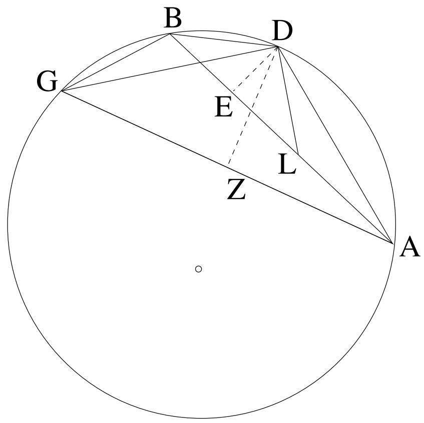
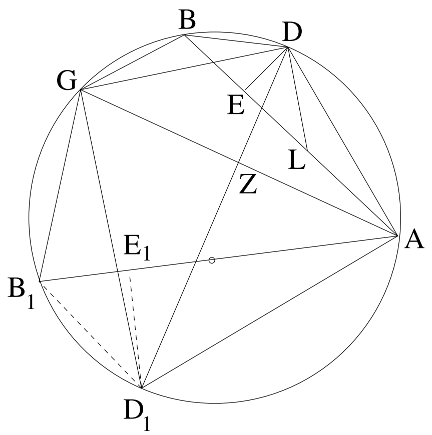
An Iranian proof of the formulas of Heron and Brahmagupta.

We consider a triangle ABG with unequal sides, let $AG > AB > BG$, and with the circumscribed circle. D is the midpoint of arc ABG (see figure). Point L is on AB in such a way that $AL = BG$. We draw DL. We will indicate the area of a triangle PQR as ΔPQR.



1. Show: (i) the two trangles DBG, DLA are congruent
(ii) $\Delta ADG = \Delta ABG + \Delta LDB$
(iii) The two triangles ADG and LDB are similar.
Hint: if P, Q, R are three points on a circle, then angle then the magnitude of angle PQR (in degrees) is half the magnitude of the arc subtended by it (in degrees).

2. Let E be the midpoint of BL and let Z be the midpoint of AG. Note that the two triangles GDZ and BDE are also similar.
Show: (i) $DZ^2 : 2\Delta GDZ = 2\Delta GDZ : GZ^2$
(ii) $(DZ^2 - DE^2) : \Delta ABG = \Delta ABG : (GZ^2 - BE^2)$. [Give a detailed proof!]
(iii) $\Delta ABG = \sqrt{(s(s-a)(s-b)(s-c))}$, where a, b, c are the lengths of the three sides of triangle ABG and $s = \frac{1}{2}(a+b+c)$. Hint: $DZ^2 + AZ^2 = DE^2 + AE^2$.
This method was known in Greek antiquity and was probably discovered by Archimedes (ca. 250 BCE.). This proof was given by the tenth-century Iranian mathematician Abū ᶜAbdallāh al-Shannī.



3. Consider a cyclic quadrilateral $ABGB_1$. Cyciic means that the four angular points of the quadrilateral are on one circle. We will indicate the area of this quadrilateral as We try to generalize the above-mentioned proof for a triangle to this quadrilateral. Let D_1 be the midpointof of arc AB_1G We drop perpendicular D_1E_1 onto AB_1.

Show:

(i) The four triangles BDE, GDZ, D_1GZ and $D_1B_1E_1$ are all similar.
(ii) $AE^2 - B_1E_1^2 : [ABGB_1] = [ABGB_1] : AE_1^2 - BE^2$.
(iii) The area of a cyclic quadrilateral with sides a, b, c, d and half circumference $s = \frac{1}{2}(a+b+c+d)$ is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

This rule was stated (with arguments amounting to a proof) by the Indian mathematician Brahmagupta (ca. 625). The rule was also proved in the tenth century AD by the Iranian mathematician Abū ᶜAbdallāh al-Shannī. The idea of his proof is as in this exercise. The rule is not valid for quadrilaterals which are not cyclic. The proof is in Arabic, a German translation has been published.

Literature: The two Arabic proofs by Abū ᶜAbdallāh al-Shannī.were cited by Abū Rayḥān al-Bīrūnī (973-1048) and printed in India in tract 1, pp. 61-67 of *Rasa'il al-Biruni. Containing Four Tracts*, Hyderabad: Osmania Oriental Publication Bureau, tract no. 1. The German translation is found in pp. 39-42 of H. Suter, Das Buch der Auffindung der Sehnen im Kreise, *Bibliotheca Mathematica, Dritte Folge,* 11 (1910-11), pp. 11-78, These works can be downloaded from jphogendijk.nl/biruni.html work no. 8 (Chords).