Al-Birūnī and the Theory of the Solar Apogee: an example of originality in Arabic Science

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The wording of the theme which we have been invited to discuss seems simple enough at first sight. But when looking at it more closely doubt starts arising as in the heart of Doctor Faustus trying to translate the first words of St. John's Gospel into his beloved German.

At the risk of furnishing a new proof of the so-called German thoroughness, we turn to the Concise Oxford Dictionary to scrutinize the various meanings of the word "original". Existing from the first, as illustrated by "original sin", may be safely excluded; the same, probably, is true of the second definition: that has served as a pattern. On the contrary, not derived or dependent, not imitative, novel in character and style, will have to be taken into consideration, and equally what follows: inventive, creative, thinking or acting for oneself.

Islamic science, at its very outset appears to be purely receptive and derived from earlier models, hence not original according to whichever of the above definitions it may be. But already at a very early stage, there is observable a tendency to attain as comprehensive as possible a survey of the whole of Greek scientific activity and reasoning. As concerns the motives underlying this phenomenon, it may be true that practical aims prevail, and this has been emphasized time and again. But, to those who prefer reading manuscripts to re-reading what others have said about the matter, it becomes evident that the true stimulus or incentive, already in early Abūsāid times, perhaps even in the late Ummayad period, has been a genuine desire for factual knowledge and understanding. In fact, this desire, and the circumstance that Islamic scientists never got tired of quenching their thirst for knowledge, is an essential characteristic of medieval Islam.

Of the examples pertinent to our subject — their name is legion — we limit ourselves to quoting here the translations of Hunain b. Ishāq (c. A.D. 850) and the various members of his family. They certainly do not comprise only those works which could be considered of practical interest; on the contrary, we find among them the whole Galenic opus including its purely theoretical and logical writings, works by Aristotle and Plato, Euclid, Archimedes, Menelaos, and others.

By the end of the fourth century of the Hegira, about A.D. 1000, we may say without exaggeration that there was no important scientific work preserved from Greek Antiquity that was not translated into Arabic, and that Indian writings had met with due interest as well. The influence of the latter on the development of scientific thinking, though definitely lesser than the one exercised by Greek literature, can be perceived above all in mathematics; but it should not be underrated in astronomy and astrology either. In fact, never before in history has there been an activity comparable to the one displayed by the early translators.

Instead of trying to enumerate the most prominent achievements (such a list would be very long), we deem it preferable to present an analysis of one special problem, one which we believe illustrates better than anything else the perfect mastery and the extreme care with which one of the greatest Islamic scholars — at the same time one of the truly great figures in the whole history of science — treated a subject that was no less important than complex and difficult to deal with. We do not hesitate to count it among those achievements which bear witness to true originality, the word original having to be taken here in the sense of “not imitative” and “novel in character”, but certainly not in that of “not derived or dependent”; for there can be no doubt that any great accomplishment is necessarily derived from and thus dependent on the work done by preceding generations.

As is well known, Hipparchus had been the first to account for the unequal lengths of the four seasons by devising a model in which the earth no longer occupies the centre of the sun’s circular orbit. He introduced for the first time the conception of uniform circular motion round a point eccentric to the earth, which automatically yields two extreme points diametrically opposite to one another: the apogee and the perigee, and the straight line connecting them, which evidently goes through the centre of motion as well as through the centre of the earth. Without a developed trigonometrical method he arrived at the

1 We refrain from mentioning Apollonius in this context on account of the scarcity of the information available.
conclusion that the apogee is situated in the point $5\frac{1}{4}$° of Gemini and that the centre of motion is distant from the centre of the earth by approximately two-and-a-half sixtieths of the radius. His method as well as his results are discussed in extenso in Ptolemy’s Almagest (IV, 3). Ptolemy claimed to have re-observed and checked the data given by Hipparchus and found his own results to be identical with the ones obtained 300 years before by his admired predecessor.

In the Islamic period, we find that from the very beginning special attention was paid to this part of the theory of the sun. To give an idea of the errors of observation susceptible of falsifying the results, it will suffice to mention the fact that, while the transit of the sun through the points of the spring and the autumn equinoxes is relatively easily observable, a reliable observation of the moment at which the sun passes through the points of the summer and winter solstices meets with the very greatest difficulties. The reasons are obvious: evidently the speed of variation of the sun’s declination reaches its maximum (c. 24° in one day) near and at the equinoaxes, and its minimum (c. 12° in one day) at the solstices. A direct observation of the latter quantity was out of the question; even when observing with a telescope only an experienced and skilled observer will be able to attain good results.

According to al-Bīrūnī’s historical survey as offered in Book 6, Chs. 7 and 8, of his Maṣūdī Canon (al-Qamān al-Maṣūdī), the first observations were carried out in the Shāmāsīya Quarter of Baghdad, or perhaps rather near the gate carrying the same name, in the year 190 Yazdajird (A.D. 830–1). Our author states that he has drawn his information from Abū Ja‘far al-Khāzin’s commentary to the Almagest. What appears striking is that these early astronomers were independent enough to discard the method employed and described by Ptolemy, and to replace it by a method of their own (at least we do not know what other source they could have drawn from) which proves far superior to the Greek one. Instead of observing the sun’s passage through the four cardinal points of its annual course, they measured its passage through the points 15° Taurus, 15° Leo, 15° Scorpio, and 15° Aquarius. This small but significant modification enabled them to obtain more reliable measurements without, on the other hand, requiring a modification of the method of computation as well. Obviously, the drawbacks of the new method are by far outweighed by its advantages. But in spite of this methodological improvement, the practical result obtained proved completely erroneous, because it yielded a value for the longitude of the apogee which was about 20° too small.

Only one year later, Thābit b. Qurra or the Banū Mūsa, or both, re-observed two of the quarters, and obtained a far better result: 82°5 (according to the Kiṭāb fi sanāt al-shams bi ‘l-‘arsād). As this comes pretty close to the effect produced by the precession of the equinoxes (rejecting Ptolemy’s obviously erroneous confirmation of Hipparchus’s value and counting nearly one-thousand years corresponding to 15° when using Thābit’s new value of one degree in 66 years), Thābit, taking recourse to the principle which later became known under the name of Ocean’s razor, concluded that the two motions must necessarily be equal.

During the next two centuries, the phenomena in question were re-observed repeatedly; as for the methods employed, both the original Ptolemaic and the new “method of the four fuṣūl” were made use of. The following list will give an idea of the strong interest that Islamic astronomers took in the subject.

- Thābit b. Qurra or the Banū Mūsa or both, 201 Yazdajird (A.D. 842), at Baghdad (Ptolemaic method).
- al-Battānī, 251 A.Y. (A.D. 882), at Raqqā (Ptolemaic method).
- 345 A.Y. (A.D. 976), at Baghdad (method of the four fuṣūl).

This and a series of his own observations of the seasons and of the fuṣūl (carried out in the Jurjānīyya, the Western part of Khwārizm, 385 A.Y.—A.D. 1016), the accuracy of which ranks astonishingly high, is the material serving as a basis for al-Bīrūnī’s learned dissertation. He starts his investigation by re-telling Ptolemy’s account of the course of his own work. According to his habit, he presents the method in a not strictly historical way; thus he introduces sines instead of chords, he discusses variations resulting from the equation of time and he

1 In the last 12 hours before and in the first 12 hours after the solstices, the variation amounts to no more than 3°5°.
studies the effect produced by introducing an improved value of the mean solar motion gained from his own determination of the tropical year. In order to avoid errors otherwise inevitable, he defines the latter as the period contained between two subsequent passages of the sun through the autumnal point; as becomes evident in the later course of his dissertation, he is clearly aware that an arbitrarily chosen starting-point will not serve to yield a correct value of the period in question. After having dealt with everything worth knowing of Ptolemy's procedure, he discusses and analyses with the same thoroughness the results of his Islamic predecessors.

He then presents and evaluates the results of his own observations, whereby he finds the solar apogee to be situated at 84° 59' 51" 9'. In doing so he surprises his readers by developing and applying a method of his own, consisting of three essentially different variants, all three of which he shows to lead to the same numerical result. He bases himself on a theorem set forth for the first time by Archimedes and discussed by himself in a special risala, which is of methodological interest because he describes it in more than twenty different proofs. The theorem, in short, reads as follows: if a broken line is inscribed in a circular arc, and if the perpendicular is drawn from the point bisecting the arc on the (major part of the) broken line, then the broken line too is bisected by the perpendicular. It is significant to state in the present context that al-Bīrūnī was interested not only in the purely mathematical (methodological) aspects of the subject, but also recognized the applicability of new theorems to practical problems. No doubt, others before him had introduced new concepts and methods into astronomy, as for instance tangents and co-tangents as well as the spherical sine theorem; but a systematic consideration of the criteria according to which preference is to be given to one method or another obviously appears for the first time in al-Bīrūnī's writings.

From the whole of the material providing the basis for his investigation, al-Bīrūnī infers that there undoubtedly exists a continual motion of the apogee in the direction of increasing longitudes. But at this stage of his analysis he expressly refrains from making any statement as to whether such motion is regular or irregular.

To illustrate the course of the discussion from this point, we give in the following a literal translation of a passage in which al-Bīrūnī describes and analyses the introductory part of

Ptolemy's account (Almagest, III, 1) of the length of the solar year:1

Hipparchus, investigating the motion of the apogee, in a way similar to the one followed by ourselves, recognized that the revolutions in the sphere of the zodiac, which are the years belonging to the sun, are not equal2 and that the mean motion, when occurring in the sphere of the apogee, is uniform in respect of the revolutions. He therefore tried to find it [i.e. the mean motion] without having recourse to the years because of their irregularity. And he appears to have become aware that the motion which is common to the apogees is identical with the one proper to the sphere of the fixed stars. He then went in quest of the uniform revolutions referring to the sun's conjunctions with the fixed stars and its return to each one of them. Ptolemy, believing that in doing so he had been looking for the length of the solar year, tried to show as a necessary consequence that, if the solar year were defined by the sun's return to the fixed stars [i.e. one particular fixed star], anyone else beside Hipparchus would be at liberty to define it [i.e. the year] as its return to any given planet, so that there would result many different years to the sun [i.e. many different definitions of the solar year]. His [i.e. Ptolemy's] answer is that the year is much too obvious in its characteristics to leave concealed from the awareness of plants and animals and, it is unnecessary to say, of man the fact that it is the period comprising the four seasons, which are bound to the sun's return to its place in the sphere of the Zodiac. To this, somebody taking the place of Hipparchus would have to say: "First and foremost, leave the year aside, for the reason that its limitation [definition] is based on a datum [position] that depends on the moon. Then know that I have not been looking for it because it does not remain fixed to one and the same measure [length], in such a way that it could yield the mean way [motion] of the sun and its uniform revolutions, by which the motion of the sphere of the apogee falls short of that of the Zodiac. I am not in the possession of any observations that might yield the value of the motion of the apogee in regard to its position within them [i.e. the signs of the Zodiac]. Therefore I am inclined to share your opinion concerning the motion of the planetary apogees, namely, that it is identical with that of the fixed stars, although you may disagree with me on the question of the solar apogee. As to this, I by no means agree with you because, to me, its motion is beyond doubt. And because that motion comprises the apogees altogether, in my opinion the sun's revolution in the sphere of the apogee is equal to its return to a fixed star. But I do not wish to call it [the named period] a

1 Qādirī (Hyderabad, 1546–61) II, 662, 1–663, 1. 6.
2 The negation contained in MS orient. 1613 of the Preussische Staatsbibliothek is erroneously omitted in the printed edition.
year, so that you do not scold me and convict me of inconsistency. If I could find the duration [revolution] of its uniform motion from its returns to the planets, why then should I avoid doing so [literally, refrain from trying to derive it from it].

In discussing the question of the variation in the length of the year because of the motion of the apogee, al-Bīrūnī deals first with a theorem stating that the equation of solar motion reaches its maximum when, if seen from the centre of the universe, the sun stands at right angles to the line of apsides. The two subsequent theorems treat of the relation connecting the apparent motion (as observed from the centre of the universe) with the corresponding equation. The conclusion is that the sun's passage through two arcs appearing equal to the observer will be fastest in the one whose distance from the apogee is greatest.

These theorems would have sufficed to demonstrate the dependency of the length of the year on the choice of the point of departure. But al-Bīrūnī, instead of contenting himself with such a general statement, now makes this question the starting-point of a thorough investigation on the kinematic conditions of uniform circular motion observed from an eccentric point. He crowns and concludes his demonstrations with the theorem that the apogee and the perigee are the points at which the apparent velocity reaches its extreme values (minimum and maximum) and that, in passing from one to the other, a continual increase or decrease of velocity will be observed. To the best of our knowledge, this is the first time that the concept of accelerated motion was made the subject of a mathematical analysis, which fact alone would be sufficient to count al-Bīrūnī among the greatest mathematicians.

1 Contrary to what might be believed at first sight, Ptolemy's determination of the stationary points of the planets as based on “Apollonius and other mathematicians” (Almagest XII, 1, cf. O. Neugebauer's article on Apollonius's planetary theory in Communications on Pure and Applied Math., VIII, 1955, pp. 641–6) cannot be regarded as an anticipation of al-Bīrūnī's conclusions. For, evidently, Ptolemy and his predecessors operate with the conception of two given and constant angular velocities. The variable velocity, which results from these, if applied to the epicyclic or eccentric model, is not taken into consideration, but only the singular case of a change from positive to negative direction. No attempt whatever is made to introduce the idea of velocity passing through zero. The lack of a general definition of velocity becomes evident as soon as we try to generalize Ptolemy's method in order to determine the point in which the resulting velocity assumes a given value. On the contrary, al-Bīrūnī's true concern is to furnish a mathematical definition and analysis of the variable velocity of an object in each point of its orbit.

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In view of the uniqueness of his investigations, which vie in importance with the ones that led 600 years later to the discovery of the infinitesimal calculus, we give here one characteristic passage in transliteration, followed by a literal translation:

\[ \text{wa idhā kāna 'l-amr 'alā hādīhā istabāna anna 'l-buṭ' tan djanbatay al-awd} \text{ wa annahū ghāyat al-buṭ' 'indahu thumma yatanākṣa wa yadhhabu naḥwa 'l-suра' wa anna ghāyatuhā 'inda 'l-hādīd thumma yatanākṣa wa yadhhabu naḥwa 'l-buṭ' sān djanbatayhi li-anna 'l-taḥāšū' wa 'l-isrā' yākunūn bihāsab tāżayūn al-taʃfūd 'fi 'l-iaɾ-}\]
\[\text{diṭāt wa tamarṣūṣi. [In English]: This being the case, it is obvious that slowness occurs on both sides of the apogee, and the maximum of slowness in it. Then it [i.e. slowness] diminishes and passes over towards rapidity, the maximum of which is [reached] in the perigee. Then it [i.e. rapidity] diminishes and passes over towards slowness to both sides of it, because retardation and acceleration occur according to the increase and the diminution of the differences in the equations.}

The phenomenon of accelerated motion, so far demonstrable and put in evidence only by way of geometrical considerations of a rather general character, thus becomes accessible to a strictly mathematical treatment. Using modern terminology for the sake of brevity and clarity, we summarize al-Bīrūnī's method as follows.

The apparent motion, \( \sigma \), is composed of the mean motion, \( M=\omega t \), having the constant angular velocity \( \omega \), and of an additional term \( \chi(t) \), the 'equation' dependent on the time \( t \), or we may say as well, on the mean motion, to be added to or subtracted from the latter: \( \sigma=\omega t+\chi(t) \). In forming, for the arbitrarily chosen time intervals \( \Delta_0 t=\Delta_1 t=\Delta_2 t=\ldots \), the differences \( \Delta_0 \sigma, \Delta_1 \sigma, \Delta_2 \sigma \ldots \), we have to take into account only the \( \Delta_0 \chi, \Delta_1 \chi, \Delta_2 \chi \ldots \). A geometrical demonstration then serves to show that the differences of the latter \( \Delta \chi \), in passing from the apogee to the perigee (where the equation is to be deducted), are always negative, and that for symmetrical reasons the opposite is true of the other half of the deferent.

The proof runs as follows: To each equation \( \chi \) corresponds a triangle having two sides of constant length, namely the radius of the deferent and the distance of its centre from the observer, while the length of the third side (distance sun-observer) is variable. These triangles can be arranged (by turning them round the centre of motion by the amount \( 180°-\omega t \)) in such

\[ ^{1} \text{Qāmūn II, 666, ll. 9–12.} \]
a way that the sides represented by the radius vector as well as the corners representing the equations coincide with each other. Then the free corners will again lie on a circle concentric with the deferent, whose radius is equal to the distance of the observer from the centre of the deferent; because $Δd = Δd = Δd = \ldots$, the arcs contained between two neighbouring corners will be equal, corresponding to the angle $ωdΔt$. According to Euclid (Elements, III, 8), it is evident that $ΔdX > ΔdX > ΔdX = \ldots$, in the case of equations to be deduced from the mean motion, and that $ΔdX < ΔdX < ΔdX = \ldots$, in the opposite case.

To illustrate the perfect mastery with which al-Bīrūnī handles these wholly novel concepts, we refer to the following conclusion, which he presents at the end of his demonstration: in passing from the smallest to the greatest velocity, the moved object's motion must coincide with the mean motion, $ωt$, at one particular point. It will be the point at which the equation reaches a maximum, because it is there (and only there) that $ΔdX$ vanishes, whence we will have $σ = ωt$.

From these kinematic considerations it results that the values obtained for the length of the tropical year will necessarily vary according to the choice of the point on the zodiac from which the measurements begin, as al-Bīrūnī had claimed before. For the sake of simplicity, it is first assumed that two such measurements are carried out simultaneously, one starting at the apogee, and the other at the perigee. During the subsequent year, the lengths of the apogee and of the perigee will be increased by a small amount. Hence, by the time the sun again stands in the apogee or in the perigee, it will have passed its starting-point, in respect of the zodiac, by a certain amount $δ$, which appears identical in both cases to the observer placed eccentrically to the motion of the sun, but centrally with regard to that of the apogee. But considering the fact that near the perigee the sun passes faster through that angle $δ$ than it does near the perigee, a greater value for the length of the tropical year must result in the first case than in the second.

Al-Bīrūnī points out that, as a consequence of his demonstration, in order to obtain a reliable value for the tropical year, only two places in the circle of the zodiac can serve as starting-points, namely two places near the points at which the equation becomes a maximum (i.e. the intersecting points of the ecliptic with the perpendicular to the apse-line through the earth), because only there will the angle $δ$ be traversed with a mean velocity that is equal to $ω$.

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Here we may insert a remark about the history of one of the most important and fertile concepts in the whole of mathematics: the concept of mathematical functions. The question of its origin and development is usually treated with striking onesidedness: it is considered almost exclusively in relation to Cartesian analysis, which in turn is claimed (erroneously, we believe) to be a late offspring of the scholastic latitudines formarum. But the specific development of the concept during the course of the last two centuries clearly shows that such an interpretation is much too narrow. Because of the methods employed in astronomical computation, operating with functions had already reached a high degree of perfection by the time the first attempts were made to form a general conception of functions. In fact we find that Ptolemy had already succeeded in adapting the methods of Greek geometry to the discussion of functional relations.

Al-Bīrūnī's demonstrations, as found in the subsequent part of ch. 7 of his work, may be regarded as a model investigation of such functional relations. In treating the question of the influence of the motion of the apogee on the lengths of the four seasons, he discusses the characteristic positions of the apogee, and then proceeds to define the conditions under which the length of a season becomes a maximum or a minimum, and to investigate in which of the sections it increases or decreases. Going one step farther, he enters upon an analogous discussion of the sum of two successive seasons.

Already in the preceding chapter (ch. 6, "On the mean solar motion according to the method employed by Ptolemy"), al-Bīrūnī had tried to obtain a reliable result by choosing a particular point of the zodiac as his point of departure and by then comparing the earliest observations at his disposal with his own. Obviously the long time involved considerably reduced the probable error. On this basis, al-Bīrūnī pointed out the very great errors occurring in the observation of the solstices, even when such observations were carried out with great skill and care and with the really gigantic instruments which he describes as having been used by his predecessors. Thus we learn that, in a.d. 988, Abū Sahl al-Qūhī, observing in Baghdad, employed a special room of which the floor was given the shape of a spherical shell 15 cubits in diameter, the centre coinciding with an opening in the roof of the building. Six years later, at Raly, Abū Maḥmūd al-Khujandī used a sextant with a radius of 40 cubits. Even
then, al-Bīrūnī states, there were considerable differences in the results. But even in observing the equinoxes, he adds, the errors occurring should not be under-rated. Thus the latitude of the place of observation is not always known with sufficient accuracy. For instance, the values for the latitude of Baghdad, as indicated by two of his immediate predecessors, Abū Ḥāmid al-Ṣaghānī and Abu 'l-Wafā', vary by 14' (here al-Bīrūnī mentions the exact situation of the places of observation within the city of Baghdad; their actual distance evidently excluded discrepancies of this magnitude). He finally shows that the consequences of such small errors can be considerable, and that in particular a difference of only a few minutes of the day (1 day = 60 minutes of a day) may lead to a difference of one or several degrees in determining the longitude of the apogee.

After a short excursion on al-Ṭahrīrī’s view of the motion of the apogee, which he refutes with sarcastic remarks, he concludes as follows: there can be no doubt that its value comes very close to that of precession. But because of the circumstance that the observations at his disposal are not accurate enough, he is unable to make any definite statement as to its exact value: “Truly, we despair of the possibility of finding the value of that motion in this way, having at hand no other observations than those to which we have referred.”

Thus there remained for al-Bīrūnī only one possibility, to assume as a hypothesis, which for the time being can neither be proved nor refuted, that the motion of the apogee is identical with that of precession. If we remember that Thābit b. Qurra had postulated the same identity on the basis of a purely formal teleological principle, the difference of approach becomes particularly striking.

No less significant is what follows. Al-Bīrūnī, not contenting himself with an approximate determination, now tries to establish a value as accurate as possible for the motion of precession. Because of his extraordinary skill, he goes far beyond the rough estimate of 1° in 100 years found by Ptolemy. In this context it will be well to note that al-Bīrūnī refrained from using any of Ptolemy’s own observations but preferred to have recourse to the earliest observations made by Timocharis. We may safely conclude from this that he was perfectly aware of the deficiency of Ptolemy’s observations. Al-Bīrūnī derives his own value from a comparison of one observation of Spica made by himself with a corresponding one made by Timocharis, then checks his value by comparing the observations of Regulus

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made by Hipparchus with those carried out by earlier Islamic astronomers.

By comparing the printed text of the Canon with a manuscript formerly preserved in the Preussische Staatsbibliothek (MS orient. 1619), we gain a remarkable insight into al-Bīrūnī’s working technique. We thereby obtain two series of independent values, which is due to the circumstance that al-Bīrūnī operates with two different initial data: one series is based on Timocharis’s observation of Spica as related by Ptolemy, namely the longitude of Virgo 22° 20', the other, on the value 20° 36' 55'' 43''. The reason of this discrepancy becomes obvious when we remember that the former value represents a reconstruction, because what Timocharis had actually observed in 294 B.C. was an occultation of Spica by the moon. Evidently, al-Bīrūnī started operating with the longitude as calculated by Ptolemy, but then replaced it by the value obtained from his own lunar theory.

As for the latter value, he says in the introduction to his catalogue of fixed stars (Bk. IX, ch. 3, sect. 3) that he has taken into account the parallax as well as the difference of longitude between Alexandria and his own place of observation, Ghazna. For his own observation of A.D. 1009, we find in both series a similar discrepancy: in the first, he makes it Libra 9° 24', in the second, 9° 30'. Undoubtedly, this too was due to a later correction, the reason of which, however, remains unknown.

From these considerations al-Bīrūnī obtained a value for the motion of precession far superior to the Ptolemaic value: he states that the longitudes increase by one degree in 68 years and 11 months (the modern value is c. 71° 7½'). In accordance with his hypothesis, al-Bīrūnī then deduces the value of the daily motion of the apogee from the sun’s daily motion as derived from his own value for the length of the tropical year. He thus obtains a value for the sun’s mean daily motion that is to be regarded as well founded.

Considering the fact that all his calculations so far had been based on the provisional mean motion resulting from the inaccurate length of the tropical year, he then once again re-calculates, with the improved value, the position of the apogee and the amount of eccentricity. It need hardly be said that in doing so he again employs new methods specially devised for the purpose. We point out in particular that he proceeds to a gradual generalization of his method, thus far applicable only to one-quarter of the zodiac, which finally
furnishes a no less perfect than elegant method for computing the values in question from three arbitrarily chosen successive observations. By a final correction, al-Bīrūnī takes account (ch. 9) of the circumstance that it is strictly speaking illicit to dect the daily motion of the apogee from the sun's daily motion in the ecliptic, because they move around different centres. His new model of solar motion furnishes the position of the sun in the sphere of the apogee for the very same two autumnal equinoxes which he had used as starting-points for his determination of the tropical year, and from those positions he then derives the mean daily motion in the sphere of the apogee.

A particularly good opportunity to check all these investigations would offer itself at the moment the sun, when passing through the autumn equinox, has a mean anomaly of 90° ("a mean motion of 90° in the sphere of the apogee"). Already in ch. 8 al-Bīrūnī had shown how the moment can be computed (it will have been reached, according to modern astronomy, about the year 1250). Apart from the new methods mentioned for finding the necessary parameters, al-Bīrūnī proposes to observe the altitude of the culminating sun on consecutive days about the probable time of its passage through the apogee, and to determine ex post facto the situation of the latter from the increase and decrease of corresponding intervals of longitude.

Al-Bīrūnī's investigations concerned above all the future course of events. Is the centre of the sphere of the apogee really carried around in the circumference of a circle whose centre coincides with the centre of the universe? Al-Bīrūnī employs this as an hypothesis, which, however, "is valid only as long as nothing else becomes so evident that we must accept it, be it in our life-time, or be it in a later epoch, in which others will be living than ourselves".