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AL-KHWĀRIZMĪ ON THE JEWISH CALENDAR[1]

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THE material here presented is from an Arabic treatise [3] attributed to Muḥammad ibn Mūsā al-Khwārizmī, the ninth-century scholar whose works on arithmetic, algebra [5], and astronomy [7] were influential in medieval Europe. Most of the techniques described in the tract are well known through the writings of Maimonides [2], al-Bīrūnī [1], and others. However, internal evidence in the text, adduced below, supports the attribution of authorship to al-Khwārizmī. Hence, as a source antedating those above, it seems worthwhile to sketch its contents and to describe fully those of its procedures which are otherwise unknown.

The author introduces his subject ([3], 2:1–4:2) by noting the scriptural injunctions upon which the calendar rests; he cites the nineteen-year cycle with its seven intercalated months, and since, he says, knowledge concerning it is possessed by only a few of the Jews, he has written an explanation for anyone who has occasion to use it.

There follows (4:3–5:4) an enumeration of the month names and the number of days each contains. The list begins with Nisan as the first month, as in the Bible and in the Syrian calendar, although it is clearly implied by the rules which follow that it is the determination of the first of Tishri which is of interest. The embolismic years of the cycle are noted, and it is stated that the (mean synodic) month is 29 days and $12\frac{793}{1080}$ hours. This well known parameter, 29;31,50,8,20 days ([1], transl., p. 143; [2], p. 27), is from the Babylonian System B ([6], vol. i, pp. 70, 78). Each of the 1080 equal parts of an hour is called in Hebrew a *ḥēleḳ*. Converting the above into these units we obtain 765,433 *ḥalāḳīm* for the length of the mean synodic month.

The next passage (5:5–7:10) gives the complicated rules governing the day of the week on which the first day of Tishri shall fall and whether Marḥeshwan and Kislew of that year shall be both full, or both defective, or one defective and one full. (Cf. [1], transl., pp. 150–152; [2], p. 32.) The numbers 489 in 5:7 and 494 in 5:10 should be restored to 589 and 595 respectively. The editor of the printed version has mistaken the old Arabic numeral form of 5 for the modern form of 4. (Cf. [4], p. 4.) At 6:2, at the end of a line in the MS, the number 11,200,000,000 has been written, seemingly interjected into the sentence without meaning or motive.

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The mean solar year is given as 365 days $5\frac{3791}{4104}$ hours in 7:11. This year length of 6,5;14,48,33,35,... days is known from [1] (transl., p. 143) and [2] (p. 40).

Then comes a statement (7:12–14) that between the creation of Adam and the year 1135 of the Two-Horned (i.e. Alexander) is 4582 years. Regarding the Alexander date as a current year, we have for the number of complete years between the two eras, 4582 − 1134 = 3448. This is precisely the difference which al-Bīrūnī ([1], transl., p. 141) reports as Jewish doctrine for the interval between Adam and Alexander (here the Seleucid era, 1 October −311 = 1 October 312 B.C.). The correctness of these numbers having been established, we note that the Seleucid date given, which it is reasonable to regard as that of the composition of the treatise, falls in 823/824 A.D., which is nicely within the life span of al-Khwārizmī, ca. 780–850.

Then come (7:15–8:7) the three sets of planetary positions shown below:

	The first of the days of Adam, a Friday	*The building of the Temple*	*The beginning of the years of the Two-Horned*
mean sun	5^s 26°	5^s 26°	6^s 13;31,18°
mean moon	5 26	5 26	4 6;45,49
lunar apogee	1 5	9 26;40,16	7 26;17,19
Saturn	8 15	10 22;9,1	8 8;24,6
Jupiter	6 5	3 7;42,34	3 12;52,18,13
Mars	1 6	1 15;26,21	8 12;14,46
Venus	4 25	7 12;11,47	2 1;22,3
Mercury	— —	1 13;19,19	7 10;1,18
lunar node	0 14	4 26;34,11	4 23;41,27

Presumably all entries are mean rather than true positions, and those for the inferior planets are anomalies rather than mean longitudes, otherwise the elongations from the sun are impossibly large.

Note that the mean positions of the sun and moon are the same for the first two epochs, showing that both occur at the beginning of a year, and a lunar month, and both at the same point in the nineteen year cycle. Al-Bīrūnī remarks ([1], transl., p. 168) that some of the Jews claim that the "autumnal equinox took place at the end of the third hour of the day of Wednesday, the 5th of Tishri" after the Creation. Our text has the mean sun, at epoch, four degrees before the autumnal equinox, which is consistent with the above. But beyond this the numbers for the first two sets of positions seem to make little sense. Dr. David Pingree has shown that the positions as given in the third set correspond well with the mean planetary longitudes and anomalies at the zero point of the Seleucid era in −311. In any event, the author was concerned only with the sun and moon in the remainder of his discourse.

Now comes a rule (8:7–9:3) for the computation of the mean longitudes of the sun and the moon at any given time, as follows:

Add nine to the number of complete years of the era of Alexander for the time in question, and determine the residue of the result, modulo nineteen. Convert this into days (and fractions), the *small base* (*al-aṣl al-ṣaghīr*, say p) and multiply it by the *cycle* (*al-dawr*, c_s and c_m for sun and moon respectively) of whichever of the two bodies is involved. Divide the result by the *base for the days* (*aṣl al-ayyām*, b). The quotient will be years. Discard the integer part. Multiply the remainder by twelve and divide by b. The integer part of the result will be zodiacal signs. Multiply the new remainder by thirty and divide by b. The integer part of the result will be degrees. Multiply its remainder by sixty and divide by b. The result will be minutes. Continue, if desired, for additional sexagesimal parts. The signs, degrees, and minutes thus found are to be added to the mean position at the epoch of Alexander to obtain the desired mean longitude.

The constants are given as

$$b = 35{,}975{,}351, \quad c_s = 98{,}496, \quad \text{and} \quad c_m = 1{,}016{,}736,$$

the numbers being written out in words.

The first two of these are found in al-Bīrūnī ([1], transl., p. 164), and with the same names used by al-Khwārizmī. He says that the first, b, is the length of the solar year measured in fractions such that 4104 of them make an hour. This can be verified by expressing in these fractions the year length given above. The second, c_s, is as he states, the "fractions of one nychthemeron", 24 × 4104, a day measured in the same units.

The purpose of the first sequence of steps in the operation is to cast off and disregard complete nineteen year cycles from the Creation to the time in question. Nine is added to the completed years of Alexander because

$$3448 = 9 \text{ modulo } 19.$$

The rule then says, form

$$\frac{pc_s}{b} = \frac{p}{b/c_s}.$$

But b/c_s is simply days per year, the Jewish year length given above, so that we are dividing days by days per year to obtain years. Let

$$\frac{pc_s}{b} = Y + \frac{r_1}{b}$$

where Y is an integer and $r_1 < b$. Then r_1/b is a proper fraction of a year, but in terms of the solar mean motion it can be regarded as the portion of the ecliptic which the sun shall have traversed from the end of the last tropical year (at which time it crossed Aries 0°) until the time in question. It remains to evaluate this fraction in signs, degrees, and minutes. For this purpose Y is irrelevant, and we forget about it, as the text prescribes. Since there are twelve zodiacal signs in the ecliptic we find an integer s and an $r_2 < b$ such that

$$\frac{12r_1}{b} = s + \frac{r_2}{b}.$$

So s complete signs have been traversed. There being thirty degrees in each sign, the additional degrees traversed will be the integer d satisfying the expression

$$\frac{30r_2}{b} = d + \frac{r_3}{b},$$

where $r_3 < b$. In like manner, the number of minutes, m, satisfies the expressions

$$\frac{60r_3}{b} = m + \frac{r_4}{b}, \quad r_4 < b,$$

and so on.

This settles the solar mean longitude determination. The procedure is the same for the moon but we cannot confirm the lunar cycle, c_m, from an independent source. Let us proceed to compute it independently, arguing from the analogy of the solar case. For it we saw that the rule calls for the division of p by the time in days required for one rotation of the mean sun. Hence for the moon we should divide by the time required for a mean lunar rotation, i.e., by the length of a mean sidereal month, measured in days.

Since there are $235 = (19 \times 12) + 7$ synodic months in the nineteen year cycle, in one mean synodic month the mean sun moves $\frac{19}{235}$ of a revolution. Hence the mean moon in the course of one synodic month moves $1\frac{19}{235} = \frac{254}{235}$ revolutions. Hence in 235 synodic months, or 19 years, there are 254 sidereal months. So the length of one mean sidereal month is

$$\frac{19b}{254c_s},$$

and this must be the b/c_m which, according to the rule, must be divided into p. Equating the two expressions, cancelling the b's, and solving for c_m gives

$$c_m = \frac{254c_s}{19} = \frac{254 \times 98{,}496}{19} = 1{,}316{,}736.$$

Comparison of this with the value from the text discloses a missing digit in the former. Where it should have read, "a thousand thousand and *three hundred thousand and* sixteen thousand and seven hundred and thirty-six", some scribe inadvertently omitted the Arabic equivalent of the italicized words.

The last section of the treatise (9:4–9:12) is a rule for the computation of the time since the last (mean) conjunction, i.e., the last mean new moon, to the time in question. The rule says: multiply the small base (b) by 25,920 and divide the result by "seven hundred and sixty-five days, four hundred and thirty-three". The quotient will be the months passed from the first of the *maḥzor* (the nineteen year cycle). Divide the remainder by 25,920 to obtain the days passed of that month. Divide the new remainder by 1,080 to obtain the hours. The days and hours thus determined will be the elapsed time since the last conjunction.

An explanation of this rule is straightforward when we recall the *ḥalāḳīm*, 1080th parts of an hour noted in the beginning of the paper. Since $24 \times 1080 = 25{,}920$, the latter number is the measure of a day in *ḥalāḳīm*. In like manner multiplication of b by it expresses in *ḥalāḳīm* the elapsed time from the beginning of the current nineteen year cycle to the time in question.

Again, recollection that the number of *ḥalāḳīm* per mean month is 765,433 enables us to drop the "days" in the "765 days 433" of the text as a scribal error. It is now clear that the first division is for the purpose of casting off from $25{,}920b$ as many integer multiples of 765,433 as it contains. The remainder is the time in *ḥalāḳīm* from the last conjunction, and the rest of the operation converts it into days and hours.

By and large, all these manipulations insure that only operations with integers will be required. More generally, we note that as between the three sources, Maimonides, al-Bīrūnī, and al-Khwārizmī, although each has material not found in the other two, a good deal is common to all three, and nothing in one conflicts with anything in the other two. It is safe to conclude that by the early ninth century a body of doctrine concerning the cyclic religious calendar was widely accepted in the Jewish community, of which we have thus far recovered only a part.

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