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ISTANBUL MANUSCRIPTS OF AL-KHWÂRIZMÎ’S TREATISES

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The existence of Istanbul manuscripts of two astronomical treatises of Al-Khwârizmî (≈ 783–≈ 850) was pointed out for the first time by F. Sezgin in the sixth volume of his “Geschichte des arabischen Schrifttums” [1, p. 143]; enumerating the manuscripts of Al-Khwârizmî’s astronomical works F. Sezgin mentioned the manuscript of “Witty ideas in the actions of Muḥammad ibn Mūsâ al-Khwârizmî: the determination of the azimuth using the astrolabe” and “The construction of hours on the plane sundial by Muhammad ibn Mūsâ al-Khwârizmî” in the codex of the Süleymaniye Libearary of Istanbul, registered under Ayasofya No. 4830, f. 198b-199b and f. 231b-235a. One of the co-authors of this article J. ad-Dabbâgh in 1983 recopied these two treatises, translated them into Russian and published them with commentaries in [2, p. 216-234].

§ 1. Codex Ayasofya 4830

The codex No. 4830 of Ayasofya Library was studied by M. Krause in [3] and Y. Dold-Samplonius in [4]. This codex contains numerous unique manuscripts: the Arabic translation of “Book on the section of a line in a given ratio” of Apollonius (f. 2a-52b), three mathematical treatises of the famous philosopher Al-Kindî “Book on the supreme art”, “Treatise on the determination of thought of numbers” and a treatise on the determination of distances to inaccessible objects (see [5, vol. 2, p. 67]) (f. 53a-86b, 204b-210b), an anonymous treatise on geometric algebra (f. 86b-89a), “The Book of Assumptions” of Aqâtun, published with English translation in [4] (f. 89b-102b), a part of the “Spheres” of Theodosius (f. 102b), an anonymous treatise on the theory of numbers (f. 103a-108b), the treatise of Thâbit ibn Qurra on amicable numbers, translated into Russian by G. P.

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Matvievskaya from another manuscript [6] (see [5, vol. 2, p. 86-87]) (f. 110a-121b), eight mathematical, astronomical and logical treatises of Ahmad ibn as-Surâ (see [5, vol. 2, p. 332-334]) (f. 122b-160a), eight geometrical treatises of Wijjan al-Kûhî (see [5, vol. 2, p. 189-193]) (f. 161a-182a), the treatises we are going to talk about in more detail (f. 183a-199b and 228b-235a), the treatise of Muḥammad aṣ-Ṣabbâḥ on the sundial (see [5, vol. 2, p. 59]) (f. 200a-204a), the above-mentioned treatise of Al-Kindî on the determination of distances to inaccessible objects and treatises of Abû Bakr ibn ʿĀbis, An-Nairizi and Ar-Rayyî on the same question (see [5, vol. 2, p. 117, 257 and 296]) (f. 210b-227b). The manuscript was copied in 1229—1232, but the majority of treatises in it were written in the 9-10th centuries the latest —the treatises of Ibn as-Surâ— in the first half of the twelfth century. Neither Krause nor Dold-Samplonius mentioned the treatises of Al-Khwârizmî; Krause considers the treatises in f. 228b-235a to be an anonymous treatise.

2. Al-Khwârizmî’s treatises in the Codex Ayasofya 4830

The name of Al-Khwârizmî appears only on the two treatises mentioned above. The fact that all the group of treatises that are in f. 182a-199b and f. 228b-235a of this codex belongs to Al-Khwârizmî was established by D. A. King [7] on the basis of closeness of the language and the topics studied in these treatises, to which it should be added such a distinctive feature for the time of Al-Khwârizmî as the value of the “the complete sine” (the radius of the trigonometric circle) equal to 150 which was borrowed from Brahmagupta (a similar thing exists in the Berlin manuscript of Al-Khwârizmî’s treatises MS No. 5793 of the Library Preussischer Kulturbesitz (West Berlin) which contains treatises or parts of treatises of Al-Khwârizmî on the construction of astrolabe, on the use of astrolabes, on the sundial, and on the sine quadrant, from which the name of Al-Khwârizmî was written only in the treatise on the use of astrolabes, published in German translation by J. Frank [8] and in Russian translation by G. P. Matvievskaya [9, p. 255-266]—see [7, p. 23-31] and [2, p. 151-154].

D. A. King established as well the fact that the manuscript of the Al-Birûnî Institute of Oriental Studies of the Academy of Science of the Uzbek SSR (Tashkent) No. 177/3 (f. 148a-151a)[7,
p. 12-13] also was written by Al-Khwārizmī. He found a textual coincidence between this anonymous treatise and the Istanbul manuscript of Al-Khwārizmī on the determination of the qibla. The Tashkent manuscript was copied about 1250.

The manuscript of the Ayasofya Library, as well as the Berlin manuscript, is divided into “chapters” of different lengths. Following J. Frank in this publication of the Berlin manuscript we give a number to each of these chapters.

The first 19 chapters concern the determination of the azimuth of the sunrise (rising amplitude) in a given day and the determination of the azimuth of the Sun at a given instant. As we mentioned above, the text of Al-Khwārizmī’s treatises is placed on f. 183a-199b and on f. 228b-335a of codex No. 4830 of the Ayasofya Library. However, chapters 1 and 2 and the beginning of chapter 3 are placed on f. 228a, while the end of chapter 3 and the following chapters are on f. 183a and the following folios. Chapter 1 is entitled: “The determination of the azimuth of the sunrise at any city according to what Ptolemy had done for the diameter of the celestial sphere to be equal to 120 degrees.” The following chapters are: (2) “The determination of azimuth from altitude for the southern zodiacal signs” (f. 228b), (3) “The determination of the azimuths for the northern zodiacal signs” (f. 228b, 183a), (4) “The determination of the operation with the azimuth, the shadow, and the altitude” (f. 183a-183b), (5) “The determination of altitude from the equal as well as the temporal hours” (f. 183b), (6) “The determination of the azimuth” (f. 183b), (7) “Also the determination of the azimuth” (f. 183b-184a), (8) “The determination of the azimuth for every hour and every city” (f. 184a), (9) “The determination of the azimuth of sunrise of any city” (f. 184a), (10) “Another chapter [on this topic]” (f. 184a), (11) “Another chapter [on this topic]” (f. 184a), (12) “Another chapter [on this topic]” (f. 184a), (13) “The determination of the azimuth for the altitude of given magnitude” (f. 184a-184b), (14) “The procedure of determining the azimuth by another method” (f. 184b), (15) “The procedure of determining the azimuth of the sunrise” (f. 184b), (16) “The description of determining the azimuth from the ascendent” (f. 184b), (17) “The determination of the azimuth of any hour you need for any degree you wish from all zodiacal signs” (f. 184b-185b), (18) “The construction of the hours by the azimuth” (f. 185b-186a), (19) “Also the construction of the altitude for the hours” (f. 196a).
The following six chapters concern, mainly, the determination of the azimuth of the qibla. These are: (20) “The determination of the direction of the qibla in any city you wish” (f. 186a-186b), (21) “The determination of the meridian line” (f. 186b), (22) “The determination of the qibla also for any latitude” (f. 187a), (23) “Another chapter on the determination of the qibla” (f. 187a-187b), (24) “One more short chapter on the determination of the qibla” (f. 187b), (25) “The determination of the azimuth of the qibla from the table” (f. 187b-188a). After the table of chapter 25 follows chapter 26 which contains “A table of shadows and inclinations” (f. 188b).

The chapters 27-31 are devoted to the description of different clock instructions: (27) “The construction of the binkâns, i.e., the troughs, or the tubs for the measurement of hours, equal as well as temporal” (f. 189a-190a), (28) “The construction of the binkân that shoots pebbles” (f. 190a-190b), (29) “The construction of the horary lifting wheel” (f. 191a-192a), (30) “The construction of the conical sundial for the [determination of] hours” (f. 192a), (31) “Operating with the [ritual] sundial [that is called] miknasa” (f. 193a).

Then follows chapter 32 “Geometrical construction of the azimuth of sunrise you wish for [any] zodiacal sign in the latitude you wish” (f. 193b). The following three chapters are devoted to the determination of latitudes and longitudes of cities: these are (33) “The determination of the latitude of a city” (f. 193b-194a), (34) “The geometrical determination of the latitude of a city”, (f. 194a) (35) “Cities situated in all climates” (a table of longitudes and latitudes of 163 cities) (f. 194b-196b).

The next three chapters are again devoted to the description of astronomical and mathematical instruments: (36) “Operating with a horary quadrant” (f. 196b-197a), (37) “Pair of compasses for a camp” (f. 197a-197b), (38) “A board for determining the [lunar] eclipse without the appropriate instruments” (f. 198a). After that follows (39) “Witty ideas in the construction of Muḥammad ibn Mūsā al-Khwārizmī: the determination of the azimuth using the astrolabe” (f. 198b-199a) and (40) “The determination of the latitudes of climates” (f. 199a-199b).

The last 19 chapters (the beginning of the first of them is not there) are devoted to the measurement of time. These are: (41) The
The determination of "times of prayers from the table" (f. 229a-229b), (42) The determination of "the midday altitude in any day for any latitude" (f. 229b), (43) The determination of "the surplus of the day" from the table (f. 229b), (44) Determining "when [the Sun] is in the zenith" (f. 229b), (45) Determining "when [the Sun] is for the second time in the zenith" (f. 229b), (46) The determination of "the shadow of the hour you wish with the aid of a gnomon" (f. 229b-230a), (47) "The determination of shadows for the times [of prayers]" (f. 230a), (48) "[The determination of the midday] altitude in any day and any latitude" (f. 230a), (49) "More about the altitude in any day and the time of 'asr and zuhr" (f. 230b), (50) "The midday altitude for the latitude 34 degrees from the verified inclination" (a table) (f. 230b), (51) "The altitude for the latitude 33 degrees" (a table) (f. 231a), (52) "The construction of the hours in the plane of the sundial by Muhammad ibn Mūsâ al-Khwārizmī" (f. 231b), (53) "This sundial—a sundial for Baghdad [in which the shadows and the azimuths appear] every one sixth [of every hour for the beginnings of all zodiacal signs]\) (a table) (f. 231b-232b), (54) Tables of the first and the last azimuths of the time of 'asr for the beginnings of Capricorn, Aries, and Cancer and the first and the last shadows of 'asr (f. 232b), (55) "The sundial for the latitudes 15, 18, 21, 24, 27, 30, 33, 36, 38, 40" (a table of the altitudes, azimuths, and shadows for the hours 1, 2, 3, 4, 5, 6 for the signs Cancer and Capricorn) (f. 233a-234a), (56) "The sundial for Surra man Ra‘a (Samarra) [the latitude 34]" (a table of the azimuths and shadows every half an hour for the same zodiacal signs) (f. 234a), (58) "The sundial for the place which has no latitude, i.e., for the equator" (f. 234a-234b), (59) "The construction of the sundial by geometric method for any latitude" (f. 234b-235a).

Chapters 39-40 and 52-59 were published in [2]. As we see the majority of chapters are devoted to two problems: The determination of the azimuth of some directions and the determination of time.

The Tashkent manuscript is entitled: "The determination of the direction of the azimuth of the qibla in any city that you wish" (Ma‘rifat taqwîm samt al-qibla fi‘ al-ayyî balad shî‘a). This manuscript consists of five chapters, the last four of which coincide with chapters 22-25 of the Istanbul manuscript.
§ 3. Treatise on the determination of the qibla

First of all let us consider some chapters of these treatises concerning the determination of azimuths. Azimuth (samt, literally "direction") is the angle between the direction of a certain point on the horizontal plane and some initial direction (the origin of the word "azimuth" is the Arabic word as-sumūt —the plural of as-sama). The azimuth of a celestial body or a point of the celestial sphere is the azimuth of the orthogonal projection of this point on the plane of the horizon, the azimuth of a celestial body with its altitude — the angle between the direction of this body and the plane of the horizon — form one of the spherical coordinates on the celestial sphere, in which the circle of the horizon stands for the equator and the zenith and nadir for the poles. The qibla is the direction of Mecca—the main holy place of the Muslims. The knowledge of the direction of the qibla was necessary in the building of mosques since Muslim worshippers should face Mecca when they pray; therefore the determination of the azimuth of the qibla was one of the important practical problems in Muslim countries. The azimuth of the qibla for a given city means the angle between the direction to Mecca and the direction of the meridian in this city. The azimuth of sunrise or rising (ortive amplitude, si'at al-mashriq) means the arc of the horizon circle from the east point to the point of sunrise in the given day. In all cases, the problems of determining the azimuths can be solved by means of spherical trigonometry.

In chapter 20 of the Istanbul manuscript and chapter 1 of the Tashkent manuscript approximate methods of determining the azimuth of the qibla are given. At first two cases in which the latitudes or the longitudes of the given city X and Mecca M are equal are considered. In these cases Al-Khwârizmî takes as the direction of the qibla respectively the direction of east-west and north-south. The second of these rules is exact but the first one is rough and approximate. Then an approximate construction of the zenith of Mecca on the celestial sphere of the given city is suggested: find the difference $\lambda_X - \lambda_M$ of these cities and subtract it from half of the day arc for the day corresponding to 4° of Gemini, i.e., approximately to the day in which the sun in Mecca is in the zenith. Take the difference you get as the altitude by which you determine the azimuth according to what was described in the previous chapters. In the Tashkent manusc-
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script an other approximate representation of the city $M$ is drawn by the latitudes and longitudes of the cities $X$ and $M$ in a circle with center at $X$ where the direction of the line $XM$ is taken as the direction of the qibla. We give here the trigonometric solution of this problem in chapter 22 of the Istanbul manuscript and in chapter 2 of the Tashkent manuscript. Here is the Arabic passage:

عمل القبلة أيضا في كل عرض

تأخذ ما بين عرض مكة وعرض البلد الذي تريد. فاجعله جيبا ثم اضربه في مثله، ثم خذ فضل ما بين طول مكة وطول مدينتك، فاجعله جيبا واضربه في جيب تمام عرض مكة، واصفه على قن انسب، فاخرج فسه في القدر المشروط في مثله الذي حفظه. فاصل فده جذره فاحظه، ثم اضرب القدر المشروط في قن، واقسمه على الجذر. فاصل فسه واحفظه. ثم أدر في المسجد الذي تريد دائرة هندية ثم اعرف خط الزوال. فان كانت مدينتك غربة عن مكة فقد مثل تلك القوس عن خط الزوال الجنوبى الى ناحية الشرق. فحجب انتهت فيه ووضع القبلة من وضع العود الذي في وسط الدائرة. فان كانت مدينتك شرقية عن مكة فقد من ناحية الغرب وافعل كما فعلت اولا.

Its English translation is as follows:

(22) “The determination of the qibla also for any latitude. Take [the difference] between the latitude of Mecca and the latitude of the city in which you are, take its sine, multiply it by itself and keep it in mind. Then take the difference between the longitudes of Mecca and your city, take its sine multiply it by the sine of the complement of the latitude of Mecca and divide this by 150. The thing you get is ‘the corrected sine.’ Multiply it by itself and add it to that product by itself which you kept in mind. Extract the root of the sum and keep it in mind. After that multiply the ‘corrected sine’ by the greatest sine and divide it by the root. Take the arc of the quotient and keep it in mind. Then draw an Indian circle in the mosque that you want [to build] and determine the meridian line. If your city is to the west of Mecca, cut off what is equal to this arc to the south of the meridian line to the east direction. The place where the arc reaches will be the place of the qibla with respect to the gnomon in the middle of the circle. If your city is to the east of Mecca, cut off the arc in the west direction and act in the same way as in the beginning.” (F. 187a).
If we denote Mecca, the given city, and the north pole of the earth, by $M$, $X$, and $N$, respectively; then the azimuth of the *qibla* is the angle $NXM$ in the spherical triangle $NXM$ (fig. 1). “The corrected sine” (al-jaib al-mu‘addal)—the result of the “correction” of the sine line, $\sin (\lambda_X-\lambda_M) = R \sin (\lambda_X-\lambda_M)$, i.e., its product by $\cos \phi_M R$ is “the complete sine”, the radius of the Earth which is taken in the treatise equal to 150; in the Tashkent manuscript the number 150 is replaced by the words *jumlat al-jaib* — “complete sine”. $\cos \phi_M$ is $\cos \phi_M / R$: this multiplier indicates the number that must be multiplied by the linear dimensions of the great circle of the terrestrial globe in order to get the corresponding linear dimensions of the parallel of Mecca. The method given here is based on plane trigonometry, i.e., it is assumed that the part of the terrestrial surface, with the cities $M$ and $X$ is plane. The arc $MY$ of the parallel of Mecca is identified with “the corrected sine”

\[
\sin \eta = (\sin [\lambda_X-\lambda_M] \cos \phi_M) / R
\]

(in its computation the sphericity of the surface of the earth is taken into account). The arc $XY$ is identified with sine line $\sin (\phi_X-\phi_M)$, and the hypotenuse $\sqrt{\sin^2 (\phi_X-\phi_M) + \sin^2 \eta}$ of the right angled triangle with legs $MT$ and $XY$ is taken for the arc $MX$. Further, in the triangle $MXY$, taken as a right triangle with straight sides, the sine of the angle $A$ is found by the formula
SinA = (MY/MX). R. “The Indian circle” found by Indians for determining the meridian line (“midday line”) is a circle drawn on the horizontal plane with vertical gnomon in the centre: the meridian line joins the centre of the circle with the midpoint of its arc joining the points of intersection with the line described by the end of the shadow of the gnomon.

In chapter 25 of the Istanbul manuscript and chapter 5 of the Tashkent manuscript there is a table with 20 columns and 20 rows on each of which the numbers 1 to 20 are marked the numbers 1, 2, ..., 20 above the columns denote the differences of latitudes $\varphi_x - \varphi_M$ (above these numbers the world “latitude” is written), the numbers 1, 2, ..., 20 on the right side of the rows denote the differences of longitudes $\lambda_x - \lambda_M$ (above these numbers from up to down the word “longitude” is written), in the squares at the intersection of columns and rows corresponding to these differences, the azimuth $A$ of the qibla in these differences $\varphi_x - \varphi_M$ and $\lambda_x - \lambda_M$ is indicated. We note that a table very similar to this is given in the manuscript of the library Taymur in Cairo No. 103/2 (f. 101a-102a) under the title “Table of twenty (al-jadwal al-‘ashrîni)” of Abû Ja‘far Muḥammad ibn Mûsâ al-Khwârizmî”, this gives an additional argument for ascribing this treatise to Al-Khwârizmî. There is a similar table in a zîj in the MS No. 9116 of the British Library (London), written in Yemen in the 13th c. [7, p. 15-16].

The determination of the azimuth of the qibla by applying spherical trigonometry to the spherical triangle MXN is carried out in “The book on the use as astrolabe” of Abdu‘r-Rahmân aš-Šûfî (10th c.) [10, p. 297-298] and in “al-Qânûn al-Maṣûdi” of Al-Birûnî [11, p. 425-427].

§ 4. The first treatise on the determination of the rising amplitude

Chapter 1 “The determination of the azimuth of the sunrise in any city” begins as follows. Here is the Arabic passage:

معرفة سعة المشرق في كل بلد ...
Its English translation is as follows:

“If you want to determine this take one fourth of the diameter, this is 15 [degrees], and multiply this by itself. Multiply the degrees of declination taken as a whole by themselves. Then add these and extract the roots of this sum. The thing you get is the azimuth of the sunrise of this city.” (F. 228b). If we denote the celestial equator by $EF$, and the circle of the horizon by $ES$ (fig. 2), then the point of the sunrise is the point $E$ of intersection of these two circles. If the Sun on the given day is at the point $S$ of the day circle which is at a spherical distance $\delta$ from the celestial equator (the arc $\delta$ is called the declination of the Sun), then the azimuth of the sunrise $\theta$ is the hypotenuse of the right angled spherical triangle $ESF$ with right angle $F$; the leg $SF$ of this triangle is equal to the declination $\delta$, and the angle $E$ is equal to the complement of the latitude $\varphi$ of the given city to $90^\circ$. Therefore, by virtue of the spherical sine theorem

$$\sin \theta / R = \sin \delta / \cos \varphi,$$  \hspace{1cm} (1)  

i.e.,

$$\sin \theta = R \left( \sin \delta / \cos \varphi \right).$$  \hspace{1cm} (2)

In the title of the chapter Al-Khwârizmi refers to Ptolemy. The problem of determining the azimuth of the sunrise for the day of the solstice when the declination $\delta$ of the Sun is equal to the maximal declination, i.e., to the angle between the ecliptic and the celestial
equator was solved by Ptolemy in chapter 3 of Book II of the “Almagest” [12, p. 65], where he gives a rule equivalent to the formula (1). Al-Khwârizmî uses here the division of the radius into 60 parts that had been used by Ptolemy (he mentions this in the title of the treatise, since he usually uses the division of the radius into 150 parts): Al-Khwârizmî calls here the 60th parts of the radius “degrees”. In the above mentioned problem Al-Khwârizmî, as well as Ptolemy, finds the azimuth of the sunrise for the day solstice, i.e. for $\delta = \varepsilon$. In this case he considers the triangle ESF as a plane triangle, and the latitude of the city equal to the latitude of Baghdad $\varphi \approx 33^\circ$. Therefore, $90^\circ - \varphi \approx 57^\circ$, $EF \approx SF / \tan 57^\circ \approx 24/1,6 = 15$; that is the reason why Al-Khwârizmî considers this leg equal to 15 “degrees”. Al-Khwârizmî defines the azimuth of the sunrise $\theta$ as the hypotenuse $ES$ of the plane triangle $ESF$, i.e., by the formula $\theta = \sqrt{EF^2 + \delta^2}$.

After that Al-Khwârizmî solves the same problem for each zodiacal sign, i.e. for any month. In the case when the Sun is in the ecliptic with eclipitical longitude $\lambda$, it follows from the spherical sine theorem for the right angled spherical triangle $\gamma SF$ (fig. 3) that

$$\frac{\sin \lambda}{R} = \frac{\sin \delta}{\sin \varepsilon}$$

Al-Khwârizmî approximates the ratio $\sin \delta / \sin \varepsilon$ replacing it by the ratio $\delta / \varepsilon$ and considers $\sin \lambda = R(\delta / \varepsilon)$. In this case he takes
the hypotenuse $ES$, of the triangle $ES_1F_1$ with a leg $S_1F_1$ equal to $\delta$ as the azimuth of the sunrise (fig. 4). Therefore

$$\theta_1 = ES_1 = ES(\delta/e) \approx \theta \cdot \sin (\lambda/R),$$

here Al-khwârizmî assumes $R = 150$.

§ 5. *Treatise in the determination of the azimuth of the Sun*

In chapter 2 Al-Khwârizmî finds the azimuth from the altitude of the sun, and in solving this problem he uses not plane but spherical trigonometry. We produce the rule of Al-Khwârizmî for the south zodical signs. Here is the Arabic passage:

معرفة السمت من قبل الارتفاع للبروج الشمالي

إذا اردت ذلك فخذ الارتفاع فاقله من تسعين، فما يبي فاجعله جيب، فإكان فهو جيب تمام الارتفاع. فقسم عليه جملة الجيب، فإخرج فهو الأصل الأول. ثم اقسم جيب عرض البلد الذي تريد على جملة تمامه فإخرج فاضره في جيب الارتفاع الذي تعمل له السمت. فإخرج فهو الأصل الثاني. فرز عليه جيب سعة المشرق للبرج الذي تعمل له أو الدرجة، فإم بلغ فاضره في الاصل الأول. فما بلغ فهو السمت.

Its English translation is as follows:

"If you want [to find] this, take the altitude and subtract from ninety, then turn the remaining into sine. The thing you get is the sine of the complement of the altitude. Divide the complete sine by it, the thing you get is the 'first base.' Then divide the sine of the
latitude of the city you are in by the sine of the complement [of it]. Multiply the thing you get by the sine of the altitude for which you find the azimuth, the thing you get is the ‘second base.’ Add to this the sine of the azimuth of rising of the zodiacal sign or the degree. Multiply the sum by the ‘first base’, the thing you get is the sine of the complement of the azimuth.” (F. 182b).

For solving the same problem for the north zodiacal signs Al-Khwārizmī recommends in chapter 3 to find the same “first and second bases” for the cases of south zodical signs. Here is the Arabic passage:

معرفة السمت للبرج النهائية

والآن نعمل الأصل الأول والثاني على ما أريدت. ثم خذ الأصل الثاني فانقصه من جيب سعة المشرق. فان كان أكثر من جيب سعة المشرق، فالجبر سعة المشرق منه. فا بقي بعد ذلك فاضبه في الأصل الأول، فا يبلغ قوته. فا خرج فهو السمت. فإذا أردت أن تعلم جهة هذا السمت شمالي هو أم جنوني فانظر (فان كان الأصل الثاني أكبر من جيب سعة المشرق فالسمت جنوني)، وإن كان الأصل الثاني مثل جيب سعة المشرق سوا فالسمت على خط المشرق والمغرب، وإن كان الأصل الثاني أقل من جيب سعة المشرق فالسمت شمالي.

In English the passage reads as follows (leaving out the first sentence):

After that — Al-Khwārizmī writes — “take the ‘second base’ and subtract it from the sine of azimuth of sunrise; if the ‘second base’ is greater than the azimuth. Then subtract the sine of azimuth. Multiple the difference by the ‘first base’ and take the arc of the product, you get the [south] azimuth. . . . If the ‘second base’ is exactly equal to the sine of the azimuth of the sunrise, then the azimuth is on the east-west line. If the ‘second base’ is smaller than the azimuth of the sunrise, then the azimuth is northern.” (F. 182b-183a).

If we consider the spherical triangle of the celestial sphere the vertices of which are the Sun $S$ in one of the south zodiacal signs, i.e., on the arc of the ecliptic from the beginning of Libra up to the beginning of Aries, the north pole of the world $P$ and the Zenith $Z$ (fig. 5), then in this triangle $ZP = 90° - \varphi$, $ZS = 90° - \lambda$, $PS = 90° + \delta$ and by virtue of the spherical cosine theorem
\[-\sin \delta = \sin h \sin \varphi + \cosh \cos \varphi \cos A\]

and

\[|\cos A| = (\sin h \sin \varphi + \sin \delta) / \cosh \cos \varphi.\]

Al-Khwârizmî finds \(\sin(90^\circ - h) = \cosh\), his “first base” is
\[R / \cosh = 1 / \cos h,\] his “second base” is
\[(\sin \varphi / \cosh) \sin h = \tan \varphi \sin h.\]  
In finding the azimuth of the sunrise \(\theta\) above, he found that
\[\sin \delta / \cosh = \sin \theta,\] therefore the rule of Al-Khwârizmî is equivalent to the rule we mentioned above which follows from the spherical cosine theorem.

In the case when the Sun is in one of the north zodiacal signs, i.e. on the arc of the ecliptic from the beginning of Aries up to the beginning of Libra (fig. 6), \(\angle P = 90^\circ - \varphi, \angle S = 90^\circ - h, PS = 90^\circ - \delta\) and by virtue of the spherical cosine theorem of cosines for the triangle \(\angle P S\)

\[\sin \delta = \sin h \sin \varphi + \cosh \cos \varphi \cos A\]

and

\[|\cos A| = (\sin \delta - \sin h \sin \varphi) / (\cosh \cos \varphi)\]

which is equivalent of the rule of Al-Khwârizmî for this case. The case \(h = 0\) corresponds to the case when the “second base” \(\tan \varphi \sin h\)
together with \( \sin \theta \) coincides with \( \sin \theta \), the case \( h > 0 \) — to the case when \( \tan \phi \sin h + \sin \theta > \sin \theta \) and the case \( h < 0 \) — to the case when \( \tan \phi \sin h + \sin \theta < \sin \theta \). In these three cases the Sun is, respectively, on the horizon, above the horizon and below the horizon.

A rule equivalent to the spherical cosine theorem for the triangle \( \angle PS \) was known to the Indian astronomer of the 5th c. Varāhamihira [13, p. 200-201] and apparently Al-Khwārizmī came to know this rule from an Indian source. Apparently Thābit ibn Qurra (9th c.) knew this rule through Al-Khwārizmī and produced it in his “Book on the clock instruments called sundial” [6, p. 252-266] and from him Regiomontanus came to know it through the “Ṣâbian zij” of Al-Battânî. Since the Europeans knew this theorem of spherical trigonometry through Al-Battânî, this theorem is often called in Europe “Albategnius’ theorem”.

The above given rule of Al-Khwārizmī for determining the azimuth of the Sun from the altitude of the Sun, the latitude of the city and the azimuth of the sunrise is only equivalent to the same theorem of spherical trigonometry, that was equivalent to the rule of Varāhamihira in which the altitude of the Sun is determined from the day arc, the midday altitude of the Sun in the given day and the hour angle at the moment for which the altitude of the Sun is deter-
minded; however Al-Khwārizmī gives in chapter 4 the rule of Varahamihira itself. Here is the Arabic passage:

معرفة عمل السمتم والظل والارتفاع

...نستخرج أولًا نصف قوس النهار لذللك اليوم، وجب نصف قوس النهار المنكس، وجب ارتفاع نصف نهار درجة الشمس، وجب تمام الارتفاع لنصف درجة الشمس لذللك اليوم، وجب سعة مشرق درجة الشمس لذللك اليوم. فاذدا عرفت هذه كله فأنظر الساعة التي تريد أن تعرف سميتها وارتفاعها وظلها. فإن كانت مستوى فاضتها في يه، وكان كانت معوجة فاضتها في أجزاء ساعات ذلك اليوم. فما اجتمع قالفه من جيب نصف قوس النهار المنكس كما بقي فهم الخصبة، فاحفظه. ثم اضرب هذه الخصة في جيب ارتفاع نصف نهار درجة الشمس، فما بلغ فاقمه على جيب قوس النهار المنكس. فما خرج فهو جيب ارتفاع تلك الساعة. فقوسه، فما كان فهو الارتفاع.

Its English translation is as follows:

"First of all find half of the day arc, the sine of the midday altitude of the degree of the Sun, the sine of the complement of the midday altitude of the degree of the Sun on this day and the sine of the azimuth of the rising of the degree of the Sun in this day. If you had found all these, look at the hour for which you want to find its azimuth, its altitude and its shadow. If this hour is equal, multiply it by 15, if it is temporal, multiply it by the parts of hours of this day. Subtract the versed sine of the product from the versed sine of half the day arc. The thing you get is 'the argument', keep it in mind. Then multiply this 'argument' by the sine of the midday altitude of the degree of the Sun. Divide the product by the versed sine of half the day arc. The thing you get as quotient is the sine of the altitude of this hour. Take its arc you get the altitude." (F. 183a).

"Equal hours" — astronomical hours that are equal to $1/24$ of the day, the product of these hours by 15 is the hour angle in degrees, since the celestial sphere rotates by $15^\circ$ for each "equal hour". "Temporal hours" are equal to $1/12$ of daylight or night. In the medieval East all the civil life and the Muslim prayers were determined in "temporal hours". The word "part" in the expression "parts of the hours" means the degrees of the celestial equator, "the parts of the hour" — the number of degrees of the celestial equator by which
it rotates for 1 “temporal hour”. The product of the number of “temporal hours” by “the parts of the hour” is also equal to the hour angle in degrees. “The versed sine” (in Europe sinus-versus) for angles $\alpha < 90^\circ$ is the complement of the cosine line $\cos \alpha$ to $R$, and for $\alpha > 90^\circ$ — the sum $R + \cos \alpha$ to $2R$, “the day arc” — the arc of day circle of the sun above the horizon; it is equal to the number of degrees of the celestial equator corresponding to the day-light. Since the hour angle is counted from the meridian, therefore half of the day arc is equal to the maximum value of the hour angle. The rule of Varâhamihira himself can be written in our notation in the form

$$\sinh = (\sin \text{versa} - \sin \text{vers t}) \cos \phi \cos \delta.$$

From it we obtain the rule of Al-Khwârizmî, if we put

$$\sin \text{versa} = 1 + \tan \phi \tan \delta.$$

but the tangent function is not mentioned in this rule.

These rules are equivalent to the spherical cosine theorem for the same triangle $ZPS$; and for the angle $t \leq ZPS$ (not for the angle $A \leq PS$).

§ 6. The second treatise on the determination of the rising amplitude

Al-Khwârizmî applies the rules equivalent to rules of spherical trigonometry for solving the problem of determining the azimuth of the sunrises in chapter 32. Here is the Arabic passage:

عمل سعة أي مشرق شنط من البروج

في أي عرض شنت بالهدنة

خط دائرة عليها أجج وقسم قوس أصص جزأ وعدد من نقطة عليه قوس الدائرة بقدر

عرض البلد وتخرج منه خط إلى المركز وهو خط ز. ثم عدد من نقطة أ على قوس الدائرة بقدر

ميل البرج الذي نريد أن تعمل له، وتخرج منه خط يوازي خط آه، وهو خط ح ج، ينتهي إلى

خط د. الذي هو قطر. ونظر أن يقطع من خط ز، فكانه قطعه على نقطة م. ثم نأخذ منه

وتثبت رأس البرك في نقطة المركز وهي نقطة 8 ونظر الرأس الآخر أن يقطع من خط د.،

فكانه قطعه على نقطة 8. فنخرج من نقطة ل، عموداً ينتهي إلى قوس آه وهو عمود ل، فحيث

انتهى فقد منه الإ نسبة 8، فما كان فهو سعة المشرق للبرج الذي اردته.
Its English translation is as follows:

*The geometric construction of any azimuth of sunrise you wish for any zodiacal sign in any latitude.* “Draw a circle ABCD (fig. 7) and divide its arc AD into 90° parts. From the point D on the circle lay off the arc DG having the magnitude of the latitude of [your] city and draw a line from this [point] G to the centre [of the circle], i.e., the line EG. Then from the point A on the circle lay off the arc AH having the magnitude of the declination of that zodiacal sign for which you want [to make] the construction. From [the end of] this [arc] draw a line parallel to the line AE. It is the line HF, that reaches the line BD which is a diameter. Then look where it cuts the line EG; let it be the point M. Then take a pair of compasses with a spread ME, put its leg in the centre point, i.e. in the point E, and find the place where the other leg cuts the line EG. Let this be the point N. From the point N raise a perpendicular reaching the arc AG, it is the perpendicular ON. From [the point O] where [the perpendicular] reaches count out [the arc] to the point A. The thing you get is the azimuth of the sunrise for the zodiacal sign that you wanted.” (F. 193b).

![Fig. 7](image-url)
ISTANBUL MANUSCRIPTS OF AL-KHWÂRIZMÎ'S TREATISES

Since the arc $DG$ is equal to $\varphi$, while the arc $AH$ is equal to $\delta$, therefore the line $EF = \sin \delta$ and the line $EM = R (\sin \delta / \cos \varphi)$ and if we denote the line $EN$ which is equal to it by $\sin \theta$, then by formula (2) that we get by virtue of the spherical sine theorem for the right angled spherical triangle $ESP$, the arc $AO$ is equal to the azimuth of the sunrise $\theta$.

In chapter 33 the declination of the Sun $\delta$ is determined from the azimuth of the sunrise $\theta$ and half of the day arc $\alpha$ of the Sun by a rule equivalent to the formula

$$\cos \delta = R (\cos \theta / \sin \alpha)$$

which is equivalent to “the spherical Pythagorean theorem”

$$\cos \theta = \cos \delta \cdot \sin \alpha$$

for the spherical triangle $EFS$ (fig. 4) with a leg $EF$ equal to $90^\circ - \alpha$, a second leg equal to the declination $\delta$ and a hypotenuse equal to the azimuth of the sunrise $\theta$.

The chapters about the determination of azimuths show that Al-Khwârizmî had many rules equivalent to the theorems of spherical trigonometry and it is very likely that some of these rules are intermediate links between the rules of Indian astronomers and the rules of later Islamic scientists that had played an essential role in the development of spherical trigonometry. The fact that in some chapters Al-Khwârizmî replaced the rules of spherical trigonometry by the corresponding rules of plane trigonometry is probably an evidence that these chapters were written before those ones in which spherical trigonometry is applied.

§ 7. Geographical tables

The table of geographical coordinates of cities that Al-Khwârizmî gave after the treatise on the determination of the azimuth of the sunrise by geometry contains 163 names—3 cities to the south of the equator, 22 cities in the first climate, 14 in the second one, 18 in the third, 48 in the fourth, 43 in the fifth, 11 in the sixth and 4 in the seventh climate. The coordinates indicated here, in general, coincide with the longitudes and latitudes of cities in the well-known Al-Khwârizmî’s “Book of the picture of the Earth” [14] and they are given here, apparently, in connection with the problem of the determination of the azimuth of qibla (the majority of the cities given here relates to the countries of Baghdad’s caliphate).
Probably the section on the determination of the latitudes of climates with the table of the midday shadows in the second half of Al-Khwārizmī's treatise on the determination of the azimuth by the astrolabe (f. 199a-b) [2, p. 217-219]) should be an introduction for this table. The main part of this treatise can be considered as an addition to Al-Khwārizmī's treatise on the use of the astrolabe [8, 9] where the problem of determining the azimuth is considered too.

§ 8. Treatise about drawing a circle about a camp

Chapter 37 "Pair of compasses for a camp" (Birkār al-hilla) is quite interesting. Here is the Arabic passage:

بِرْكَارِ الْحَلْة

إذا أردت أن تدير دائرة لا تخرج في سعة البركاء... فَأَنْظِرُ إلى الموضع الذي تريد أن يكون مركزاً للدائرة، فَتَفَكِّرْ في قِناةٍ على دائِبٍ مَرَاهُ... أن كان ذلك ناهراً، فإن كان بلا نسيباً على رأس القناة شُعِينًا ضاففًين. ثُمَّ تَنْتَخِذ عِضاَدة كَبِيرةٍ بَكَّريَّين كَبيريَّين وَبَنْطَين وأُسْعَين مَسْوَطين. ثُمَّ نَظِرُ إلى الموضع الذي تريد أن يكون مُحيطَ الدائرة، فَنَنْتَخِذ عِضَادة ثُمنَتِرة قَائِمَةٍ عَلَى كُرِسٍ وَتَنَلَّقَ عِضاَدة عَلَى رأس المُسْطَرة عَنْ عَمُودٍ يَنْتَخِذ حَيْثُ يَنْتَخِذِ المَرْي وَتَنِزُفُهُ حَيْثُ يُنْتَخِذِ الْعَمَلَ الَّذِي يُنْتَخِذ فَوْقَ القِناة وَهُوَ مَرْكَزُ الدائرة. فَهُدَّى خُرْجَ فَمِنْ أَيْضَارَا خُطِّينٍ نَدَاذِنَ نُنَدَاذِنَ عِضَادَة الْعَمَل.

وَبَيْضَارَارَ نِعْمَ اَنَا ان قَدْنَا عِضاَدة مِنْ مَوْضُعَا نِحْوَ المَرْكَز أو أَخْرَجْناها عَنْ نِسْبِها وَلَمْ نَشِبُ المَرْي لَمْ يَنْتَخِذ عِضاَدة الْعَمَل. فَإِذَا رَأَيْنَا الْعَمَل فَمِنْ مَوْضُعَا فَناََّ نَدَرَى المَرْكَز وَلَا لَا مَرْكَزٌ عَلَيْهَا كَأَ وَهُوَ وَلَنْ نَثْبِقَ ذَلِكَ بَلْ أَفْشَلَهُ. فَإِن لم يَجِبِ الْعَمَل مِنْ ثَقِيْلِ العِضاَدة وَالْبَصْرَة فِي مُحيَّطِ الدائرة لِأَنْ غَابَ الْعَمَل عَنْ ثَقِيْلِ العِضاَدة وَعَنْ الْبَصْرَة وَلَمْ نَتَقْدِمَ عَنْ مَوْضُعَا وَلَا تَأْخَذَهُ وَالْعَمَل فَأَنَا أَنَّ ذَلِكَ لَا نَخَافُهَا عَنْ سَطْحِهَا الأَوْلِ أوِ لَا نَخَافُهَا عَنْ مَوْضُعَا الْعَمَل. فَيَرَى رَأْبَا الْعَمَلَ وَكَذَلِكَ تَأْخَذُ كَلَّ قُدْرَنَ بَعْدِيَنَا أو مَا أَرَدْنا.

Its translation is as follows:

Pair of compasses for a camp.

"If you want to draw a circle that can not be drawn by the spread of [ordinary] pair of compasses then you look at the place you want to make the centre of this circle, install a vertical post in it and put on its top a small flag or a similar thing if it is day time and put on the top of the post a burning candle if it is night. Then take a big
alidade with two big 'thrones' and two wide plane openings. After that you look at the place you want the circle to pass through. Take a vertical ruler installed on the throne and hang the alidade at the top of this ruler. Put the indicator [of the alidade] and raise it until you see the flag that you put on the of the post, i.e., in the centre of the circle. Then a ray goes out of our eye and [reaches] the flag passing through both the openings of the alidad. We certainly know, that if we move the alidade away of our place towards the centre or towards us, without changing [the position] of the indicator, then we don’t see the flag that we wanted [to see] at this height, since these two lines are different. If we see the flag from our place, then we turn the alidad with the indicator fixed on it, without changing its [position] and we look carefully: if the flag doesn’t leave [the line] of the two openings of the alidad and our eye, then we are surely on the circle. If the flag leaves [the line] of the two openings and our eye while we don’t move closer or farther from our place, then it will
be clear for us that the reason is that we moved up or down with respect to our first plane or our height. Then we look at the place in which we lost the flag and at our original place. If they are the same place then we [again] see the flag. In the same way we determine [the distance between] any two objects that are far from each other or [similar things] that we want.” (Fig. 8) (f. 197a-197b).

The author describes here a method of drawing circles with a big radius, for example, circle encircling a military camp. In fig. 8 you see a post with “a flame of fire”, a big alidade with two sighting holes on a carriage with wheels, which is called in the text “throne”. On this carriage a vertical ruler is installed with an alidade hung in its top. You can see also in the figure a round limb by which the indicator of the alidade can move. The person looking at the top of the post through both the sighting-holes is at a constant distance from the centre of the circle, i.e. from the foot of the post in the case when the alidade is inclined to the plane of horizon by the same angle.

§ 8. Treatises on horary instruments

The horary instruments described by Al-Khwârizmî need a special detailed study. In chapter 27 a description of the “bînkân”, i.e. the water clock is given. It represents, as is mentioned in the title of the chapter, a trough or a tub, i.e., a wide circular vessel with a hole in the middle. On the radii of these vessels there is a graduation in “temporal hours” for each month, the graduation for the signs of Aries and Libra serve as well as a graduation in “equal hours”. The word bînkân is the Arabic transcription of the Persian word parandân which means a cup, a vessel, and a water clock. This indicates the origin of this instrument (in later Arabic treatises this clock is called bînkâm; this term was used by Ibn al-Haitham [5, vol. 2, p. 255]). In chapter 28 a description of the “bînkân that shoots pebbles” is given. In this clock the lowering of the water level sets in motion a gear releasing pebbles falling on a metallic plate and producing a ringing sound which indicates the expiry of the hour. The word bundâq— “pebble” in the name of this clock also is the Arabic transcription of the Persian word fundâq — “hazel-nut”. In chapter 29 a description of “the horary lifting wheel” is given. It is an original combination of a water clock with the celestial globe. The word dîlîb — “lifting wheel” in the name of this clock is also Persian. In chapter 30 a description of “the mukhula
for determining hours” is given. It is a conical sundial with a horizontal
gnomon in its top part. The name mukhula which literally means a
vessel for storing antimony (kuḫl) can be explained by the shape of
a truncated cone of the clock which is the same shape of this vessel.
In chapter 31 a description of “the operation with the sundial that
is called miknasa” is given. Here and in other places we translate
the word rukhāma which literally means marble board by the word
“sundial”. A kind of plane sundial with a vertical gnomon is described
here. The name miknasa which is from the word kanīsa—“a church, a
pagan temple” points out the ritual meaning of this sundial. In chap-
ter 36 a description of the operations with the horary quadrant
is given. On one of the rectilinear sides of the quadrant two diopters
are placed by the aid of which this side is directed to the Sun, in this
case the thread with a load that is fixed in the centre of the quadrant
cuts on its round side an arc equal to the altitude of the Sun and
intersects an arc concentric with this side and corresponding to the
month in a point on the hour line $t = $ constant that corresponds to the
hour of observation. In chapters 52-59 the construction of the hour
lines on the ordinary horizontal sundial with vertical gnomon is
described. The tables concerning the sundial show the polar coordi-
nates i.e., “the shadow” and “the azimuth” of the endpoint of the
shadow of the gnomon on the plane of the sundial as a function of the
time $t$ of the day and the ecliptic longitude $\lambda$ of the Sun that corre-
sponds to different zodiacal signs (months). The lines $\lambda = $ constant are
arcs of a hyperbola (for the equinoctial days — straight line segments),
Al-Khwārizmī considers the line $t = $ constant “hour lines” straight.
More detailed tables are given for the latitudes $33^\circ$ (Baghdad) and
$34^\circ$ (Samarra).
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