The Taylor formula for the sine was discovered in India

Cultural explanation on the history of Taylor series.
Written in 1996, taught until 2018 to first-year mathematics and physics student at Utrecht university.
Translated into English in 2024

Kerala, South-India, ca. 1400

Kerala, South-India

Taylor formula for modern sine

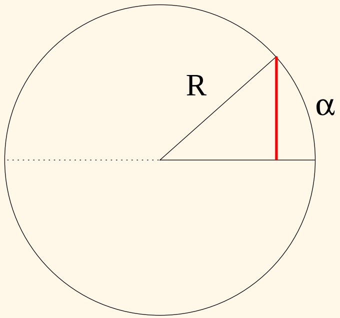
Taylor formula for the modern sine (with x in radians):

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

The Sine was invented in ancient India

Indian Sine (in red) is length of the line segment in a circle with circumference 360×60 arc minutes and with fixed radius R, for Madhava $R = \frac{360 \times 60}{2\pi} \approx 3438$ minutes.

You can show that Madhava's Sine of x arc minutes is $R \sin \frac{x}{R}$ (where $\frac{x}{R}$ is the arc in radians)



Modern Taylor formula for Madhava's Sine

$R \sin \frac{x}{R} =$

$= R(\frac{x}{R} - \frac{1}{3!}(\frac{x}{R})^3 + \frac{1}{5!}(\frac{x}{R})^5 - \frac{1}{7!}(\frac{x}{R})^7 + \frac{1}{9!}(\frac{x}{R})^9 - \frac{1}{11!}(\frac{x}{R})^{11} \ldots)$

$= \alpha - (\frac{x^3}{2 \cdot 3 R^2} - (\frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5 R^4} - (\frac{x^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 R^6} - (\ldots))))$
[expressed in minutes]

School of Madhava (Kerala, ca. 1400). Compute the sine of a given number of minutes of arc

The student has to recite and memorize Sanskrit verses:

nihatya cāpavargeṇa cāpam tattatphalāni ca
haret samūlayugvargais trijyāvargahataiḥ kramāt
cāpam phalāni cādho 'dho nyasyopary upari tyayet
jīvante sa<u>n</u>graho 'syaiva vidvān ityādinā kṛtaḥ

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Having multiplied the arc and the results of each (multiplication) by the square of the arc, one ought to divide by the squares of the even (numbers) together with the (square) roots, multiplied by the square of the Radius, in order. Having put down the arc and the results, one below the other, one ought to subtract going upwards. At the end is the Sine. A summary is made by (the verse) beginning "*vidvān*"

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this means: $x - (x \cdot \frac{x^2}{(2^2+2)R^2} - (x \cdot \frac{x^2}{(2^2+2)R^2} \cdot \frac{x^2}{(4^2+4)R^2} -$
$(x \cdot \frac{x^2}{(2^2+2)R^2} \cdot \frac{x^2}{(4^2+4)R^2} \cdot \frac{x^2}{(6^2+6)R^2} - (\ldots))))$
$= x - \frac{x^3}{2 \cdot 3R^2} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5R^4} - \frac{x^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7R^6} \ldots$

School of Madhava (Kerala, ca. 1400). Compute the sine of a given number of minutes of arc

vidvāṃs tunnabalaḥ kavīśanicayaḥ sarvārthaśīlasthiro
nirviddhāṇganarendraruṅ nigaditeṣv eṣu kramāt pañcasu
ādhastyād guṇitād abhīṣṭadhanuṣaḥ kṛtyā vihṛtyantimasy-
āptam śoddhyam upary upary atha ghanenaivam dhanuṣy antataḥ

School of Madhava (Kerala, ca. 1400). Compute the sine of a given number of minutes of arc

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"The wise ruler whose army has been struck down gathers the best of advisers and remains firm in his conduct in all matters, then he shatters the king whose army has not been destroyed.

School of Madhava (Kerala, ca. 1400). Compute the sine of a given number of minutes of arc

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"The wise ruler whose army has been struck down gathers the best of advisers and remains firm in his conduct in all matters, then he shatters the king whose army has not been destroyed.
When these five (numbers) have been spoken in order, (starting) from the one which is at the bottom multiplied by the square of the given arc, ehwn one has divided (by the square of the arc of 90 degrees), the quotient (of each) is to be subtracted, proceeding upwards, (but the quotient) of the last (is multiplied and divided) by the cube, (the quotient is to be subtracted) from the end of the arc."

What does this mean?

vidvāṃs tunnabalaḥ kavīśanicayaḥ sarvārthaśīlasthiro nirviddhāṇganarendraruṅ
"The wise ruler whose army has been struck down gathers the best of advisers and remains firm in his conduct in all matters, then he shatters the king whose army has not been destroyed.

These are five (encoded) numbers (katapayadi system)
vv=44 tnbl=6033 kvśncy=145061 sv(th)śl(th)r= 7475372 nv(dh)gnrrr=04930222.
They mean $p = \frac{44}{60^2}$, $q = \frac{33}{60} + \frac{06}{60^2}$, $r = 16 + \frac{05}{60} + \frac{41}{60^2}$,

$s = 273 + \frac{57}{60} + \frac{47}{60^2}$, $t = 2220 + \frac{39}{60} + \frac{40}{60^2}$.

The five numbers

$$p = \frac{44}{60^2},\ q = \frac{33}{60} + \frac{06}{60^2},\ r = 16 + \frac{05}{60} + \frac{41}{60^2},$$

$$s = 273 + \frac{57}{60} + \frac{47}{60^2},\ t = 2220 + \frac{39}{60} + \frac{40}{60^2}$$

are rounded values of

$$p = \frac{5400}{11!}(\frac{\pi}{2})^{10},\ q = \frac{5400}{9!}(\frac{\pi}{2})^{8},\ r = \frac{5400}{7!}(\frac{\pi}{2})^{6},$$

$$s = \frac{5400}{5!}(\frac{\pi}{2})^{4},\ t = \frac{5400}{3!}(\frac{\pi}{2})^{2}.$$

The five numbers: wolframalpha.com

$$\frac{5400}{3!}\left(\frac{\pi}{2}\right)^2 = 2220.660990\ldots = 2220 + \frac{39}{60} + \frac{39}{3600} + \frac{34}{60^3}\cdots$$
$$\frac{5400}{5!}\left(\frac{\pi}{2}\right)^4 = 273.963069\ldots = 273 + \frac{57}{60} + \frac{47}{3600} + \frac{3}{60^3}\cdots$$
$$\frac{5400}{7!}\left(\frac{\pi}{2}\right)^6 = 16.094685\ldots = 16 + \frac{5}{60} + \frac{40}{3600} + \frac{52}{60^3}\cdots$$
$$\frac{5400}{9!}\left(\frac{\pi}{2}\right)^8 = 0.551556\ldots = \frac{33}{60} + \frac{5}{3600} + \frac{36}{60^3}\cdots$$
$$\frac{5400}{11!}\left(\frac{\pi}{2}\right)^{10} = 0.012372\ldots = \frac{44}{3600} + \frac{32}{60^3}\cdots$$
$$\frac{5400}{13!}\left(\frac{\pi}{2}\right)^{12} = 0.000196\ldots = \frac{42}{60^3}\cdots$$

What should the student do, after these five numbers have been introduced?

When these five (numbers) have been spoken in order, (starting) from the one which is at the bottom multiplied by the square of the given arc, when one has divided (by the square of the arc of 90 degrees, i.e. the square of 5400), the quotient (of each) is to be subtracted, proceeding upwards, (but the quotient) of the last (is multiplied and divided) by the cube, (the quotient is to be subtracted) from the end of the arc."

the student has to compute:

$$x-(\frac{x}{5400})^3\{t-(\frac{x}{5400})^2\{s-(\frac{x}{5400})^2\{r-(\frac{x}{5400})^2\{q-(\frac{x}{5400})^2p\}\}\}\}.$$

A worked example

We use $\sin(30^o) = \frac{1}{2}$ as worked example
30^o is 1800 minutes so $\alpha = 1800$. Madhava's Sine of 1800 minutes is exactly $R/2$, with $R = (360 \cdot 60)/2\pi \approx 3437.746771$ so $R/2 \approx 1718.87339$ is the value that we should approximately find.

First approximation: $\alpha = 1800$.
Second approximation
$\alpha - (\frac{\alpha}{5400})^3 t$
$= 1800 - (\frac{1}{3})^3 \cdot (2220 + \frac{39}{60} + \frac{40}{60^2}) = 1717.7532922\ldots$
Third approximation
$\alpha - (\frac{\alpha}{5400})^3 \{t - (\frac{\alpha}{5400})^2 s\} =$
$1800 - (\frac{1}{3})^3 \cdot ((2220 + \frac{39}{60} + \frac{40}{60^2}) - (\frac{1}{3})^2 (273 + \frac{57}{60} + \frac{47}{60^2})) =$
$1718.8807122\ldots$ etc.

Madhava's method is correct!

$$R\sin\frac{x}{R} = \alpha - \frac{x^3}{3!R^2} + \frac{x^5}{5!R^4} - \frac{x^7}{7!R^6} + \frac{x^9}{9!R^8} - \frac{x^{11}}{11!R^{10}} =$$

$$x - (\tfrac{x}{5400})^3\{t - (\tfrac{x}{5400})^2\{s - (\tfrac{x}{5400})^2\{r - (\tfrac{x}{5400})^2\{q - (\tfrac{x}{5400})^2 p\}\}\}\}$$

where

$R = \frac{360\times 60}{2\pi} = \frac{5400}{\pi/2}, t = \frac{5400}{3!}(\frac{\pi}{2})^2, \quad s = \frac{5400}{5!}(\frac{\pi}{2})^4, \quad r = \frac{5400}{7!}(\frac{\pi}{2})^6, \quad q = \frac{5400}{9!}(\frac{\pi}{2})^8, \quad p = \frac{5400}{11!}(\frac{\pi}{2})^{10}.$

(Madhava's numbers)

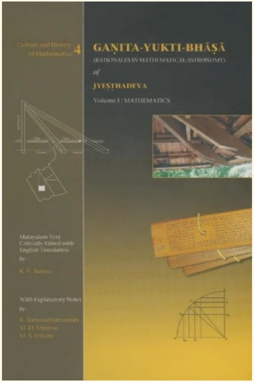
How did Madhava know - could he explain his reasons?

Yes! See the handout on the webpage of the calculus course on the internet.

(see now www.jphogendijk.nl/mampad.html for English translation)

We now have Yuktibhasa of Jyeṣṭhadeva (ca. 1530) in Malayalam

Published: Gaṇita-Yukti-Bhāṣā: Rationales in Mathematical Astronomy,
Springer 2008,
by K.V. Sarma with commentaries by other scholars
vol. 1 Mathematics,
vol. 2 Astronomy,
more than 1000 pp.



Yuktibhasa of Jyeṣṭhadeva (ca. 1530) is easier accessible (K.V. Sarma p. 102)

"the following is the procedure to be adopted. The required arc is the first result. When this is squared, halved and divided by the radius, the second result is got. Keep this second result separately. Now multiply the second result also by the arc and divide by 3 and also by radius. Place the result got below the first result. Then multiply this also by the arc and divide by four and the radius. Keep the result below the second result. In this manner, derive successive results by multiplying the previous result by the arc, and dividing by corresponding successive numbers 1, 2, 3 etc. and by radius."

First result x

Second result $\frac{x^2}{2R}$

Third result $\frac{x^3}{2 \cdot 3 R^2}$

Fourth result $\frac{x^4}{2 \cdot 3 \cdot 4 R^3}$

n-th result $\frac{x^n}{n! R^{n-1}}$

Yuktibhasa of Jyeṣṭhadeva (ca. 1530) is easier accessible (K.V. Sarma p. 102)

"Now, place below the first result the odd results, viz., the third, the fifth etc., and place below the second result the even results, viz., the fourth, sixth etc. Then subtract successively the bottom result from the one above it, the remainder from the one still above it. Ultimately, in the first column, the resultant first result will be left and in the second column the resultant second result will be left. These will be the required Rsine and Rversine (=R-Rcosine)"

First result x	Second result $\frac{x^2}{2R}$
Third result $\frac{x^3}{2\cdot 3R^2}$	Fourth result $\frac{x^4}{2\cdot 3\cdot 4R^3}$
Fifth result $\frac{x^5}{5!R^4}$	6-th result $\frac{x^6}{6!R^5}$
...	...

Yuktibhasa of Jyeṣṭhadeva, meaning of the quoted text

We get

$$R\sin\frac{x}{R} = x - (\frac{x^3}{2\cdot 3R^2} - (\frac{x^5}{2\cdot 3\cdot 4\cdot 5R^4} - (\ldots))))$$

$$R - R\cos\frac{x}{R} = \frac{x^2}{2R} - (\frac{x^4}{2\cdot 3\cdot 4R^3} - ((\frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6R^5} - (\ldots))))$$

Yuktibhasa of Jyeṣṭhadeva , Part of Malayalam text for series for sine and cosine (K.V. Sarma p. 454)

5. v. പഠിതജ്യാക്കൾ ഇല്ലാതെ സൂക്ഷ്മ ജ്യാശരാനയനം

ഇങ്ങനെ ഫലങ്ങളെല്ലാം ജ്യായോഗത്തിങ്കന്ന് ഉണ്ടാക്കേണ്ടൂ എന്നിരിക്കുന്നേടത്തു ചാപയോഗത്തിങ്കന്ന് ഉണ്ടാക്കയാൽ സംസ്കാരഫലങ്ങളെല്ലാം വാസ്തവഫലത്തിങ്കന്ന് ഏറ ഉണ്ടായിരിക്കും. എന്നിട്ടു മീത്തെ മീത്തെ സംസ്കാരഫലം നടേത്തെ നടേത്തെ സംസ്കാരഫലത്തിങ്കന്നു കളയേണം, എന്നാലിവണ്ണം വേണ്ടൂ[63] ഇവിടുത്തെ ക്രിയാക്രമം. ഇഷ്ടചാപം നടേത്തെ രാശിയാകുന്നത്. ഇതിനെ വർഗ്ഗിച്ച് അർദ്ധിച്ച് ത്രിജ്യകൊണ്ടു ഹരിച്ചതു രണ്ടാം രാശിയാകുന്നത്. രണ്ടാം രാശിയെ വേറെ ഒരിടത്തു വെപ്പൂ. പിന്നെ ഇതിനേയും ചാപംകൊണ്ടു ഗുണിച്ച് മൂന്നിലും ത്രിജ്യകൊണ്ടും ഹരിപ്പൂ. ഈ ഫലത്തെ[64] പ്രഥമഫലത്തിന്റെ കീഴെ വെയ്പ്പ്. പിന്നെ ഇതിനേയും ചാപംകൊണ്ടു

Yuktibhasa of Jyeṣṭhadeva , Part of Malayalam text for series for sine and cosine (K.V. Sarma p. 454)

ക്രിയാക്രമം. ഇഷ്ടചാപം നടേത്തെ രാശിയാകുന്നത്. ഇതിനെ വർഗ്ഗിച്ച് അർദ്ധിച്ച് ത്രിജ്യകൊണ്ടു ഹരിച്ചതു രണ്ടാം രാശിയാകുന്നത്. രണ്ടാം രാശിയെ വേറെ ഒരിടത്തു വെപ്പൂ. പിന്നെ ഇതിനേയും ചാപംകൊണ്ടു ഗുണിച്ച് മൂന്നിലും ത്രിജ്യകൊണ്ടും ഹരിപ്പൂ. ഈ ഫലത്തെ[64] പ്രഥമഫലത്തിന്റെ കീഴെ വെയ്പ്പ്. പിന്നെ ഇതിനേയും ചാപംകൊണ്ടു ഗുണിച്ച് നാലിലും ത്രിജ്യകൊണ്ടും ഹരിപ്പൂ. ഫലം ദ്വിതീയഫലത്തിന്റെ കീഴെ വെപ്പൂ. ഇങ്ങനെ അതതു ഫലത്തിങ്കന്നു ചാപംകൊണ്ടു ഗുണിച്ച് വ്യാസാർദ്ധംകൊണ്ടും ഒന്ന് രണ്ട് തുടങ്ങിയവറ്റിൽ മീത്തെ മീത്തെ വറ്റെക്കൊണ്ടും ഹരിച്ചാൽ[65] മീത്തെ മീത്തെ ഫലങ്ങളുണ്ടാകും. ഇവിടെ മൂന്നാമത് അഞ്ചാമത് എന്നു തുടങ്ങിയുള്ള ഓജഫലങ്ങൾ പ്രഥമരാശിയുടെ പങ്ക്തിയിൽ കീഴെ കീഴെ വെപ്പൂ. നാലാമത് ആറാമത് തുടങ്ങിയുള്ള യുഗ്മഫലങ്ങളെ ദ്വിതീയരാശിയുടെ പങ്ക്തിയിൽ കീഴെ കീഴെ വെപ്പൂ. പിന്നെ എല്ലായിലും കീഴേത് അടുത്തു മീത്തേതിൽ കളയൂ. ശിഷ്ടം അടുത്തു മീത്തേതിൽ, ഇങ്ങനെ ഒരു പങ്ക്തിയിൽ പ്രഥമരാശി ശേഷിക്കും. മറ്റേ പങ്ക്തിയിൽ ദ്വിതീയരാശി ശേഷിക്കും. അവ ഇഷ്ടജ്യാശരങ്ങൾ.

Thanks for your attention: it was great to work with you, and we hope to see you again!

See the webpage
www.jphogendijk.nl/mampad.html
for all presentations and handouts.