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THE TRIGONOMETRY OF AL-BIRUNI

by

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The Trigonometry of Al-Bīrūnī

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[1] The trigonometry of al-Bīrūnī is well presented to the scholarly world through Carl Schoy's Die trigonometrischen Lehren des al-Bīrūnī (Hanover, 1925). One concept herein involved was also expounded by Schoy in his Al-Bīrūnī's Method of Approximation of Chord 40°, (American Mathematical Monthly, 33 (1926), pp. 95-96). Both, as well as M.A. Kazim's Al-Biruni and Trigonometry, published in the commemoration volume of 1951, were built upon sections of al-Bīrūnī's monumental work, Al-Qānūn al-Mas'ūdī. The present paper is a study of a work by al-Bīrūnī devoted fully to trigonometry, namely, Istikhrāj al-Awtār fī al-Dāira bi-Khawāss al-Khaṭṭ al-Munḥanī fīhā = Deriving the chords in a circle by the properties of the line broken in it. Probably, this work is the first ever devoted to trigonometry as such. It seems be the background of the trigonometrical sections in al-Bīrūnī's Canon. It is the work referred to in Das Buch der Auffindung der Sehnen in Kreise, ubersetzt und mit Kommentar versehen von Heinrich Suter, (Bibliotheca Mathematica, Vol. II, pp. 11-78, 1910).

To review a work already reviewed and commented upon by an excellent scholar like Suter cannot probably be passed without justification. In the present case, the justification is not difficult to present. Suter reviewed ms. Leyden, 1012, where-as the present study reviews the ms of Bankipore, Patna published by the Hyderabad Publication Bureau, in 1948. The former manuscript is in 21 folios, whereas the latter fills 226 pages of the Hyderabad publication. The two are so alike that they cannot be but one and the same work, and so different that the difference cannot be a scribal mistake. It seems that the Hyderabad copy gives the work in full, and the Leyden copy gives an abridgement thereof by the pen of al-Bīrūnī himself. In this abridgement, some details at the beginning, and the more complicated parts at the end are omitted and the intermediary substance is rather better arranged and better worded.

It should be pointed out that more than half the 226 pages in the hyderabad publication do not belong to the work under consideration. Some of these form parts of Rasā'il Ibn Sinān published by the same Bureau, and some are parts of at least two works by al-Bīrūnī hitherto thought to be lost. To republish these works properly arranged and annotated is a scholarly rewarding project that the pressure of other duties has prevented me from accomplishing. The part from page 1 to page 108 and that from page 224 to the end of page 226 cover, almost in full, the treatise we here review.

[2] Like all medieval mathematicians, al-Bīrūnī conceived chords in a circle as sines of angles. If θ is an angle at the centre of a circle of radius r, and crd θ is the chord subtended by this angle, then it is easy to see that

$$\text{crd } \theta = 2r \sin \frac{\theta}{2} = d \sin \frac{\theta}{2}$$

This is how the Greeks, the Indians and the Muslims understood the sine function. Al-Bīrūnī was the first to concentrate on the circle where d=1, and thus to him crd $\theta = \sin \frac{\theta}{2}$.

Finding out the length of crd θ therefore meant finding out $\sin \frac{\theta}{2}$. The angle was often denoted by its measure in degrees, but also as a fraction of the complete revolution. Thus crd $\frac{1}{4}$, crd $\frac{1}{9}$ denote respectively crd 90°, crd 40° and equal the values of $\sin 45^\circ$ and $\sin 20^\circ$. Apart from 30°, 45°, 60° whose sines can be readily given by elementary knowledge, crd θ and sin θ were obtained by utilizing Prolemy's theorem, namely, in a cyclic quadrelateral, the product of the diagonals is equal to the product of one pair of opposite sides added to the product of the other pair. Six chords are here involved. Taking these in special cases, the rules for $\sin (90^\circ - \theta)$, $\sin (\pm \alpha)$, $\sin 2\alpha$ can be derived. Of the theorems in Euclid's Elements, Ptolemy's is the only one which leads to these results, which are inevitable for obtaining the sine table, of course with approximation.

[3] Al-Bīrūnī finds the journey from Ptolemy's theorem to the sin table too long and suggests another method for:-

(i) deriving the essential trigonometric formulas, and

(ii) obtaining more easily and probably more accurately, the values of sines.

He suggests as alternative to Ptolemy's theorem four geometrical relations which he proves to be equivalent, in the sense that if on proved, the others follow from it. He does not claim discovery of these theorems. On the contrary, the greater part of his work is a collection of the different proofs thereof suggested by others, mos Moslims, but including Archimedes.

Giving several proofs to one and the same theorem looks stran not only to us, but also to al-Bīrūnī's contemporaries. In his intr tion, he objects that Ibn Sīnā reckoned such a collection "fudhūl" means "redundant" and implies the sense of being a waste of time ar effort. Al-Bīrūnī defends himself by stating that collecting these (i) gave him chance to stay in company with great minds and (ii) redundant since it has led to results that are the poles on which science of astronomy stands.

From our point of view, what led to these results are the fo theorems and not the multiplicity of the proofs thereof. But we app this multiplicity because it gives names of mathematicians that are otherwise unknown to us and proves the existence of some books tha otherwise we may doubt that they ever existed.

Here are the four theorems:

1) If in a circular arc, a straight line is broken to form two u chords, and, from the midpoint of the arc, a perpendicular is dropped to the line, it will bisect it.

Thus in figure 1, ABC is an arc, and ABC is a booken line forming the two chords AB,BC. D is the midpoint of arc ABC, and DH⊥AB. According to the theorem, AH = HB + BC.

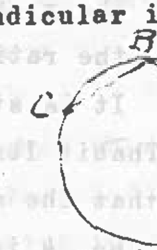


Fig. I

2) As in the line ABC, (Figure II), if D is the midpoint of AC, then $AB.BC + \overline{DB}^2 = \overline{AD}^2$,

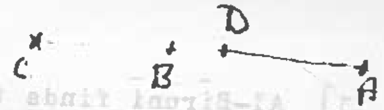


Fig. II

So it is in arc ABC, (Figure I), D being its midpoint:

chord AB. chord BC + $(\text{chord } DB)^2 = (\text{chord } AD)^2$

This result is expressed in sine form as follows:

$\sin \text{arc } AB.\ \sin \text{arc } BC + \sin^2 \text{arc } DB = \sin^2 \text{arc } AD$

This means that an arc is measured by the angle it subtends at the circumference.

3) If an arc is bisected and another arc is adjoined to it, their chords have this same property.

Thus if in Figure III, D is the midpoint of arc AB, and BC is adjoined, then chord AC. chord BC + $(\text{chord } BD)^2 = (\text{chord } DC)^2$

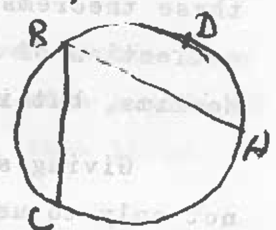


Fig. III

4) DH. HB = $\Delta ADC - \Delta ABC$ (Figure I)

It is pointed out that of all triangles standing in a circle on the same base, the isosceles has the greatest perimeter.

[4] By the middle of page 49, al-Bīrūnī is through with these theorems and the several proofs of them. Now he starts the second part of his work, namely constructions that are made easy by these theorems.

1 to 4: From two given points to draw two lines intercepting a given angle and having:

(1) the sum of their lengths equal to a given length,
(2) the difference between their lengths equal to a given length,
(3) the product of their lengths equal to a given area,
(4) the ratio between their lengths equal to a given number.

It is stated that Menelaus gave a lengthy method for no.1, and so did Thābit Ibn Qurra, that Abū Saʿid al-Sijizi gave a very simple method, and that the method of al-Bīrūnī is claimed to be as simple.

No. 4 is said to have no bearing upon the theorems under consideration but is appended here because with the first three it makes a sequence.

5. In a given circle to construct a triangle with a given perimeter.

6. A proof of Archimedes rule of altitudes of triangles, namely, if a,b,c are the lengths of the sides of ΔABC, and a perpendicular is dropped from A to BC, then c is divided into two segments and

$\frac{1}{2}\left(c + \frac{a^2 - b^2}{c}\right)$ = the greater segment, and

$\frac{1}{2}\left(c - \frac{a^2 - b^2}{c}\right)$ = the smaller segment

7. A proof of Archimdes rule for the area of the triangle, namely, if a,b,c are the sides, $s=\frac{1}{2}(a+b+c)$, Δ = the area, then,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

8. A proof of the Indian rule for the area of the cyclic quadrilateral namely, if a,b,c,d are the sides, s the semiperimeter and A the area, then $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$

9-15. New proofs of certain theorems related to, and new methods of calculating, some astronomical constants, like the greatest inclination (of the sun), the area of the eclipsed portion of the heavenly body (sun or moon), etc.

16,17 Two problems that "often appear in books of algebra". The first is the problem of the rod: A rod stands upright on level ground; required to break it so that when it is bent it reaches the ground at a given distance from its foot; where should it be broken

The second is the problem of the two palm trees. These are of different heights and at opposite sides of a river of given width. Two birds stand on their summits. A fish appears in the water and the birds dash to catch it. They reach it at the same time. Where in the river does the fish appear ?

This part of the work ends by pointing out that there may be other methods to solve or prove the foregoing problems and theorems, but the methods presented in the book aim at utilizing the four theorems stated in the early pages.

[5] Discussion of chords as such starts towards the end of page 84. The following points are established:

1) Chords vary according to their arcs. If one chord is given all others can be expressed in terms of it. They are bound in greatness by the diameter; which is the greatest chord. But there is no bound to their smallness. We therefore express them as fractions of the diameter.

In what follows, d=diameter; α, β, γ are arcs whose chords are a, b, ℓ; crd $\frac{1}{m}$ = chord of the arc which is $\frac{1}{m}$ of the circumference.

2) crd $\frac{1}{6} = \frac{d}{2}$. This is the first chord to know and the only rational one.

3) crd $\frac{1}{10} = \sqrt{(\frac{d}{2})^2 + \frac{1}{4}(\frac{d}{2})^2} - \frac{d}{4}$

This is certainly the same as $\frac{r}{2}(\sqrt{5} - 1)$, but the first form agrees literally with the wording of the author.

4) crd $(\frac{1}{2} - \alpha) = 2\sqrt{\frac{d+a}{2}\,(d - \frac{d+a}{2})}$

This is said to be the same as, but easier to calculate than, $\sqrt{d^2 - a^2}$, for this involves squaring two quantities.

5) Applying (4) to (2) and (3) makes two more chords known. He obviously means crd $(\frac{1}{2} - \frac{1}{6})$, crd $(\frac{1}{2} - \frac{1}{10})$

6) crd $2\gamma = \left\{\ell\sqrt{(d^2 - \ell^2)\cdot\frac{1}{4}} \div \frac{d}{2}\right\}\cdot 2$

7) crd $\frac{1}{2}\gamma = \sqrt{(\frac{d - \sqrt{d^2 - \ell^2}}{2})^2 + (\frac{\ell}{2})^2}$

8) Thus, with crd $\frac{1}{6}$ known, the following can be calculated:

crd $\frac{1}{3}$, crd $\frac{1}{2}\cdot\frac{1}{6}$, crd $\frac{1}{4}\cdot\frac{1}{6}$, ...

And with crd $\frac{1}{10}$, crd $\frac{4}{10}$ and crd $\frac{1}{5}$ known, their multiples and Sub-multiples can be calculated,

Also from crd $\frac{1}{2}$, crd $\frac{1}{4}$ is obtained, which is $\sqrt{\frac{d^2}{2}}$, and thus crd $\frac{1}{8}$ can be obtained.

crd $\frac{1}{8} = \sqrt{(\frac{d}{2})^2 - (2\,\text{crd}\,\frac{1}{4} - \frac{d}{2})\frac{d}{2}}$

Thus crd $\frac{1}{2}\cdot\frac{1}{8}$ can be obtained.

$$\text{crd}(\alpha+\beta) = d^2 - \left[\left\{\frac{d}{2} - \left(\left|\sqrt{\frac{d^2-a^2}{4}} - \sqrt{\frac{d^2-a^2}{4}}\right|^2 + (\frac{a+b}{2})^2\right) \div d\right\} \times 2\right]^2$$

$$\text{crd}\,\tfrac{1}{2}(\alpha+\beta) = \sqrt{\left(\sqrt{(\frac{d}{2}+\frac{b}{2})(\frac{d}{2}-\frac{b}{2})} - \frac{d^2-a^2}{4}\right)^2 + (\frac{a+b}{2})^2}, \quad \beta < \alpha$$

$$\text{crd}(\alpha-\beta) = \sqrt{d^2 - \left\{d - \left[\left|\sqrt{\frac{d^2-a^2}{4}} - \sqrt{\frac{d^2-b^2}{4}}\right|^2 + (a - \frac{a+b}{2})^2\right] \div d\right\}^2}$$

Let $C_1 = b.\frac{1}{2}\,\text{crd}(180 - \alpha) \div \frac{d}{2}$,

$C_2 = \sqrt{a^2 - |C_1^2 - b^2|}$,

then crd $(\alpha+\beta) = C_1 + C_2$

crd $(\alpha-\beta) = C_1 - C_2$

Let $C = b.\frac{1}{2}a - \frac{d}{2}$

then crd $(\alpha+\beta) = \sqrt{|a^2-c^2|} + \sqrt{|b^2-c^2|}$

crd $(\alpha-\beta) = \left|\sqrt{a^2-c^2} - \sqrt{b^2-c^2}\right|$

b.crd$(180-\alpha) \div d = C_1$

crd$(\alpha-\beta) = \sqrt{a^2-b^2+C_1^2} - C_1$

a.crd$(180-\beta) \div d = C_2$

crd$(\alpha+\beta) = C_1 + C_2$, crd $(\alpha-\beta) = |C_1 - C_2|$

$\frac{a^2-b^2}{\text{crd}(\alpha-\beta)} = \text{crd}\,(\alpha+\beta)$

These results are attributed to Abū Nasr, Ibn Iraq.

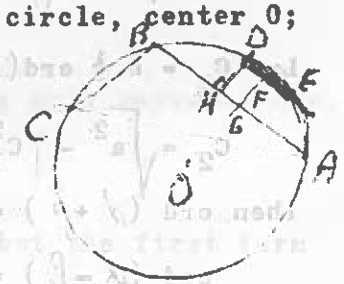
$\frac{a^2 - (b^2 + \text{crd}^2(\alpha-\beta))}{\text{crd}(\alpha-\beta)} + \text{crd}(\alpha-\beta) = \text{crd}(\alpha+\beta)$

It is probably unfair both to al-Biruni and to the history of science to claim that the rules 9 to 16, expressed by my alphabetic symbolism, represent the statements of the great author. They certainly do not, but are what his results amount to if expressed in this symbolism. What al-Bīrūnī did was invariably:

i) to show by Euclidean reasoning that the required chord can be calculated, and

ii) to formulate the process of calculation built upon that reasoning. To show more clearly how this is done, I give here, in some detail, al-Bīrūnī's enunciation of rule no.10, viz. $crd(\alpha+\beta)$:

In the figure, AB,BC are given chords in a circle, center O; required to find chord AC.



From D, the midpoint of arc ABC,DH is dropped ⊥ AB. → AH = ½(AB + BC), according to theorem 1.

AD = crd(½ arc ABC). Thus if AD is found, AC can be derived.

But $\overline{AD}^2 = \overline{AH}^2 + \overline{DH}^2$.

To calculate DH,DE is drawn // AB and OGF is dropped ⊥ DE. OG=½crd(180-arc AB), OF=½crd(180-arc BC).

Therefore AD and hence AC can be found. The process is as follows: "Subtract the product of each one of the two chords by itself from the product of the diameter by itself, and take the square root of each remainder. Multiply the difference between the two square roots by itself, and add to that the product of half the sum of the two chords by itself. Divide the result by the diameter and subtract the quotient from the radius. Double the result and multiply this double by itself; subtract this from the diameter. The square root of the remainder is the chord of the sum of the two arcs".

Of course, our symbolism may reduce these steps considerably, but we here study how al-Biruni presented the probelm.

6) On pages 101-104, the following are solved:

1) To find crd (180- x), given crd γ = a, d+crd(180-α)=b, but d unknown. The problem is said to be like that of the rod. The formulas here given are:

(i) $\frac{1}{2}b + \frac{a^2/4}{\frac{1}{2}b} = d$

(ii) $\frac{1}{2}b - \frac{a^2/4}{\frac{1}{2}b} = crd(180 - \alpha)$

2) If a,b are two chords in a circle, d unknown, but $crd(180 - \alpha) + crd(180 - \beta) = C$, then

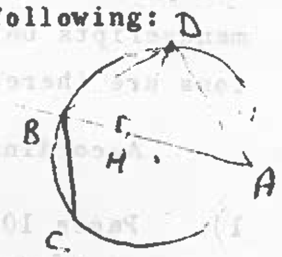
$$C + \frac{|a^2-b^2|}{C} = crd(180 - \beta)$$

$$C - \frac{|a^2-b^2|}{C} = crd(180 - \gamma)$$

3) If AC = d, AB + BC = K, to find AB, BC. Let $\sqrt{\frac{1}{4}d^2 - (\frac{1}{2}k)^2} = C$, then, the two chords are $\frac{1}{2}K \pm C$

4) If AC is known, and AB:BC is known, to find AB, BC.

[7] At this stage, the Leyden ms is concluded with the statement that the rules available yield crd 3° and crd $\frac{3^\circ}{2^n}$, but not crd 1°. MS Hyderabad however, as from page 105, gives the following:

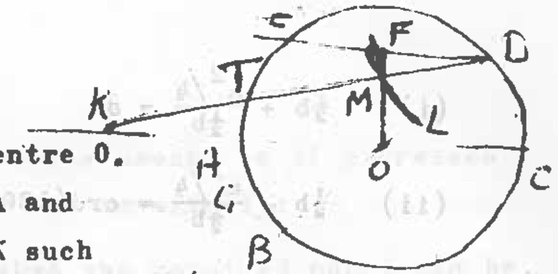


1) The problem is to find crd $\frac{\theta}{3}$, given crd θ. Let ADB be an arc whose chord AB is known. If arc BD = arc BC = ½ arc AD, then according to al-Biruni's four theorems, the problem can be solved, if the HD perpendicular is such that BD = BC.

Again if Z is taken along AB, and BZ = BC = BD, it is proved that ZD is tangent to the circle. IF Z can be taken such that BZ = BD, the problem is solved.

2) No one has been able to solve the problem geometrically. Al-Kindī and the ancients solved it mechanically, and later mathematicians used the properties of the hyperbola.

3) The author gives a method for approximate trisection of an angle and thus for finding crd $\frac{\theta}{3}$. This covers pages 107,108 and 224 to 226, where the book ends. It runs as follows:



Let AB be a given arc of circle centre O. Draw the diameters AOC, BOD,. Produce CA and draw DE//CA, OF⊥ DE. Manage to draw DMTK such that TK is equal to the radius. Then DK trisects ∡ ODE, and if OG//DK, OC trisects arc AB. These results are proved, and the following is given to find crd θ/3. If A$\widehat{O}$B = θ, OF = ½ crd 2θ. Take OM a little greater than ⅔ OF. OM is the required chord. But OM:MF must be equal to KM:MD, if they are not, we have to change the value of OM.

8 What about the other pages in the text ?

Over ten years ago, I published a note in Islamic Culture showing that the substance in the two hyderabad publications, Rasā'il al-Bīrūnī and Rasā'il Ibn Sinān are badly mixed. In that note I managed to rearrange the pages according to content, and pointed out that the final word comes after seeing the manuscripts and drawing evidence from page ends. During the past ten years I have succeeded in obtaineng a microfilm of these manuscripts but the critical pages are unfortunately unreadable. My conclusions are therefore still according to content:

Accordingly, it seems to me that

1) Pages 109-137, 154-180 make up the greater part of a work by al-Bīrūnī, which I take to be the so-called Hall al-Ta'dil. It is probably no.11 in Boilot's collection (institut Dominicain d'Etudes Orientales, 1955). It is complete except for the opening lines and probably a few pages at the end.

2) Pages 139-153 are parts of a work by Ibn Sinān named Masa'il Handasiya.

3) Pages 180-184 seems to be part of a work by al-Bīrūnī on asymptotic behaviour. It is probably no. 67 in Boilot's collection, namely: Maqāla fi ... amr al-Khattain allazain yaqrubān wa lā yaltaqiyān.

4) Pages 185-201-218 (to be rearranged) should go to Ibn Sinān's Harakat al-Shams.

5) Pages 219-223 seem to be the first pages of another work by al-Bīrūnī, probably Boilot's no.4, namely: Takmīl Zīj Habash bi al-ʿIlal.

I take leave to say again that a critical edition of the two Hyderabad publications is highly rewarding.

The trigonometry of al-Bīrūnī

The paper makes a study of a work by al-Bīrūnī devoted wholly to trigonometry. This is the first of the Rasā'il published by the Hyderabad publications Bureau, in 226 pages. But more than half these pages are a misplacement and have little to do with al-Bīrūnī's trigonometry.

Al-Bīrūnī states that ~~eq~~ sine tables are formed by utilizing Ptolemy's theorem of the cyclic quadrilateral. This makes derivation of the required formulas rather long. He therefore suggests four geometrical relations that he proves to be equivalent. From these relations he derives the formulas for $\sin(90-\theta)$, $\sin(\alpha \pm \beta)$, $\sin 2\alpha$ and $\sin \frac{1}{2}\alpha$.

He finds that starting from $\sin 60^\circ$, $\sin 30^\circ$, $\sin 15^\circ$,... can be found, and starting from $\sin 36^\circ$ which he derives by Euclidean methods, $\sin 18^\circ$, $\sin 9^\circ$,... can be found. From $\sin 18^\circ$ and $\sin 15^\circ$, $\sin 3^\circ$ can be found. He thus finds that he can obtain $\sin 3^\circ$. But what about $\sin 1^\circ$?

This leads him to the question of the trisection of an angle to which he gives an approximation.

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