avoided the operation with parts in adding them and subtracting them and converting them into whole numbers. They contrived this by administering numbers to the parts which they derived from the denominators of the parts and by multiplication they made these numbers the denominators of the parts in such a way that these were about whole numbers for the beginner. The masters among the able scholars we find doing the same. (M) For those who occupied themselves with astronomical calculations, as Ptolemaeus and others, did not stick to the parts, but passed from them and their kind into newly devised denominators. They divided unity into sixty parts in order to replace what relates to one half and one third and what is yet more awkward than these and the operation of its multiplication and addition and the like by whole numbers. And likewise, when they needed such parts they divided them also into a whole number of parts, every unit into sixty parts and they created names as minutae, secondae, tertiae, etc.,

After mentioning this, let us now remind the introduction containing the conditions by means of multiples which Euclid used and let us sum them up. Often we will alter Euclid's order in them because of the accordance we think there is in it with the object we intend to gain. So we say: When the ratio of the first to the second is like the ratio of the third to the fourth, then any equimultiples found of the first and the third and any equimultiples found of the second and fourth are only found with the property that when the multiple of the first exceeds the multiple of the second the multiple of the third too exceeds the multiple of the fourth, and when the multiple of the first is equal to the multiple of the second the multiple of the third too is equal to the multiple of the fourth, and when the multiple of the first falls short of the multiple of the second the multiple of the third too falls short of the multiple of the fourth. And also when we convert this we say: When there are four magnitudes and any equimultiples found of the first and the third and any equimultiples found of the second and the fourth are only found with the property that when the multiple of the first exceeds the multiple of the second, the multiple of the third too exceeds the multiple of the fourth, and when the multiple of the first is equal to the multiple of the second, the multiple of the third too is equal to the multiple of the fourth, and when the multiple of the first falls short of the multiple of the second, the multiple of the third too falls short of the multiple of the fourth, then the ratio of the first to the second is like the ratio of the third to the fourth.

What remains for us in this writing is to mention the ratio which is greater than another ratio, So I say: When there are four magnitudes, the first containing more parts of the second than the third contains of the same parts of the fourth, then the size of the first as compared with the second is greater than the size of the third as compared with the fourth. This wants no proof, because what contains more parts is a greater
ratio than what contains less parts. Also I say that this sentence is convertible too, viz. that when the size of the first as compared with the second is greater than the size of the third as compared with the fourth, some parts may be found of the second and fourth so that the first (N) contains more parts of the second than the third of the fourth.

Some things of this kind that we can very well do without are presented among what is necessary. So I say that when there are four magnitudes and any parts found of the second and the fourth are not found, unless if the parts of the second exceed the first magnitude, the parts of the fourth exceed the third magnitude, and if the parts of the second fall short of the first magnitude, the parts of the fourth fall short of the third magnitude, then the magnitudes are proportional even if we do not mention the condition of equality. For to mention equality or to leave it out comes to the same thing. For when they are fixed (enclosed) by exceeding and falling short at a time, the condition of equality is necessarily existant by force of logic, and the proportionality in the magnitudes exists. For instance: When we assume four magnitudes AB, C, DE and F, and suppose that any parts agreeing in number and denomination taken from C and F are only found with the property that when the parts of C exceed AB, then the parts of F exceed DE too, and when the parts of C fall short of AB, then the parts of F fall short of DE too, equality not being mentioned, then I say that parts of C equal to AB are only found if also the parts of F are equal to DE. The proof of this is that it can not be otherwise. For if that could be, let AB be exact parts of C, and let DE contain less or more of such parts of F, let DE be more. Let DG contain as many parts of F as AB contains parts of C. Now you must not stop dividing F until you reach the first part of it that is less than EG. Let that part be H. This is less than EG. Let us take a similar part of C, say K. Then we subtract from DE the parts equal to H that it contains and no doubt their remainder will fall between G and E, because GE is greater than H. Let those pieces be DL, LM, MN. Then we subtract from AB the pieces like K that it contains. From it can be subtracted as many pieces as from DG can be subtracted, pieces like H, because we have supposed that the ratio of AB to C is as the ratio of DG to F, for either contains the same number of parts of its companion. Let these parts be AO, OP, PQ. Now AQ is greater than AB, just as DN is greater than DG. Therefore AQ is parts of C, and DN is similar parts of F in the same number. However AQ is greater than AB and DN is less than DE, so that some parts of C and F are found and the parts of C are found to exceed AB and the parts of F to fall short of DE. And we had supposed that no same parts of C and F would be found so that the parts of C would be found to exceed AB, unless the parts of F would be found to exceed DE. This is a contradiction, it is impossible. Therefore if parts of C are not found equal to AB, unless similar parts of F are found
equal to DE, then the three conditions of being equal, exceeding and falling short are soundly found, and therefore the proportion (O) is sound, and that is what we wished to demonstrate (14).

Also I say that when the ratio of the first to the second is greater than the ratio of the third to the fourth, same parts of the second and the fourth are only found with the property that the parts of the second are found to exceed the first magnitude if the parts of the fourth too are found to exceed the first magnitude. If for instance the ratio of AB to C is greater than the ratio of DE to F, I say that all equal parts taken from C and F have the property that when the parts of C exceed AB, the parts of F too will exceed DE. The proof is that it can not be otherwise than we said. For, if that could be so, viz. if some parts of C and F had the property that the parts of C were found to exceed AB and the parts of F were not found to exceed DE, let the parts of C exceeding AB be AG and such parts of F that do not exceed DE, DH. Now DH and AG are same parts of F and C, and DE contains more parts of F than AB contains parts of C, and so the ratio of DE to F is greater than the ratio of AB to C; but the ratio of AB to C was the greater. This is a contradiction, it is impossible. Therefore no parts of C are found exceeding AB, unless the same parts of F exceed DE; and this is what we wanted to demonstrate.

After mentioning all this, I say that when of four magnitudes the ratio of the first to the second is greater than the ratio of the third to the fourth, some parts can be found of the second and the fourth with the property that the parts of the second fall short of the first magnitude and the parts of the fourth do not fall short of the third magnitude. If, for instance, the ratio of A to B is greater than the ratio of C to D, then I say that it is possible to find some same parts of B and D with the property that the parts of B fall short of A and the parts of D do not fall short of C. The proof is that this is found of some parts taken from B and D, for otherwise parts of B falling short of A are not found, unless the parts of D fall short of C. And also, because we have supposed that the ratio of A to B is greater than the ratio of C to D, it is certain that whatever equal parts we find of B and D, the parts of B are only found to exceed A if the parts of D too exceed C, as we have demonstrated in the preceding proposition. Therefore A, B, C, D are four magnitudes and all parts taken from B and D have the property that if the parts of B exceed A the parts of D exceed C, and when the parts of B fall short of A the parts of D fall short of C. Therefore the ratio of A to B is as the ratio of C to D. We
supposed however that the ratio of \( A \) to \( B \) was greater than the ratio of \( C \) to \( D \). This is a contradiction, it is impossible. Therefore it is not (true) that whatever parts taken from \( B \) are found less (\( P \)) than \( A \) the parts taken from \( D \) are necessarily less than \( C \) too. It is rather possible to find parts of \( B \) falling short of \( A \), while the parts of \( D \) do not fall short of \( C \), and this is what we wished to demonstrate (15).

From the point of view of the multiplicity on the other hand it is now sound too that when the ratio of the first to the second is unlike the ratio of the third to the fourth, it is possible that equimultiples are found of the first and the third and equimultiples of the second and the fourth with the property that the multiple of the second is found to fall short of the multiple of the first and the multiple of the fourth is not found to fall short of the multiple of the third, in the same way as we found parts of the second falling short of the first and the parts of the fourth not falling short of the third. And likewise as we say that the multiple of the second falls short of the multiple of the first and the multiple of the fourth does not fall short of the multiple of the third, we say too that the multiple of the first exceeds the multiple of the second and the multiple of the third does not exceed the multiple of the fourth.

Let us now mention things separately as Euclid mentions them. So we say: There are four magnitudes and it is possible to find equimultiples of the first and the third and equimultiples of the second and the fourth so that the multiple of the first is found to exceed the multiple of the second and the multiple of the third not to exceed the multiple of the fourth, then the ratio of the first to the second is greater than the ratio of the third to the fourth. And as to its converse, of which we demonstrated the way of its finding, it is this: When the ratio of the first to the second is greater than the ratio of the third to the fourth, then it is possible to find equimultiples of the first and the third and equimultiples of the second and the fourth so that the multiple of the first is found to exceed the multiple of the second and the multiple of the third not to exceed the multiple of the fourth.

We thus demonstrated in this writing the soundess of what Euclid says about the multiples and that he does not choose the explanation of ratio by means of the multiples without the reason of their sound connection with ratio. It is even impossible to find anything that is more soundly connected with ratio and stronger in rigour and exactness as to its sections and properties than the multiples, because nobody can doubt that ratio is not but a comparison of a magnitude with another magnitude in order to know the size of one of the two as compared with the other. And the comparison of the less as compared with the greater is only known by parts of the greater, and the multiples are only an enlargement of the parts, and the multiples include the
comparison of the less with the greater as well as the comparison of the greater with the less, and with parts it is only usual to compare the less with the greater, although in this writing, while mentioning the parts, we have paid no attention to what place we attach to them because of the opinion (Q) we have given about facilitating the reasoning and the verification of it in behalf of our need of its frequent repetition and use. For the more it is used in the reasoning, the more there is occasion for relief and abbreviation. And as to finding the case of the greater ratio and the equimultiples taken of the first and the third and the equimultiples taken of the second and the fourth with the property that the multiple of the first exceeds the multiple of the second and the multiple of the third does not exceed the multiple of the fourth, how is the way to produce these multiples, as this condition is not found in all multiples necessarily, but only in some multiples, as we have mentioned, well, we know a means by which it is produced, but only after we know how to produce a fourth magnitude having a ratio to three given and known proportional magnitudes. We want to find another, fourth, magnitude which will be the fourth of them and fitting them in such a way that the ratio of the first given one to the second given one is as the ratio of the third given one to the fourth sought one. For if we do not know how to produce this fourth magnitude to the three given magnitudes then we will not know the means to get at finding the multiple in the aforesaid condition either. In the sixth book (is written) how to produce to two magnitudes a third proportional and it is also within the scope of that book how to produce a fourth (proportional) to three. However this only applies when the magnitudes are all lines, but as to angles, or surfaces, or solids we do not see this. We have seen people trying to produce this aforesaid multiple and exerting themselves in finding this fourth magnitude without a proper method of working which might enable them to find it. For me there is no difference between him who is exerting himself in finding this fourth magnitude and (him who is) exerting himself in finding the multiple with the aforesaid property. And on this ground we think, and the Lord knows better, that Euclid uses this multiple in the twelfth proposition of the fifth book (16) without it being mentioned before how is the way to find it. Something of the kind does not weaken the proof and does not detract anything from it, because, if the thing is necessarily existent, it can do no harm that its existence is used in the argumentation and that it is used in proving other things with it. For if this were not allowed in a proof we were no more permitted, for instance, to believe that the side of the septagon inscribed in the circle is less than the side of the hexagon and greater than the side of the octagon both inscribed in one circle, nor that the side of the plane hendecagon is less than the side of the decagon and greater than the side of the plane dodecagon both in one circle. For the construction of the septagon-
chord is not possible in the geometrical way. But when we want to show that it is
greater than the side of the octagon (R) we draw a chord and suppose it to subtend the
seventh part of the circle, by assumption only, not in reality, and then we properly
arrange the proof, and the lack of practicability does not harm in this theoretical matter.
It only harms in the performance of the drawing of it, e.g. when we intend to construct
the side of the plane regular tetrocoidecagon inscribed in the circle, we are not allowed
to draw a chord and to suppose, at variance with the truth, that it subtends the seventh
part of the circle, then to divide the arc into two equal parts and to finish the figure.
The like is not allowed and the distinction is clear. For that the like is allowed in
establishing a proof and is not fit for practical performance of a drawing is evident,
if God, who is lofty, is willing.
And as our words have carried us so far and we have mentioned what we hoped
to mention, it is the moment for us to break off our reasoning and to conclude it to
the praise of the Lord, Whose glory is great, hallowed be His names. We pray to Him
for forgiveness and to excuse our error and stumbling and to guide (us) to what
pleases Him of truth in word and deed.

Finished is the commentary on ratio according to the reasoning of the fakih
Abū 'Abd Allāh Muhammad ibn Muṣād al-Djallānī,
God have mercy upon him.
III.

OTHER COMMENTARIES ON RATIO.

1. Al-Dajjâni wrote his commentary (1) in defence of Euclid's work because some people were not satisfied with it and tried to make it complete or clear according to their own thinking. In particular in view of Euclid's definition of proportional magnitudes he remarks: For many think that Euclid approaches the explanation of ratio from a door other than its proper door, and introduces it in a wrong way by his definition of it by taking multiples, and in his separating from its definition concerning its essence that which is understood by the very conception of ratio; and they judge that there is no obvious connection between ratio and taking multiples.

2. Before verifying as to how far other manuscripts bear out this statement of Al-Dajjâni's, I am bound to discuss the Greek text that was subjected to the said criticism, confining myself as much as possible to Euclid's conception of ratio and his definition of proportional magnitudes.

For a better understanding of what could be said about the authenticity of the Greek text one must take into account the part that Theon of Alexandria acted in its tradition. About this Dijksterhuis (2) writes the following:

"The way in which the text of the Elements has reached us has been greatly influenced by the version of it written by Theon of Alexandria (end of the 4th century A.D.). Theon's object seems to have been not so much to render the text as pure as possible as to make the contents as clear as possible; he rectified errors, or what he thought to be errors, modernized the mode of expression, inserted propositions, corollaries and cases that Euclid apparently had thought superfluous, and elucidated the course of proofs where he thought it liable for possible difficulties. The consequence of this opening up the work was that later Greek authors used Theon's text almost exclusively, and perhaps one never would have found out how much he altered in the other, wordings, if Peyrard had not discovered in 1810 a manuscript of the text which was older than Theon's. This so-called manuscript P was used by Heiberg in composing the famous text-edition that by this time generally underlies the study of Euclid."

3. The first seven definitions in Book 5 according to Heiberg are:

α'. Μέρος ἄτι μέγεθος μεγέθους τὸ ἐκλασαν τὸ μείζων, ὅταν κατα-

μετρῇ τὸ μεῖζον.

β'. Πολλαπλάσιον δὲ τὸ μείζων τοῦ ἐλάττονος, ὅταν καταμετρήται ὑπὸ τοῦ ἐλάττονος.

γ'. Λόγος ἄτι δύο μεγεθῶν ὀμογενῶν ἢ κατὰ πτηλικότητα ποικ. σχέσις.

1. A magnitude is a part of a magnitude, the less of the greater, when it measures the greater.

2. The greater is a multiple of the less when it is measured by the less.

3. Ratio is some state of two magnitudes in connection with size.

δ'. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἢ δύναις πολλαπλασιάζο-

μενα ἄλληλων ὑπέρεχειν.

ε'. 'Εν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρότον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καὶ τρίτου ἴσας πολλα-

πλάσια τῶν τοῦ δεύτερου καὶ τε-

τάτου ἴσας πολλαπλασίων καθ' ὑποιονοῦν πολλαπλασιασμοῖν ἕκα-

τερον ἕκατέρου ἢ ἀμα ὑπέρεχη ἢ ἀμα ἢ ἀμα ἀκείπει αἰρθέντα κατάλληλη.

ζ'. Τὰ δὲ τῶν αὐτῶν ἔχεντα λόγον μεγέθη ἀνάλογον καλείσθω.

ζ'. 'Οταν δὲ τῶν ἴσας πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπέρεχη τοῦ τοῦ δεύτερου πολλα-

πλασίου, τὸ δὲ τοῦ τρίτου πολλα-

πλασίου μὴ ὑπέρεχη τοῦ τοῦ τετάρτο-

του πολλαπλασίου, τότε τὸ πρώτον πρὸς τὸ δεύτερον μείζων λόγων ἔχειν λέγεται, ὕπερ τὸ τρίτον πρὸς τὸ τέταρτον.

These must be completed by the following additions:

a. in γ': πρὸς ἄλληλα (ποικ. σχέσεις),

to be translated as: (some state) with regard to each other, and

b. after σχέσει:

ἀναλογία δὲ ἢ τῶν λόγων ταυτότης,

to be translated as: Proportion is the identity of ratios;

4. Magnitudes are said to have a ratio
to each other that can be multiplied
so as to exceed each other.

5. Magnitudes are said to be in the
same ratio, the first to the second
and the third to the fourth, when any
equimernal multiples of the first and
third simultaneously exceed, are
equal to, or fall short of the equi-

numeral multiples of the second and
fourth taken in corresponding order.

6. Let magnitudes which have the same
ratio be called proportional.

7. When of the equimernal multiples
the multiple of the first exceeds the
multiple of the second, the multiple
of the third however does not exceed
the multiple of the fourth, then the
first is said to have a greater ratio
to the second than the third has to
the fourth.

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c. or after 5\textdegree: ἀναλογία δὲ ἐπίν ἣ τῶν λόγων ὑμεῖς, to be translated as: Proportion is the conformity of ratios.

4. On these definitions Thābit ibn Qurra (3) had given his opinion, with reference to which al-Māhānī (4) wrote a treatise (5). In this he remarks that the former for a better understanding of ratio had referred to numerical ratios (iumkinu līl-\textsuperscript{ī}l\textsuperscript{ī}nāni an iqaṭa ʿalāhī min qibāli ʿl-maʿrīfāt bi-l-nisbatā ʿalā ʿl-sūbitā ʿl-sadādijātā). Under the heading "ratio" (fi ʿl-nisbatā) he writes: "The ratio of two homogeneous magnitudes and also that of two numbers is the state (al-ḥālū) of either of them when measured by the other (ʿinda taqdiqīhī li-sāhībīhī) or conversely. Three cases must be distinguished:

1. the less takes up the greater entirely (istaghaqqahu) so that no remainder of it is left;
2. the less does not take up the greater entirely but of the greater (something) is left less than the less; when the less is measured by this (remainder), then it takes it up entirely or (something) is left less than the first remainder; when now the first (remainder) is measured by this (second remainder) then it takes it up entirely, or a remainder is left; when now the second (remainder) is measured by this (third remainder) and when this is done continuously, then a remainder is arrived at that takes up the foregoing rest entirely, or
3. one does not arrive at a remainder taking up the foregoing entirely.

The first two cases belong to the cases of comparative measurement found with numbers as well as with magnitudes, the third is only found with magnitudes." After this al-Māhānī writes under the heading "being proportional" (fi ʿl-tanāsūbi): "This is that the state of one magnitude (al-mansūbū) of the first pair with regard to the other (al-mansūbū ilāhi) as to taking up entirely in measuring, and also the number of times (quotient) in that measuring, is like the state of one magnitude of the second pair with regard to the other as to measuring and the numbers of times (quotients)." The author remarks that this holds good likewise when the magnitudes of each pair are interchanged, and besides that being proportional in the two other cases is in accordance with this one, i.e. that if the order of measurement is maintained the numbers of times (quotients) are the same.

Under the heading "proportion" (fi ʿl-munāsābāti) he writes: "The magnitudes that have one ratio are those having the property that when the first and third are measured by the second and fourth, or conversely, the numbers of times (quotients) in both measurements are the same; and that, if of both (magnitudes) (reminders) are left less than the two less magnitudes, and if the less magnitudes are measured by them, the numbers of times (quotients) in these measurements are the same; and thus ad infinitum." So far al-Māhānī.

5. The well-known commentary (6) of al-Nairīzī (7) is also important in view of the tradition of the text, as it is added to the translation of the Elements by al-Hadīdijādī (8). The definitions begin as follows: "The less magnitude is a part (dājuz\textsuperscript{ā}ni) of the greater when it measures the greater (kāna ʿaṣādādāri)." To this is added in the margin: "Part: such is a magnitude of a magnitude, the less of the greater, when it measures the greater\textsuperscript{rī}.

Also the greater is a multiple of the less when it is measured by the less" and further: "Ratio is a certain relation (liḍṭātān) as to measure (qadrūn) between two magnitudes of the same species\textsuperscript{s}, to which is added in the margin: "Ratio is the identification (aijātātān) (9) of the measurement of two homogeneous magnitudes, each of them with regard to the other; whatever the measure might be." Then follows that "ratio is the state of one quantity (qadrūn) with regard to another quantity of the same species." This state is "a relation of one of two magnitudes with the other, I mean that the first is measured by the other". Of this state two varieties are found, viz. the state of commensurability (al-ṣīṣīrākū) and the state of incommensurability (al-tābāḫūn).

As to the state of commensurability, it implies for the two magnitudes the existence of another state of commensurability that measures both entirely (jaʿuddūhumā ẓāmīn\textsuperscript{ā}n), or that one of them measures the other. If one of them measures the other the state of the less with regard to the greater is the state of the part (al-dājuz\textsuperscript{ā}ni) and the state of the greater with regard to the less is the state of the multiple (al-adāf\textsuperscript{ā}nū). If however the greater a remainder is left less than the less magnitude, then this remainder is bound to measure the less magnitude and to take it up entirely (jaṣtaghāraqūhū) in measuring (bi-l-\textsuperscript{ī}l\textsuperscript{ī}dādūni), or of the less a remainder is left less than the first remainder. When this measures the less magnitude and takes it up entirely in measuring, then this remainder is the third magnitude that measures the two commensurable magnitudes. If however a remainder is left less than the first remainder, then this second remainder is again bound to measure the first remainder and to take it up entirely in measuring, unless a remainder is left less than the second remainder. If it measures this (second remainder) and takes it up entirely in measuring, then this remainder, I mean the third remainder, is the third magnitude that measures the two commensurable magnitudes entirely (dāmjīn\textsuperscript{ī}n). If however a remainder is left less than the third remainder, then there are two possibilities (faʿāla inna ḫādīhī l-ḥālā mina ʿl-tādūdū laisat tāḥālū min dīhātānī), either the measuring of the remainders arrives at a remainder that measures the foregoing and takes it up entirely, so that this remainder is the third magnitude that measures the two magnitudes and the state of the less with regard to the greater.
is the state of parts (hâlu 'l-adjâlî), whilst that remainder is one of the parts of the greater, or the affair does not arrive at a remainder that measures the foregoing entirely, but the process continues ad infinitum. This last state of one magnitude with regard to the other is the state of incomensurability."

After this the author remarks that Euclid says: "Being proportional is similarity of ratios", and that this similarity implies "a comparison (muqâṣāsatun) of the state of two magnitudes (with regard to each other) and the state of two (other) magnitudes (with regard to each other) that are in the same ratio ('alâ 'l-nisbatîhî)." "When this state of the first two magnitudes is also the state of the other two magnitudes, then it is said that this state is the state of similarity of ratios. When however matters otherwise, then this state is not the state of similarity and there is no being proportional. He (i.e. Euclid) says similarity, because this state is a quality (kâ'ilâtun) and not a quantity (kâmînjatun). For when the state of the first two magnitudes (with regard to each other) is the state of equality, then the state of the other two magnitudes (with regard to each other) is the state of equality too; and when the state of the first two magnitudes is the state of multiple, or the state of part, or the state of parts, or one of the other states of ratio that are found in the case of commensurable magnitudes (aw sâ'iru abhâdî 'l-nisbatî 'llâtî hija 'l-maqâdîrî 'l-muṣâratikanî), then the state of the other two magnitudes with regard to each other is the same state too. This is a quality and not a quantity. And in accordance with this is the state of incomensurables (with regard to each other)." The author then elaborates this point giving due attention to the numbers of times (quotients) that each remainder measures the foregoing, a less remainder always being left being. Thereupon he says: "When this sequence (al-taṣâwâ'âl) in measuring the remainders never ceases to turn out equally ad infinitum, then it is said that the magnitudes, their state being as mentioned, are proportional, and this state is the state of similarity of ratios."

After this the author first gives an example and extends his reasoning to the state of being greater in ratio, whereupon he says that Euclid gives the definition: "Four magnitudes are said to be in one ratio (fi 'l-nisbatin wâhidatin), the first to the second and the third to the fourth, when equinumeral (al-mutasâwîjatî 'l-marrâtî) multiples of the second and third, either unlike exceed, or unlike are equal to, or unlike fall short of equinumeral multiples of the second and fourth, when compared consecutively (idhâ qâsit 'alâ 'l-wâli) one with the other. The author expresses his opinion 'that this is a matter that belongs to the principles for him who has advanced so far. For every book (10) has its principles according to its place in the range of books (inna hâdhâ sajam laisâ juhâdû 'alâhi 'llâtû burhânî li-anâhû mina 'l-awâ'îli 'inda man intâhâ 'llâhâ 'l-mawdî) idhâ kâna li-kulli maqâlâtîn awâ'îli

bîhasbi mortobati tilka 'l-maqlâhatî)." Then follow Euclid's words: "Let magnitudes that are in one ratio be called proportional (muṭânâšibatun)" and "When the equinumeral multiples are such that the multiple of the first of them exceeds the multiple of the second, and the multiple of the third does not exceed the multiple of the fourth, then it is said that the ratio of the first tot the second is greater than the ratio of the third to the fourth."

So far al-Nairizi.

6. Ibn al-Haiitham (11) in his commentary on Euclid's premises (12), so far as Book 5 is concerned, writes the following: "Ratio is a certain relation, and when there is a certain relation it is only found with two things." Then he divides ratio into the general ratio (al-nisbatu 'l-mudâlûtu) and the definite ratio (al-nisbatu 'l-mu'a'ajtronatu). The first contains assertions about greater and less, or something like such part, etc. The definite ratio however is subdivided into three species: 1. the ratio of equality; 2. the double, or a certain multiple, or the half, or a certain part, or certain parts, or what is composed of these; 3. when we say: the ratio of this magnitude to this other magnitude. These three species are combined in two species: "the numeral ratio (nisbaton 'addajatun) and the ratio that is not numeral (ghairu 'addajitun)." Numerical ratios are the ratio of equality, the multiple and the parts, a non-numerical ratio is the ratio that can be found but cannot be expressed (hija mā jumkin ân 'uṣûra wa lâ jumkin ân juntaa bi-hâad). This ratio is found with the continuous quantities (fi 'l-kâmînjatî 'l-muṭâsâhiratî), but it may also be found with numbers, for instance where irrational roots are concerned. On the ratio found with continuous magnitudes the author says that this ratio is subdivided into two species, the numeral ratio and the non-numeral one. "The numeral ratio in the case of continuous magnitudes, one of them to the other, is like the ratio of a number to a number, and this is clear. For among magnitudes (couples) are found one (magnitude) of which is a multiple of the other, or a part of it, or parts of it, or equal to it. All these ratios are found with numbers too." As to the magnitudes with a non-numeral ratio the author remarks that it is characteristic of each pair of magnitudes of this kind (min chōṣât kulli miqâlarâîn 'alâ hâdhîhi 'l-sifati) that when one is measured by the other, etc., the successive remainders do not arrive at a remainder measuring the foregoing one. After having expatiated on the numeral ratio the author says: "The definition of ratio found in all species of the continuous quantity, which comprises all ratios used by the ancients, is that ratio is the identification of the measure (aijîjatu qadri) of one of two magnitudes with regard to the other. A paraphrase (tafsîrân) of it is that when the quantity of magnitudes is in question, ratio is the very notion inquired (al-nisbatu hija
"I-mātānā 'l-lādhī fi kammījātī 'l-maqādīrī 'l-lādhī jis'ṭalū 'anhu bi-a'ījīn. More recent authors defined ratio with other words (bi-a'llāzin ghai'īrāh ĥādīhīhīh) and this is what they said: It is a certain relation (iḍāfātun mā) of two homogeneous magnitudes as to magnitude (fi 'l-miqādārī). Euclid mentions the definition of ratio in Book V; in some manuscripts (ft ba'ḍī 1-nusachi) it is found in the first wording, in other (manuscripts) in the second one." Ibn al-Haitham's opinion is that both are correct.

Then he enters into detail about parts and multiples, bringing up some geometrical questions and one of linguistic nature too when saying that "di'īfun really means "equal"" (al-dīfu huwa 'l-mathalū). Then follows the definition: "The magnitudes that have a ratio to one another are those that multiplied (iḍāhā dāfīfāt) are capable of exceeding one another." The author demonstrates that it is pointed to be in one ratio, the first to the second and the third to the fourth, when equimultiples of the first and the third and the second alike exceed equimultiples of the second and the fourth, or alike are equal to them, or fall short of them, when taken in (due) order, whatever the multiples may be." The author elaborates on this point, after which he quotes Euclid's words: "Let magnitudes which have one ratio be called proportional." He then recapitulates the foregoing by saying that proportional magnitudes are those having the said characteristic (chāṣātun). And, says Ibn al-Haitham, "Upon my life, it is an indispensable characteristic (chāṣātun jāzmūtan) of proportional magnitudes. However, when only stated it is not clearly understood (illā annāhā lāsat zāhiratīn li 'l-fahmi bi-mudjārādī 'l-qaulī), but needs a proof." To the proof he gives I will return later on (13).

7. Of the commentary (14) of 'Umar al-Chajāmī (15) the second chapter deals with "ratio and being proportional and the true essence of either." It begins as follows: "The element-writer says about the essence of ratio that it is the identification of the measure (aiqātun qudrī) of two homogeneous magnitudes, one with regard to the other." Then the author explains the meaning of homogeneous by means of Euclid's next definition (the 4th), after which he returns to the foregoing one. He develops the succession of possibilities in the usual way, viz. arguing that the less is either a part of the greater and takes it up entirely when they are related to one another, or it is parts of it, or yet something else ('ala wadžhīn āchāra). After this he says that "it is characteristic of quantity to be submittable to the consideration of equal or unequal. Now ratio is this consideration when two homogeneous magnitudes are related to one another and of the consideration of something else connected with it, and this is the magnitude (miqādārū) of that ratio when it is a ratio of magnitudes (nisbatun miqādārijatun). This is more clear in the case of numeral ratios (al-adadījāt). The first thing found as to ratios is found with numeral ratios, viz. that numbers are related to one another, and it is found that they are either equal or unequal. Then unequals are compared and it may be found that the less measures (ja'uddu) the greater, as three (measures) nine. Then the number of times (quotient) is looked for that three measures nine, and three is found; therefore three measures nine three times. Of this the name is deduced in accordance with linguistic rules, and it is said to be the third part; hence the ratio is one third. The author extends this consideration to the case of parts, after which he continues: "Then one looks for the ratio with magnitudes too and finds with them, besides these two cases, a third case in consequence of the fact that magnitudes are not composed of indivisible parts, and that their division has no definite end as is the case with numbers (wa dhālikā annā 'l-maqādīra ghaīru murakkhabat minā 'l-adjażūi 'llātī lā tāndżāzuwa la wāsīla li-'nāšāmīhīmā nihājatun mahādūdatun kamā 'l-muāstādī). For numbers are composed of indivisible parts, each of which being the unit." After this the author discusses how the less is subtracted from the greater until a remainder is left less than the less, and how this process leads to an end in the case of numbers. For this he refers to Book 7. With magnitudes however this process is not bound to lead to an end, which Euclid discusses in Book 10.

"However", 'Umar declares, "we do not need this in our reasoning. When this is how matters stand it does not follow as a matter of course that of two arbitrary magnitudes the less is either a part of the greater, or parts of it, but yet another possibility is to be observed, not in the way of numbers, but in the way of magnitudes. If now somebody should say that these three cases do not present themselves at all, but only the two cases that occur with numbers, then we reply and say that it does not give us any trouble to view the rules of ratio and proportion from these three angles. Should this subdivision prove to be absurd in the course of the reasoning, well, we would not be to blame for it. Should it prove however not to be absurd, then we have mentioned it at least and have exhausted all possible cases. For this is a mystery from which other very deep logical mysteries are to be learned. Therefore, try to grasp it!"
continues as follows: “These are his (Euclid’s) words about being proportional, and we will call it the common way of being proportional (al-tanásbu ‘l-maṣṣīḥūn). As for ourselves we will deal with the true way of being proportional (al-tanásbu ‘l-haqiqī), whilst the whole 5th book deals with the common way.”

Then the author develops his own idea, the first part of which he recapitulates as follows: “When there are four magnitudes and the first is a part of the second, and the third the same part of the fourth; or the first is parts of the second and the third the same parts of the fourth, then the ratio of the first to the second will not fail to be like the ratio of the third to the fourth. This is the numeral ratio (al-nisbatu ‘l-ta‘ādādījatu).” After this follows an exposition of what is found when the magnitudes are incommensurable and the succession of remainders in the well-known process does not come to an end. If under these circumstances the number of times (quotient) that a remainder measures the foregoing (remainder) in the relation of the first (magnitude) to the second magnitude is equal to the number of times (quotient) that the corresponding remainder measures the foregoing (remainder) in the relation of the third (magnitude) to the fourth magnitude, then he says that the ratio of the first to the second is like the ratio of the third to the fourth, “and”, he says, “this is the true way of being proportional in the geometrical kind.”

So far ‘Umar al-Chajjāmī.

### IV. DISCUSSION OF THE DIFFERENT COMMENTARIES AND EVALUATION OF THEIR PURPORT.

1. The Arabian commentaries quoted in the foregoing chapter show a striking similarity. As opposed to Euclid’s way of dealing with the subject matter of Book 5 they propose another method, which is in all four cases essentially the same. Hence the question arises whether we are entitled to take this fact for a symptom of a manner of thinking typical of the scientists of the Muslim period. All Arabian mathematical activity however was so largely dependent on the Greek tradition that we cannot help being very suspicious of conclusions about essentially new ideas.

In recent years the birth and growth of Greek mathematics were thoroughly investigated (1). The notion we are now able to form of its evolution, on the authority of Dijksterhuis a.o., shows a twofold crisis at the end of the 5th century B.C. (2). Leaving aside the difficulties with regard to continuity and infinity I will pay due attention to the other stumbling-block: the discovery that irrationality is a general phenomenon, viz. that “the implied assumption that each pair of homogeneous magnitudes has a ratio of a number to a number” is incorrect. In this connection we must bear in mind that the Greek notion of number comprised only the integers 2, 3, 4, etc.. An excellent solution of the difficulties arising from the said discovery was incorporated in the way Euclid dealt with proportions in Book 5 of the Elements. However the question remains to be seen whether the Greeks could have cleared their reputation of having based their notion of proportionality on the aforesaid error in a way more in line with their original conception of ratio.

2. It is more than likely that attempts of the kind have been made by the Greeks presumably almost at the same time in which Eudoxus moulded his theory of proportion in the concrete form known to us from the preface of Book 5 of the Elements (3). The following short exposition may support the conjecture formulated at its end.

If two homogeneous magnitudes are commensurable they have a common measure, viz. a magnitude exists going into both precisely when they are measured by it. Let us suppose for a moment that the common measure of two magnitudes a and b goes 35 times in a and 16 times in b, which numbers however are unknown to us. Now we measure the greater by the less and find that is goes 2 times, leaving a remainder \( r_1 \), less than b. This remainder is 3 times the common measure, but this too is unknown to us. Then we measure the less by this first remainder and find that it goes 5 times in it, leaving a second remainder \( r_2 \), less than \( r_1 \). This second remainder is the common measure itself, but we do not know this beforehand. We therefore measure the first remainder by the second remainder and find that it goes 3 times in it precisely. Hence we have:

\[
\begin{align*}
  r_1 &= 3r_2 \\
  b &= 5r_1 + r_2 = 15r_2 + r_2 = 16r_2 \\
  a &= 2b + r_1 = 32r_2 + 3r_2 = 35r_2.
\end{align*}
\]
In this way we find the ratio of two magnitudes together with their common measure. Euclid practised the method in the Books 7 and 10, using the term ἀνθυματίζειν to denote the alternate measurement of the successive remainders.

Now we take two different pairs of magnitudes and try to find the ratio of either pair in the aforesaid way. Each pair for itself must be homogeneous but no more is required. The relation of the magnitudes of one pair “in respect of size” is effectively characterized by the series of numbers (quotients) indicating how many times each magnitude or remainder goes into the foregoing (i.e. next greater) one. Therefore it is not far-fetched to say that it would have been in perfect accordance with the original (numeral) Greek notion of ratio to call the ratio of the first pair equal to the ratio of the second pair if the said series of quotients in both cases is the same.

However, at the end of the 5th century B.C. the Greeks were well aware of the fact that far from all pairs of magnitudes have a common measure. Hence they also knew perfectly well that the series of quotients in many cases does not arrive at a definite end. For if it does a common measure is found and the ratio will be numeral.

If now, finally, two pairs of magnitudes produce the same infinite series of quotients, neither has a ratio in the original (numeral) sense. Nevertheless it would not have been a large step for the Greeks to declare: Two pairs of magnitudes have the same ratio if the series of quotients in one pair is the same as the series of quotients in the other pair, irrespective of whether these series are finite or infinite.

“That this method has played a part in Greek mathematics anyhow”, says Dijksterhuis in 1929 (4), “is confirmed by the manifold occurrence of approximations of irrational values that upon examination appear to be approximating fractions of developments into continued fractions”. In our example of this paragraph the ratio of a to b can be expressed by:

\[
\frac{a}{b} = 2 + \frac{1}{\frac{5}{1}},
\]

which in a corresponding way leads to an infinitely continued fraction in case the ratio for lack of a common measure is “irrational”, ἀποθέτειν. Moreover we have evidence yet more closely connected with the theory of ratio.

3. In Topics VIII.3, 158b, 29-35 Aristotle refers to a definition of “having the same ratio” that comes to “having the same ἀνθυματίζειν”. Heath (in 1908) realised that it was not identical with the Euclidean one, but failed to give a satisfactory explanation. This was, probably for the first time, given by Zeuthen (in 1917), but it seems to have remained unobserved until 1926 when Junge independently of the former pointed to the same obvious solution. A more elaborate study of it was then produced by Hasse and Scholz (in 1928) (3) and in particular by Becker (in 1936) (6).

Zeuthen’s solution is the following.

In his commentary Alexander says upon the passage mentioned: “The definition now of proportion that the ancients used is this: proportional are magnitudes to one another which have the same ἀνθυματίζειν. Aristotle however called the ἀνθυματίζειν ἀντανάκλασις.” It is evident that for Alexander the meaning of ἀνθυματίζειν must have been perfectly clear. From where then did he get the term? An obvious conjecture is: from Euclid, whom he cites more than once and whose work he supposes to be common knowledge. Now Euclid does not use the noun, but in 7; 12 and 10; 2,3 he uses the verb ἀνθυματίζειν in the sense of subtracting in turn one magnitude or number or a remainder of it from the other or its remainder in order to find the greatest common handled a proportion by means of an ἀνθυματίζειν, i.e. they developed it into measure. At this point of the reasoning the conclusion leaps to the eye: the “ancients” a continued fraction. They were perfectly aware of the fact that in the case of two equal ratios, “the relation in respect of size” being the same, the immediate consequence of this sameness was the appearance of two identical “chains” of quotients, whether finite or infinite. Their definition therefore covered rational as well as irrational proportions and in this respect was not inferior to the Euclidean one.

Why then, we might ask, did it fall into the background? A plausible answer would be: because of its impracticability. This solution however was disproved by Becker, who developed a general theory of proportion based on the antiphoraic definition and extended so as to cover the needs of Euclid 10 (6).

4. Having thus found by a lucky chance a single passage in Aristotle (and not even in a mathematical work) pointing to the existence in the flowering age of Greek mathematics of a current conception of ratio deviating essentially from the Euclidean one, we find in the Moslim period an avowed preference for the very ideas that the Greeks seem to have kept dark so carefully that it appears to have been done intentionally. In almost every writing on the subject the Euclidean process of finding the greatest common measure of two magnitudes is assumed as the natural source of all ideas about ratio and proportionality. Nevertheless we must not lose sight of the fact that an elaborate Arabian theory of proportions, based on the antiphoraic conception of ratio has not been found up to the present. I think it probable that this is mainly a consequence of the great perfection of Euclid’s way of dealing with the subject matter, a way that took the wind out of the sails of his rivals.

That the Greeks did not so much as mention the whole theory is not so surprising after all. At that time their aspiration for beauty and virtue, for perfection indeed, had met with the temptation of actual finality. Perfection seemed to have come within
their reach, thence they were impatient with all that could not stand its test. For Spartan warriors to dispose of weak infants was as natural as the practice of Greek mathematicians in Euclid's age never to publish anything in a state of defectiveness or unfinished development; and as long as the anthyphairesetical way of dealing with proportions could not bear comparison with the Euclidean method they were rather inclined to ignore it. They invented the synthetic way of enunciating the outcome of their investigations, that up to the present is used in most mathematical textbooks. The exception that proved the rule was found in 1906 by Heiberg, the so-called Ephod of Archimedes, a writing by the great mathematician dealing with the way in which he had deduced some theorems that he had proved elsewhere in a rigorous way by exhaustion (7).

5. A positive evaluation of Muslim mathematical productivity must take account of a difference in attitude between the Greeks and the Arabian scientists of the middle ages. The latter entertained a deep-rooted veneration for the Greek classics, taking them for perfect almost by axiom: As to their own work, however, they had no pretensions of the kind, and they revealed their train of thoughts without any diffidence, being interested at the same time in that of others. This made them view the study of mathematics from the psychological angle more than the Greeks did, who confined themselves almost exclusively to logic. The Euclidean doctrine of proportions indeed met the relentless requirements of contemporary logic, when previous ways of dealing with ratio had become discarded as a result of the discovery of irrationality as a general phenomenon. From the form in which it was presented, however, little or nothing could be deduced regarding the way in which it had come into being.

Now about ratio and proportionality, unlike most other difficult notions in mathematics, a certain amount of knowledge will originate more or less intuitively and spontaneously in the mind of any person occupying himself with them in an attentive way. If he is a trained mathematician he will no doubt be disposed to admit that a logical justification of his views is required. When, however, a justification of the kind assumes the character of a theory not reminding him in any way of his own conceptions, then his inner self will rebel against it, irrespective of his possible admission that the said theory cannot be opposed on logical grounds. The chances are that he will either try to obtain an equivalent result in a way more in line with his own thoughts, or try to build a bridge between them and the unsatisfying theory.

6. The quoted commentaries show that both attempts have been made in the Muslim world, although the first does not appear anywhere to be carried through to an extent like Becker's effort in 1933. The different methods exhibit a marked similarity. In all cases the reasoning starts from the numeral ratio, whilst the Euclidean process of finding the greatest common measure underlies all attempts to oppose a private view to Euclid's conception of ratio and his definition of proportional magnitudes. Once more I will trace this briefly in the quoted references.

Al-Mahâni (respectively Thâbit ibn Qurra) brings out clearly that for him ratio is the mutual behaviour of two magnitudes when compared with one another by means of the Euclidean process of finding the greatest common measure; and that two pairs of magnitudes are proportional when the two series of quotients appearing in that process are identical.

Al-Nârîzî develops essentially the same theory. He builds no bridge to Euclid's definition of proportional magnitudes, but expresses his opinion that the latter has the character of a principle that must be assumed and cannot be proved.

Ibn al-Haitham distinguishes between numeral and non-numeral ratios and remarks that it is characteristic of the latter that the Euclidean process of finding the greatest common measure does not arrive at an end with magnitudes having such a ratio to one another. As opposed to Al-Nârîzî's his opinion is that Euclid's definition needs a proof. This proof functions as a bridge between the two different ideas about proportionality. Through the intermediary of it Ibn al-Haitham hopes to preserve the advantages of either conception, viz. psychological satisfaction as well as logical justification. In short it comes to the following.

Three premises precede:
1. When there are four magnitudes and the ratio of the first to the whole second is the ratio of the third to the whole fourth, then the ratio of the first to a portion (8) of the second is the ratio of the third to a portion of the fourth.
2. Each magnitude is capable of being halved and each half is capable of being halved and each half of a half is capable of being halved, until the number of parts becomes greater than a prescribed number.
3. When there are two different magnitudes the less is capable of being doubled, until the multiple becomes greater than the greater magnitude.

Then the author takes four magnitudes, supposes them to be proportional and takes equimultiples of the first and third and of the second and fourth and proceeds to prove that the property of alike exceeding, falling short, or being equal is characteristic of them. He reminds the reader of the fact that the ratio of the first to the second is numeral or not numeral and begins by supposing that it is numeral. In this case he is able to prove that the magnitudes have the property in question.

When however the ratio of the first to the second is not the ratio of a number to a number the reader begins by supposing the multiple of the first to fall short of the multiple of the second. Using his premises he now takes a magnitude $M_1$, a little less
than the second, having a numerical ratio to the first. It follows that also a magnitude $M_2$ can be found, less than the fourth, having the same numerical ratio to the third. He is able to make the difference between the magnitude $M_3$ and the second magnitude as small as he desires, so small indeed that the multiple of the first magnitude is still less than the multiple of $M_1$. The latter magnitudes had a numerical ratio to one another, whence it follows that the multiple of the third magnitude too is less than the multiple of $M_2$, which is less than the fourth. But then this multiple is less than the multiple of the fourth magnitude a fortiori.

In other possible cases the reasoning runs in a corresponding way.

"Umar al-Chajami" clearly takes his stand right away. He too bases his reasoning on the numerical ratio and from there deals with non-numerical ratios by means of Euclid's process of finding the greatest common measure. His definition is: proportionality is similarity of ratios. He takes Euclid's definition for an interpretation of the former, but he is not at all content with it. Instead he is of the opinion that Euclid deviates excessively from the true essence of being proportional, embodied in the identity of the aforesaid series of quotients. This true way of being proportional is the subject matter of his treatise.

7. Coming back to al-Dijayami I wish to begin by stating that the criticism of Euclid which he so violently objected to really existed. Even the rather strong assertions as to the lack of connection between ratio and the multiples we find back in the treatise of "Umar al-Chajami", who must have been a contemporary of his.

As for himself al-Dijayami is full of admiration for Euclid. That is why he undertook to defend him. He is convinced of the correctness of Euclid's definitions and subsequent reasoning; hence he desires to convince others too. In order to ensure a reasonable amount of success he first needs a common base acceptable for both parties. He takes the line that a primitive conception of ratio and proportionality is found in the mind of every right-thinking person. From this he derives a number of truths characteristic of proportional magnitudes, which he takes for perfectly evident (9). The first is that in a proportion the first term contains as many parts of the second as the third contains of parts of the fourth. Expressly he says that this is so evident that a closer proof is superfluous. In view of the phenomenon of irrationality, however, this truth constitutes no fruitful idea to be used as a starting-point for the theory of proportions in the case of magnitudes. If however the first is a whole number of parts of the second and the third the same number of parts of the fourth, then proportionality is assured. But yet something else is not open to any doubt: if the first contains more or less parts of the second and the third the same number of parts of the fourth, then proportionality is altogether out of the question. This too needs no proof "because it appeals immediately to the mind", for in this case "the size of the first as compared with the second is not like the size of the third as compared with the fourth". This, apparently, is the view of the cited critics too when they speak of the measure of a magnitude with regard to another magnitude, whilst Euclid confines himself to "some state of two magnitudes in connection with size", a view not unknown to the said critics either but insufficiently distinguished from the former. The subtle difference is that this non-Euclidean view leads to measuring a ratio, whilst Euclid only meant to introduce a definition of the similarity of two ratios. Al-Dijayami is convinced that his statement is clear and does not need a proof, "since things that are clear and evident to the mind without need of proof are not made clearer by proximity in the explanation, because there is no method to make clear what is already clear in it".

After this the author apparently has focussed his attention on what he wanted to prove, viz. that the Euclidean condition of multiples implies the true essence of proportion. Having explained already what he means by "true essence" he now proceeds to make the connection. For that purpose he converts Euclid's multiples into parts, what comes to parting by the number of the first multiple. When this is done it is evident that magnitudes truly proportional according to his own views satisfy Euclid's condition too. For otherwise, if some parts of the second, for instance, exceeded the first magnitude while the same parts of the fourth did not exceed the third, the first would contain fewer parts of the second than the third of the fourth, and of this he had already proved the impossibility.

Now only the converse was yet to be proved, viz. that magnitudes proportional in the Euclidean way were proportional in virtue of his own considerations too. The proof he produces is an indirect one showing much resemblance to the proof of Ibn al-Haitam. The existence of a fourth proportional and the unlimited divisibility of magnitudes underlie his reasoning. The demonstration takes the following course.

If four magnitudes $AB$, $C$, $DE$ and $F$ are of such a kind that $\frac{AB}{C} = \frac{DE}{F}$ and conversely, then $AB : C = DE : F$. For if such were not true but, for instance, $AB : C = DG : F$ ($DG > DE$), we take the first part $H$ of $F$ that is less than $GE$. Let us suppose $H = \frac{1}{p} F$. Now a number $a$ can be found to the effect that $\frac{a}{p} F < DE < \frac{a + 1}{p} F < DG$. Of $C$ we take the same parts and find that $\frac{a}{p} C < AB$ according to the supposition. However $\frac{a + 1}{p} C > AB$, because $\frac{a + 1}{p} F > DE$. Therefore $\frac{a + 1}{p} C > AB$, whilst $\frac{a + 1}{p} F < DG$. This is
inconsistent with the supposition that \( AB : C = DG : F \). This supposition therefore must be rejected.

In the same way it can be proved that the supposition \( AB : C = DG : F \) leads to a contradiction if \( DG < DE \). Therefore, says al-Djājānī, it is only true that \( AB : C = DE : F \).

After having demonstrated in this way that Euclid’s characteristic of proportional magnitudes agrees with the conception of proportionality that will arise spontaneously in a man’s mind, al-Djājānī introduces the Euclidean multiples. This would come to a simple multiplication by \( m \) in our, more modern, notation, but the author, not having the disposal of our surveyable means of expression, needs four pages. Then he raises in a rather unconvincing way the “advantages of Euclid’s multiples. He closes the second part recapitulating Euclid’s “Introduction”, only changing his order. In accordance with his previous argumentation, however, this means that he replaces Euclid’s statement, which is a mere definition, by a convertible proposition.

The third part of al-Djājānī’s treatise deals with unequal ratios. The starting-point, once more not needing a proof, is that the first ratio is greater than the second, if the first magnitude contains more parts of the second magnitude than the third of the same parts of the fourth. When, on the other hand, the first ratio is greater than the second, then “some parts may be found of the second and fourth so that the first contains more parts of the second than the third of the fourth.”

Before passing on to the proof the author first introduces the following simplification. In order to be sure of proportionality, he says, it is sufficient to know that \( \frac{n}{m} B \geq A \) goes with \( \frac{n}{m} D \geq C \), without equality being mentioned. In point of fact he demonstrates that the relation of the inequalities entails the relation of the equalities. When now the first ratio is greater than the second the parts of the second magnitude will only exceed the parts of the first magnitude, if the parts of the fourth too exceed the third magnitude. For otherwise the third would contain more parts of the fourth than the first of the second, so that the second ratio would be the greater.

Then follows the proof that it must be possible to find parts of the second less than the first, whilst the same parts of the fourth exceed the third, as soon as it is established that the first ratio is greater than the second. For if, for instance, this would not apply to the magnitudes \( A, B, C \) and \( D \), then \( \frac{n}{q} B < A \) would entail \( \frac{n}{q} D < C \) without exception, while \( \frac{n}{q} B > A \) would entail \( \frac{n}{q} D > C \), because the ratio of \( A \) to \( B \) is greater than the ratio of \( C \) to \( D \). Then, however, we would have proportionality at variance with the supposition. All this is converted into multiples, after which the author replaces Euclid’s definition of a greater ratio by a convertible proposition.

8. By way of summary I now wish to remark that the study of our treatises of Moslem mathematicians betrays a mentality different from the Greek state of mind. This is, first, a consequence of the fact that they are not the creators, but only the commentators of the mathematical acquisitions under consideration. However, the stress they lay on convincing alongside of proving points to their open eye for the difficulties connected with the transfer of science, which can not be maintained by mere printing (or in their time “copying”). They kept the fire burning. In this way Ibn al-Haithom became, as a natural philosopher, the predecessor and teacher of Roger Bacon and no doubt deserves a similar honour as a mathematician. And when in later centuries the spirit gives way to erudition, the productivity of the Arabic scholars, losing its intrinsic value, dries up in a desert of drab sterility. To conclude my writing I wish to illustrate this view by the following quotation from al-Tūsī (10).

9. In his introduction to Book 5 al-Tūsī declares that measuring one magnitude by the other is the essence of ratio. He distinguishes four cases, viz. equality, part or multiple, parts, and the case of incommensurability. When magnitudes are capable of exceeding one another they are homogeneous and no doubt have a ratio to one another. If two other (or partly the same) magnitudes have a ratio to one another that in no respect is different from the former, the four (or three) magnitudes are called proportional. The way of measuring is the way of finding (or not finding) the common measure and the author’s reasoning intimates that, consciously or unconsciously, his conception of proportionality is: “having the same kind of relation to one another in finding the common measure, i.e. having the same antanimesis.”

Concerning proportional magnitudes the author states that when of the first and third any same multiple is taken and likewise of the second and fourth, the multiple of the third will exceed the multiple of the fourth if the multiple of the first exceeds the multiple of the second; and likewise in the case of equality or falling short.

Of this “definition” of Euclid’s the author gives an ingenious but less convincing “proof”, which I reproduce at full length. It runs as follows.

“Let the ratio of \( A \) to \( B \) be as the ratio of \( C \) to \( D \), and let some multiple be taken of \( A \) and \( C \), viz. \( E \) and \( F \); and of \( B \) and \( D \) some multiple, viz. \( H \) and \( K \). Then I say: if \( E \) exceeds \( F \), then \( F \) exceeds \( K \), and if it is equal it is equal, and if it falls short it falls short.
Proof: As the ratio of A to B is as the ratio of C to D, C exceeds D if A exceeds B, and is equal if it is equal to it, and falls short if it falls short. But E and F are equinumeral multiples of A and C, therefore F exceeds D if E exceeds B, and is equal if it is equal, and falls short if it falls short. But H and K are equinumeral multiples of B and D, therefore F exceeds K if E exceeds H, and is equal if it is equal, and falls short if it falls short; and this is what we wanted to demonstrate.

And when there are four magnitudes and the ratio of the first to the second is unlike the ratio of the third to the fourth, then it is impossible when some equinumeral multiples of the first and third are taken and likewise of the second and fourth, that the multiple of the first does not exceed the multiple of the second unless the third exceeds the multiple of the fourth, and is not equal to it unless it is equal to it, and does not fall short of it unless it falls short of it. For if otherwise, let the ratio of A to B be unlike the ratio of C to D, and let be taken some equinumeral multiple of A and C, viz. E and F, and let be taken some equinumeral multiple of B and D, viz. H and K. Because E does not exceed H unless F exceeds K, and is not equal to it unless it is equal to it, and does not fall short of it unless it falls short of it, and these are equinumeral multiples of A and C, therefore A does not exceed H unless C exceeds K, and is not equal to it unless it is equal to it, and does not fall short of it unless it falls short of it. And H and K are equinumeral multiples of the magnitudes B and D, therefore A does not exceed B unless C too exceeds D, and is not equal to it unless it is equal to it, and does not fall short of it unless it falls short of it. But A exceeds B and C does not exceed D, or is equal to B and C is not equal to D, or falls short of B and C does not fall short of D by assumption. This is a contradiction; therefore the sentence is well-founded and that is what we wanted to demonstrate.”

CHAPTER II. NOTES ON THE TEXT.

B. 3 I read: bi-kithra‘ tad‘ifin
B. 9 I read: jua‘alu la-hum‘u ‘l-mutoba‘ināni
B. 23 I read: al-mi‘qārānī
C. 18 I read: li-an‘anahumā
D. 5 I read: sjai‘un
D. 14 I read: wa-lā an jaqōla
and at the end: wa-lā anna
E. 4 I read: nāqisatan ‘an-hu
and half-way: aqālu mim-mā
E. 12,13 I read: tihā kānaq adjaz‘u ‘l-rābi‘i‘ tūsāwi ‘l-mi‘qāra ‘l-thāliha
E. 17,18 I read: ilā wa-adjaz‘u z nāqisatan ‘on dh
and further: fa-aqālu anna nisbata ‘ab ili ḍj
E. 19 I read: fa-in kāna hh asghara min z
E. 23,24 I read: wa-takun tilka ‘l-fuṣūlū dh ‘im mn
F. 3 I read: nisbatu ‘ab ilā dh
F. 14 I suppress the second: wāhidin
F. 21 I read: dhū ‘ad‘āfīn
F. 22 I read: wa-uqaddamu mā jūhātiq ilaihi mithālihu
H. 17 I read (at the end): li ‘l-thāni
H. 23 I read: li‘iḥdähuma
K. 4 I read: ṭu tūzdajud ‘ad‘āfū ‘l-‘awwali
K. 8 I read: fa‘ju‘chadh li-ā wa ḍj ‘ad‘āfūn mutasāwījatun
and further: wa-lī-b wa ḍ
K. 11 I read: wa-min-d dkūliqa ‘l-djuz‘ā
N. 2 I read at the end: allātī
N. 14 I read: asghara min ḍh
N. 19,20 I read: fa-jakūn ‘al af akbara min ‘ab kamā anna dīn akbaru min dh
N. 21 I read: ‘al af akbaru min ‘ab
N. 22 I read: adjaz‘u z nāqisatan
O. 19 I insert at the beginning: fa-in lam jūdjad
P. 1 I read: al-ma‘ḥūdhahū min d
P. 18 I read: fi ‘ḥādha ‘l-kitābi
Q. 4 I read: li ‘l-thāni wa ‘l-rābi‘i
CHAPTER II. NOTES ON THE TRANSLATION.

(1) I do not know any commentator of this name. The word remotely recalls σύνοδος too.

(2) The 2nd Postulate.

(3) The addition to the 18th Definition of Book I.

(4) This assertion used to be known as the 6th Postulate or as the 9th or the 12th Axiom.

Proclos already considered it an unnecessary addition.

(5) That numbers are magnitudes does not agree with the Aristotelean (and no doubt Euclidean view that magnitudes are divisible ad infinitum, while numbers as composed of indivisible units are not.

(6) I do not know a Greek equivalent.

(7) This definition assures the possibility of the following one.

(8) A striking deviation from the Euclidean view.

(9) This implies the view that a ratio is a fraction.

(10) This reasoning must have met with opposition even in al-Djajjani's time.

(11) The existence of such a magnitude DG is not called in question.

(12) If \( \frac{n}{m} A \leq B \), then \( n A \leq \frac{n}{m} m B \).

(13) If \( \frac{n}{m} B \leq A \) involves \( \frac{n}{m} D \leq C \), then \( n B \leq \frac{n}{m} m A \) involves \( n D \leq \frac{n}{m} m C \) and vice versa.

(14) If \( \frac{n}{m} C \leq A \) involves \( \frac{n}{m} F \leq D E \), then \( \frac{n}{m} C = A B \) involves \( \frac{n}{m} F = D E \).

Proof: assume \( A B = \frac{p}{q} C \), \( D E > \frac{p}{q} F \).

Now we take \( D G = \frac{p}{q} F \).

Let be \( H = \frac{1}{q} F < D E - D G \) and \( K = \frac{1}{q} C \).

Then it is possible to find \( \frac{p'}{q'} \) so that: \( \frac{p'}{q'} F > D G \).

However \( A B : C = D G : F \); therefore \( \frac{p'}{q'} C > A B \), while \( \frac{p'}{q'} F < D E \), which is incompatible with the supposition.

(15) If \( A : B > C : D \), then it is possible to find some \( p \) and \( q \), so that \( \frac{p}{q} B < A \) and \( \frac{p}{q} D \geq C \).

Proof: Suppose that such \( p \) and \( q \) could not be found, then \( \frac{n}{m} B < A \) would involve \( \frac{n}{m} D < C \) without exception.

But \( A : B > C : D \), therefore: \( \frac{n}{m} B > A \) involves \( \frac{n}{m} D > C \) anyhow, so that \( A : B = C : D \), which leads to a contradiction.

(16) Euclid 5; 12 has nothing of the kind, but 6; 12 gives the construction of the fourth proportional. The passage is not quite clear.
CHAPTER III. NOTES.

(1) Algiers 1446, 3°.


(3) No. 8 of the survey in chapter I.

(4) No. 7 of idem.

(5) Paris 2467, 16°, 197 V° (-207). Probably (partly) the same: Berlin 6009, 1°, 34 b - 38 a; Corallah 1502, 5°, 25 a - 26 b.


Codex Leidensis 399, 1. Euclidis Elementa ex interpretatione al-Hadschschadshchii cum commentaris al-Narizii. Arabice et Latine ediderunt notisque instruxerunt R.O. Besthorn et J. L. Heiberg. (Books 1 - 4; the Books 5 and 6 were later published by G. Junge, J. Raeder and W. Thomson.) 1893 and later.

(7) No. 11 of the survey in chapter I.

(8) No. 2 of idem.

(9) I have been in doubt about this translation, but I think it is right after all. The word did not occur in the dictionaries I consulted.

(10) Add.: of the Elements.

(11) No. 31 of the survey in chapter I.


(13) Page 61.

(14) Paris 4946, 4°. The same: Leyden, Cod. or. 199 (8).

(15) No. 33 of the survey in chapter I.

(16) From this I understand that "Umar al-Chajjami takes Euclid's 5th definition for an interpretation or commentary (sjarhun) of the 4th.

CHAPTER IV. NOTES.

(1) Dijksterhuis, De Elementen van Euclides, deel I, afd. I.

(2) Idem, l.c. hoofdstuk V.

(3) Idem, l.c. hoofdstuk V, 4.

(4) Idem, l.c. blz. 73.


(7) Dijksterhuis, l.c. blz. 58.

(8) The author here uses ba'dun, not diuz'un.


(10) No. 43 of the survey in chapter I.
STELLINGEN.

I.
Ten onrechte schrijft Klamroth (Ueber den Arabischen Euklid, Z.D.M.G. Bd 35, p. 292) "... und für meterin wird qadara oder 'adda gesetzt, je nachdem es sich um continuele or discrete Grössen handelt."

II.
De mening van Oscar Becker (Eudoxos-Studien I, Quellen und Studien z. Gesch. d. Math. Abt. B. 2, H. 4, 311-333, 1939) "... dass von der antihypaitetischen Theorie aus keine direkte Beweismöglichkeit von V, 16 für allgemeine Grösse besteht" (dat in een redentheorie gebaseerd op de ontwikkeling van verhoudingen in kettingbreuken de stelling, dat in een evenredigheid de binnentermen verwisseld mogen worden, niet voor algemene grootheden bewezen kan worden) is ongegrond.

III.
De bedoeling van W. de Geus om in zijn dissertatie (Continue Intrestrekening, Amsterdam, 1943) een systematische opbouw te geven van de continue intrestrekening uitsluitend op de grondslagen van de infinitesimalrekening is door een gedeeltelijk onjuiste opzet niet voldoende verwezenlikt.

IV.
De evolutietheorie voor sternren van Hoyle en Lyttleton (Proceedings of the Cambridge Philosophical Society, 1939 en 1940) vindt een onvoldoende quantitatieve basis in het hierbij in aanmerking genomen accretie-mechanisme.

V.
Dat voor het volgen van wiskundeonderwijs op de middelbare school een specifieke aanleg nodig is, is niet bewezen.

VI.
Met de woorden: "Souvenez-vous que le hasard ne favorise que les esprits préparés" (in 1854 gesproken tot de studenten te Rijssel) formuleerde Pasteur de hoofdwt van het productieve denken.
VII.
Bij het Voorbereidend Hoger Onderwijs kan en moet de wiskunde dienen als oefenstof voor het productieve denken.

VIII.
De leer van Otto Selz (Versuche zur Hebung des Intelligenzniveaus. Z. Psych. Bd. 134, 1935) die uitgaat van het grondbeginsel, dat intelligentie niet een onanalyserbare gave is, maar gedefinieerd kan worden als een structuur van specifieke psychische gedragingen, ingesteld op het verwerven en aanwenden van inzicht, is een goede grondslag voor de opbouw van een didactiek der wiskunde.

IX.

X.
Uit de betreffende literatuur (o.a. Höfler, Psychische Arbeit, in Z. Ps. 8 (1895) 44 e.v. en 161 e.v.; Binet et Henri, La fatigue intellectuelle, 1898; Lehmann, Die körperlichen Äußerungen psychophysischer Zustände II (1901) 118 e.v.; Foucault, Les lois les plus générales de l’activité mentale, in An. Ps. 19 (1913) 75 e.v.; E. L. Thordikke, Educational Psychology III, 1914; F. G. en C. G. Benedict, Mental effort in relation to gaseous exchange, heart rate, and mechanics of respiration (1933); J. Jongbloed, De invloed van geestelijke arbeid op de totale stofwisseling, in het Ned. T. v. Gen. 86 III 32 (1942) blz. 2012 e.v.) blijkt, dat vermoeidheid na geestelijke arbeid moet worden toegekregen aan de inspanning der willekeurige (d.i. door de wil gerichte) aandachtsconcentratie.

XI.
Dat de bloedintoxicatie van Mosso en de stofwisselingsverhoging van Jongbloed in direct verband staan met de eigenlijke geestelijke arbeid, is niet bewezen.

XII.
Voor de opleiding van toekomstige leraren kan de bestaande opleiding van toekomstige artsen tot voorbeeld strekken.