alpha, \omega) = \sin \max \beta (\alpha) \cdot \sin \omega,
\]

which differs very slightly from \((15)\) for the small \(\beta\) involved.

One passage, f. 23v:3 - 23v:11, is puzzling. It looks as though the author wants the line of inclination to make an angle of \(i_m\) with the equating diameter. This seems pointless. It is true that \(i_m\) is involved, but it should be added to the max \(\beta_2 (\alpha)\), angle \(RSH\), not to the line of inclination. Also, he seems to want the line of inclination cut off in length equal to the epicycle radius. This would do no harm, but it appears unnecessary.

3.1. Planetary Distances (f. 26v:13 - 27v:12)

For any given time, the distance from the earth to the moon is the length of \(DE\) in Figure 1, where this figure represents the lunar configuration at the given instant. In like manner, for any planet the earth-planet distance is the length of line \(De\) in Figure 9. These distances having been measured in sixtieths of the plate radius, to convert them into distances measured in the standard scale, sixtieths of the respective deferent-radius, it is necessary to multiply each by a proper norming coefficient. These are given by the author in the table on f. 27v and by us in Table 2, Column 2. Determination of the solar distance has already been discussed, in Section 20 above.

32. Stations and Retrogradations (f. 28r:1 - 30r:1)

When a planet's forward motion in the zodiac ceases, it is said to be \(\text{mustaqim}\), stationary. It then becomes \(\text{retrograde (rajāt)}\), having passed through the first, or \(\text{retrograde station (maqām-rajāt)}\). After a time it again becomes stationary, passing through the second or \(\text{direct station (maqām-istiqāmat)}\); thence it resumes forward motion and is said to be \(\text{direct (mustaqim)}\).

Our author devotes a chapter to the use of the Plate of Heavens in computing the time of reaching a station. He follows implicitly the directions given in the Nuzha, which is based on the theory of Ptolemy, who in turn bases his development on a proposition he attributes to Apollonius of Perga (c. 200 B.C.). In substance, this elegant theorem ([43], ed. of Halma, vol.ii, p.312) states that if the deferent center and the center of the universe coincide, then the planet (P in Figure 15) will be stationary when

\[
\frac{m}{n} = \frac{\nu_e}{\nu_p},
\]

\(\nu_e\) being the angular velocity of the epicycle center \(E\) about \(O\), \(\nu_p\) the angular velocity of \(P\) about \(E\), and \(OT\) being perpendicular to \(ET\). The \(\text{station (maqām)}\) is \(\sigma\), the value of the epicyclic anomaly when the above expression obtains. Of course this is a simplification of the actual Ptolemaic model, for in the latter the center of the universe is displaced from the deferent center by a distance \(d\). So in general OE is not a constant,
but a function of \( \lambda \), here the longitude of the epicycle center measured from the deferent apogee. We call this variable radius \( \rho(\lambda) \), and note that \( \rho(0^\circ) = R + d \) and \( \rho(180^\circ) = R - d \). Moreover, Apollonius' theorem applies only approximately to this model, and the location of the station on the epicycle is also a function of \( \lambda \), \( s(\lambda) \) say. Ptolemy computes directly only three values of this function for each planet: \( s(0^\circ) \), \( s(180^\circ) \), and \( s(v_d) = s \), now the location of the station when the epicycle center is at the mean distance. For intermediate values of the argument he uses what amounts to the interpolation scheme

\[
s(\lambda) = \begin{cases} 
\frac{\rho(\lambda) - R}{d}[s(0^\circ) - s], & 0 \leq \lambda \leq v_d, \\
\frac{\rho(\lambda) - R}{d}[s - s(180^\circ)], & v_d \leq \lambda \leq 180^\circ.
\end{cases}
\]

Thus he makes the change in \( s(\lambda) \) proportional to the change in \( \rho(\lambda) \) for corresponding \( \lambda \). (Cf. [3], vol.ii, p.246).

The arrangements in our manuscript are somewhat different, at least in appearance. On f. 29r is a table of \( s(0^\circ) \) and \( s(180^\circ) \), reproduced below together with the corresponding values determined by Ptolemy ([43], ed. of Halma, vol.ii, p.355), al-Battānī ([3], vol.ii, p.138), and Ulugh Beg [54]. It should be remembered that these numbers depend on four parameters, not only on \( d \) and \( r \), but also on the mean motions and mean anomalistic rates of the planets involved. Kāshī makes no mention or use of an independently computed \( s(v_d) \); instead he puts

\[
s(\lambda) = s(0^\circ) + \frac{[\rho(0^\circ) - \rho(\lambda)](s(180^\circ) - s(0^\circ))}{2d}.
\]

The difference between this expression (17) is more apparent than real. In fact, if \( s(v_d) = [s(180^\circ) - s(0^\circ)]/2 \), they are equivalent, and Ptolemy's values for \( s(v_d) \) are practically the mean between his extreme values.

<table>
<thead>
<tr>
<th>Planet</th>
<th>Almagest</th>
<th>Kāshī</th>
<th>Ulugh Beg</th>
<th>al-Battānī</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3° 22' 15&quot;</td>
<td>3° 22' 15&quot;</td>
<td>3° 22' 10&quot;</td>
<td>3° 22' 15&quot;</td>
</tr>
<tr>
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<td>4° 41' 0&quot;</td>
<td>4° 41' 0&quot;</td>
<td>4° 34' 0&quot;</td>
<td>4° 41' 0&quot;</td>
</tr>
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<td>5° 7' 10&quot;</td>
<td>5° 5' 30&quot;</td>
<td>5° 7' 10&quot;</td>
</tr>
<tr>
<td>D</td>
<td>5° 15' 10&quot;</td>
<td>5° 15' 10&quot;</td>
<td>5° 16' 10&quot;</td>
<td>5° 15' 10&quot;</td>
</tr>
<tr>
<td>E</td>
<td>4° 27' 10&quot;</td>
<td>4° 27' 00&quot;</td>
<td>4° 27' 10&quot;</td>
<td>4° 27' 10&quot;</td>
</tr>
</tbody>
</table>

The values laid out on the plate for \( \rho(0^\circ) \), \( \rho(180^\circ) \) and \( 2d \) are displayed on f. 28v for the convenience of the user. \( \rho(\lambda) \), the "preserved distance", is to be obtained by direct measurement with the ruler.

The first and second stations, being symmetrically disposed on the epicycle with respect to the true (epicyclic) apogee, computation of the one for a given \( \lambda \) gives the other immediately.

The final step in the determination consists simply of evaluating \( (s(\lambda) - a)/\dot{\alpha} \) where \( a \) is the anomaly for the time being and \( \dot{\alpha} \) is the rate of change of \( a \) with respect to time. The result will be the time until station is reached, in the same units used to express \( \dot{\alpha} \).

Special remarks concerning Mercury (f. 28v:5, 29r:2) are occasioned by the oval deferent of this planet (see Figure 3), the effect of which is to bring the epicycle nearest the center of the universe at two different points, not at the deferent apogee as is the case with all other planets. These two points are about 120° away from the deferent apogee, as is inferred.