The power series for the sine in Kerala (India) ca. 1400-1550.

Text 1:

निहत्य चापवर्गेण चापं तत्तत्फलानि च ।
हरेत् समूलयुग्वर्गैस्त्रिज्यावर्गाहतैः क्रमात् ।। ४४० ।।
चापं फलानि चाधोऽधो न्यस्योपर्युपरि त्यजेत् ।
जीवाप्त्यै, संग्रहोऽस्यैव 'विद्वान्' इत्यादिना कृतः ।।४४१।।

Text 2:

'विद्वांस्' 'तुन्नबलः' 'कवीशनिचयः' 'सर्वार्थशीलस्थिरो'
'निर्विद्धाङ्गनरेन्द्ररुङ्' निगदितेष्वेषु क्रमात् पञ्चसु ।
आधस्त्याद् गुणितादभीष्टधनुषः कृत्या विहृत्यान्तिम-
स्याप्तं शोध्यमुपर्युपर्यथ घनेनैवं धनुष्यन्ततः ।।४३७।।

Text 1:

nihatya cāpavargeṇa cāpam tattatphalāni ca
haret samūlayugvargais trijyāvargahataiḥ kramāt
cāpam phalāni cādho 'dho nyasyopary upari tyayet
jīvante saṅgraho 'syaiva vidvān ityādinā kṛtaḥ

Translation of Text 1: Having multiplied the arc and the results of each (multiplication) by the square of the arc, one ought to divide by the squares of the even (numbers) together with the (square) roots, multiplied by the square of the Radius, in order. Having put down the arc and the results, one below the other, one ought to subtract going upwards. At the end is the Sine. A summary is made by (the verse) beginning "*vidvān*"

Text 2:

vidvāṃs tunnabalaḥ kavīśanicayaḥ sarvārthaśīlasthiro
nirviddhāṇganarendrаruṅ nigaditeṣv eṣu kramāt pañcasu
ādhastyād guṇitād abhīṣṭadhanuṣaḥ kṛtyā vihṛtyantimasy-
āptam śoddhyam upary upary atha ghanenaivam dhanuṣy antataḥ

Translation of Text 2: The wise ruler whose army has been struck down gathers the best of advisers and remains firm in his conduct in all matters, then he shatters the king whose army has not been destroyed. When these five (numbers)[1] have been spoken in order, (starting) from the one which is at the bottom multiplied by the square of the given arc, when one has divided (by the square of the arc of 90 degrees), the quotient (of each) is to be subtracted, proceeding upwards, (but the quotient) of the last (is multiplied and divided) by the cube, (the quotient is to be subtracted) from the end of the arc.

Source: the verses are found in several works, including a work written between 1529 and 1556 by Shankara.

[1] The numbers are $2220\ 39'40''; 273\ 57'47''; 16\ 05'41''; 33'06''$ and $44''$, written in the *kaṭapayādi* system: 44 = vv, 3306 = lbnt, 160541=ycnśvk, 2735747=r(th)lś(th)vs, 22203940=rrrng(dh)vn. Two consonsnts between parentheses, such as (th), (dh), correspond to a single letter in Sanskrit. Vowels and some consonants do not have a numerical meaning here. The notation is in the sexagesimal system: $1'$ is 1 minute, that is $\frac{1}{60}$, $1''$ is 1 second, that is $\frac{1}{60 \cdot 60}$.

What is the meaning in modern notation?

Text 1 means: Sin $\alpha = R \sin \frac{\alpha}{R} = \alpha - \alpha \cdot \frac{\alpha^2}{R^2 \cdot (2^2+2)} + \alpha \cdot \frac{\alpha^2}{R^2 \cdot (2^2+2)} \frac{\alpha^2}{R^2 \cdot (4^2+4)} - \ldots$ with $R = \frac{10800}{\pi} \approx 3438$, α measured in minutes of arc, and Sin α the Indian sine, which is $R \sin \frac{\alpha}{R}$ where sin is the modern sine, $\frac{\alpha}{R}$ in radians. N.B. the minute of arc was a unit of length in India, hence $2\pi R = 60 \cdot 360$.

Text 2 means: Sin $\alpha = \alpha - (\frac{\alpha}{5400})^3(2220\ 39'40'' - (\frac{\alpha}{5400})^2(273\ 57'47'' - (\frac{\alpha}{5400})^2(16\ 05'41'' - (\frac{\alpha}{5400})^2(33'06'' - (\frac{\alpha}{5400})^2 44''))))$. The numbers $2220\ 39'40''$ until $44''$ are rounded values of $\frac{5400}{3!}(\frac{\pi}{2})^2, \ldots \frac{5400}{11!}(\frac{\pi}{2})^{10}$.

How were the power series for the sine discovered by Mādhava (ca. 1400)?

We will simplify the explanation (avoiding constant factors) by using instead of the Indian Sine and Cosine
(Cos(x)=Sin($5400' - x$) $= R\cos(x/R)$) the modern sine and cosine.

1. Geometrically one can see that for any arc p with $0 \leq p \leq 90^o$ and for a small arc h we have: $\sin(p+h) - \sin(p) \approx h \cdot \cos(p)$, and $\cos(p) - \cos(p+h) \approx h \cdot \sin(p)$.

2. We now divide an arc a with $0 \leq a \leq 90^o$ into a large number N of equal small arcs h, thus $a = Nh$. Then we see:

$$\sin(a) = \sin(a) - \sin(0) = \sum_{i=1}^{N}(\sin(ih) - \sin((i-1)h) \approx \sum_{i=1}^{N} h \cdot \cos(ih) \quad (1)$$

$$1 - \cos(a) = \cos(0) - \cos(a) = \sum_{i=1}^{N}(\cos((i-1)h) - \cos(ih) \approx \sum_{i=1}^{N} h \cdot \sin(ih) \quad (2)$$

3. We begin with the approximation $\sin(x) \approx x$.

By substituting this in (2) we obtain $1 - \cos(a) = \sum_{i=1}^{N} h \cdot \sin(ih) = h^2 \sum_{i=1}^{N} i \approx \frac{h^2 N^2}{2} = \frac{a^2}{2}$, therefore $\cos(a) \approx 1 - \frac{a^2}{2}$. Substituting this in (1) invullen we get $\sin(a) \approx \sum_{i=1}^{N} h \cdot \cos(ih) \approx \sum_{i=1}^{N}(h - \frac{h^3 i^2}{2}) = Nh - \frac{h^3}{2} \sum_{i=1}^{N} i^2 \approx Nh - \frac{h^3}{2} \cdot \frac{N^3}{3} = a - \frac{a^3}{2 \cdot 3}$. We substitute this in (2), then we get $1 - \cos(a) \approx \frac{a^2}{2} - \frac{a^4}{2 \cdot 3 \cdot 4}$, and so on.

Source: Anonymous (16th-century?) commentary on the two verses, see Gold & Pingree. References:

B. Datta, A.N. Singh, *History of Hindu Mathematics: A Source Book*, Part 1, Lahore: Motilal Banarsi Das, 1933.

D. Gold, D. Pingree, A Hitherto Unknown Sanskrit Work concerning Mādhava's Derivation of the Power Series for Sine and Cosine, *Historia Scientiarum* **42** (1991), 49-65. See https://dl.ndl.go.jp/pid/11023753/1/28

C.T. Rajagopal, M.S. Rangachari, On an Untapped Source of Medieval Keralese Mathematics, *Archive for History of Exact Sciences* **18** (1978), 89-102.

T. A. Saraswati Amma, *Geometry in Ancient and Medieval India*, Delhi: Motilal Banarsidas, 1979.

B.V. Subbarayappa, K. V. Sharma, *Indian Astronomy: A Source Book*, Bombay: Nehru Centre, 1985 (see p. 64).