Workshop: Constructing the astrolabe by traditional geometrical methods and according to the simplified methodology of Taqi al-Din *In this workshop we will first investigate the geometric contruction of an astrolabe plate and the ecliptic, and then compare our work with Taqi al-Din's treatise, in order to understand his motivation and his methodology.*

To motivate the audience we showed in the workshop a photo of an astrolabe plate that was made in Türkiye in 1120 H / 1708-1709 CE for the latitude 41^o (Istanbul), now in Oxford (no. 39955)

Part 1. Geometrical construction of the tropics of Cancer and Capricorn and the ecliptic.

See Figure 1a. The points are labeled with letters A, B, G, D etc., similar to the abjad-numbers in Figure 1b in Taqī al-Dīn's work on the astrolabe, written in 1576-1577 CE. We have added points K, X, P, lines AP, ZET, and the ecliptic circle.

In order to experience the geometrical construction, we now ask you to construct a similar figure as accurately as possible, according to the following instructions (see always Figure 1a). Draw a circle with centre E and radius exactly 6 cm at the bottom of your piece of paper in such a way that there is a margin of at least 4 centimeter all around the circle. Draw two perpendicular diameters AB, GD; thus $EA = EB = EG = ED = 6$ cm. The circle is the celestial equator and point E is the celestial north pole. Extend both diameters on both sides until the edges of the paper.

1. Find point T on arc AG of the equator in such a way that arc GT $= \angle GET = \varepsilon = 23.5$ degrees, the obliquity of the ecliptic. Draw line AT to intersect DG extended at Y.

Draw the circle with center E and radius EY. This circle is the tropic of Capricorn, the outer boundary of the astrolabe.

2. Find point Z on arc DB such that arc $ZD = \angle ZED = \varepsilon = 23.5$ degrees. Let line AZ intersect the diameter EGD at H. Draw the circle with center E and radius EH. This is the Tropic of Cancer.

Alternative and easier construction (Taqi al-Din's method): Draw TB to intersect EG in K. Then point K is also on the Tropic of Cancer, so we do not need points Z and H.

3. Find the midpoint X of line HY. With center X and radius XY draw a new circle, the ecliptic.

Shortcuts: You can also construct X either (1) by drawing $\angle EAX = \varepsilon = 23.5$ degrees; or (2) by finding point P on arc BG such that arc BP=2ε=47^o, or (3) by drawing TZ and then AX perpendicular to it.[1]

Check your work: the ecliptic must be tangent to the tropics of Cancer and Capricorn, see Figure 1a. Taqi al-Din also says that if your drawing is accurate, the ecliptic must pass through points A and B.

Questions to think about:

a. Show that this geometric construction produces the stereographic projection of the tropics of Cancer and Capricorn on the plane of the celestial equator.

b. Show that the shortcuts to find X are correct. You may need some theorems about angles and arcs in the circle, for example $\angle BAP = \frac{1}{2}\angle BEP$.

c. Measure the lengths of segments EY, EH, XH and EX, that is, the radii of the tropics of Cancer and Capricorn, the radius of the ecliptic, and the ecliptic and the distance of the centre of the ecliptic to E.

Then compute the length of these segments using trigonometry, assume $AE = BE = CE = DE = 6$ cm.

How accurate is your geometrical construction?

d. Compare your construction and computation with Taqi al-Din's values (which he presents in a table that we have not reproduced here): $91+\frac{30}{60}+\frac{55}{3600}$, and $39+\frac{20}{60}+\frac{16}{3600}$, and $65+\frac{25}{60}+\frac{36}{3600}$, and $26+\frac{5}{60}+\frac{20}{3600}$. Note that $R = 60$ so the unit is one milimeter. We introduce a notation for sexagesimal numbers with the symbol ; to distinguish the integer from the fractional part, and commas to separate sexagesimal places. In this notation, the numbers are $91; 30, 55$ and $39; 20, 16$ and 65; 25,36 and 26;5,20.

[1]Taqi al-Din does not mention these shortcuts for the horizon, but he discusses one of the shortcuts for the (mathematically equivalent) case of the prime verticai.

Part 2. Geometric construction of the almucantars on an astrolabe plate for a city with geographical latitude $\phi = 32^o$. We will only construct horizon, zenith and two almucantars, but the general idea will be clear from the construction.

We will illustrate part of our work with two of Taqī al-Dīn's figures.

Preparation: Begin with a new sheet of paper. Draw the equator as a circle of radius exactly 6 cm as above, and also the tropic of Capricorn (with radius 91.5 mm, you do not have to repeat the construction). We do not need the tropic of Cancer. We use a new notation, not the notation of Figure 1a.

See Figure 2a. Call the center E. Draw the two perpendicular diameters of the equator AB and GD, and extend them to the boundary of the plate (the tropic of Capricorn). Point E is the celestial North pole, line AEB and its rectilinear extension is called "East-West line", GED and its rectilinear extension is called the meridian. We will see that point G is above the horizon and D below it.

Construction of the horizon: First read the whole procedure before carrying out the construction. Find the diameter ZET in such a way that $\angle AEZ = \angle BET = \phi = 32^o$. Then also arcs AZ and BT are $\phi = 32^o$.

Draw AT to intersect GD at point Y and extend AZ to intersect the extended part of DG at point H (not shown in the figure). Then HY is the diameter of the horizon on the plate, so find the midpoint K and draw the circle with radius KY, this is the horizon. You only need to draw the part inside the Tropic of Capricorn.

This construction is impractical because point H is far away from the centre and may well fall outside the paper. The centre K can also be found by a shortcut, so you do not need point H (which will be outside the Tropic of Capricorn anyway). For example, you can draw AK as a perpendicular to ZT. Then draw the horizon as a circle with centre K and radius KY.

Check your work: the horizon must pass through points A and B (If it does not, Taqi al-Din says that you must find the errors in your work and correct them, before continuing).

Note that A is the point on the horizon exactly in the East, B is the point on the horizon exactly in the West. Y is the point on the horizon exactly in the North. The southernmost part of the horizon is point H which is outside the astrolabe plate.

Construction of the zenith: Find point R on the quadrant GB in such a way that $\angle GER = \phi = 32^o$, then also arc $GR = \phi = 32^o$. Draw AR to intersect the meridian GD at Q. Point Q is the zenith.

We now construct the almucantar for altitude $a = 60^o$. You may want to read the whole procedure before actually making the drawing.

First find two points P and S on circle $ABGD$ with distance $90^o - a = 30^o$ degrees from point R which was used to construct the zenith; in the figure these points P and S are joined by a broken red line, which is perpendicular to ER.

Draw two straight lines AP, AS and find the two points of intersection with the meridian GD. These two points are the endpoints of a diameter of the almucantar for altitude 60. Find the midpoint and mark it clearly for later checking. Note that the midpoint is not the same as Q. You can now draw the almucantar (the small red circle in Figure 2a) and put the number "60" near it.

Repeat the procedure to draw the almucantar for altitude $a = 30^o$. The result is also shown in Figure 2a. Begin by finding two points on the equator $ABGD$ $90^o - a = 60^o$ away from point R, and 30^o away from points Z and T which were used to construct the horizon. Then join these points to A and find the intersections with the meridian, which are the endpoints of the diameter of the almucantar for 30^o. Find the midpoint and mark it clearly for later use. Draw the almucantar and put the number 30 next to it.

Compare your work with Taqi al-Din's Figure 2b - in this figure the almucantars for every 15 degrees of altitude are drawn for $\phi = 30^o$. Taqi al-Din does not explain the shortcut to find the centre of the horizon. His figure contains the Tropic of Cancer, which we have omitted. His black lines have to be engraved on the astrolabe. His red lines should be written in a non-permanent way and erased when the construction is finished.

Questions to think about:

a. Why are the zenith and the two almucantars in Figure 2a really the stereographic projections of the zenith and the almucantars for 30 and 60 degrees altitude above the horizon?

b. Explain why the shortcut to find the centre of the horizon is correct.

Comparison with Taqi al-Din's tables. In Table A, Taqi al-Din computed for a locality with latitude $\phi = 32^o0'$ the radii **r** and distances**d** to the center E of the astrolabe, for the horizon and the almucantars for altitude **a**=1^o to 89^o, and also the distance of the zenith to the centre of the astrolabe. He assumes that the radius of the equator is 60 units, therefore his unit corresponds to 1 millimeter in our drawing. He presents **r** and **d** in units and sexagesimal fractions (thus: 13 47 means $13 + \frac{47}{60}$, we use the notation 13;47). It is therefore easy to compare your geometrical drawing with the table.

Measure KY and KE and EY in millimeters and compare them with the numbers in Taqi al-Din's table: $EY = 17;12$, $KY = 113;14$, $KE = 96;2$. Taqi al-DIn also said $EH = 209;15$, Locate these numbers in red abjad-notation in the vertical line on the left side of the table. Note that the abjad number 17 12 (in our notation 17;12) means $17 + \frac{12}{60}$.

Measure for the almucantars for 30^o and 60^o the radius and the distance to the centre E of the astrolabe in millimeters. Compare the values with the entries in abjad in the table (last line, the arguments 30 and 60 are in red and the table is arranged from right to left).

Measure QE in millimeters and compare with the distance for the zenith in the last line, left side. (Note that the radius of the zenith is written as 0 0 in the table.). If the abjad number itself is not clear, try to read the numbers above it in order to get an idea of the behaviour of the numbers.

Taqi al-Din computed the entries in the table on the basis of a table of tangents in the beginning of the treatise. There he presented in modern terms $R \tan x$ for x with intervals of 5 minutes of arc, between $0^o5'$ and $89^o55'$. These were probably based on his sine table, the computation of which is explained in detail in another work, the *Sidrat al-Muntahā.*

Question to think about:

c. Try to work out yourself the formulas for radius and distance to the centre of the astrolabe for an almucantar of altitude a at geographical latitude ϕ. Can you recompute some of the numerical values in the table for $\phi = 32^o$?

Hint: we derive the formula for the zenith. In Figure 2a we have arc $GR = \phi$, therefore arc $BR = \angle BER = 90 - \phi$ and therefore $\angle BAR = 45 - \frac{\phi}{2}$. Since EA=60 we obtain $EQ = \tan(45 - \frac{\phi}{2})$. For $\phi = 32$ we have $EQ = 60 \tan 29 = 33.2585 \ldots \approx 33;16$ as in Taqi al-Din's table.

Part 3. Geometric construction of the azimuthal circles on an astrolabe plate for a city with geographical latitude $\phi = 32^o$.

Finally we will construct three azimuthal circles (Figures 3a, 3b). You may want to draw a new figure (Figure 3a) with Tropic of Capricorn (radius 91.5 mm) and equator $AGBD$ (radius 6 cm) just as in FIgure 2a. Also add the same point R on arc GB in such a way that arc $GR = \phi = 32^o$ and draw AR to intersect DG in the zenith Q. In Figure 3a we have also drawn the horizon but this is not necessary in your figure.

We first construct the nadir: draw diameter REX and find point X on arc DA. (Thus arc $XD = \phi = 32^o$). Draw AX and extend it to meet GD extended at U. Then point U is the nadir (and as you see in Figure 3a, it is under the horizon)

Now find the midpoint V of QU and draw the circle with centre V and radius VQ. This circle is the "first vertical" which is perpendicular to the horizon. On most astrolabe plates, only the part of this circle above the horizon is drawn. According to the old conventions, points on the first vertical have azimuth zero (East or West), and points on the meridian GD have azimuth 90^o (North or South).

Check your work: the first vertical must pass through points A and B, which are the Eastern and Western points on the horizon.

All other azimuthal circles pass through points Q and U. Therefore their centres must be on a line through V perpendicular to GD. Draw this perpendicular line (broken line in Figure 3a)

To find the circle for azimuth 30^o North of the Easternmost point, find W on this perpendicular line in such a way that $\angle VQW = 30^o$. The circle with center W and radius WQ, that is the blue circle in Figure 3a. is the circle for azimuth 30^o: the part between Q and the horizon is in Northeastern direction, exactly 30^o North of East, and the other part (above Q) in Southwestern direction, exactly 30^o South of West. Draw the part of this circle inside the plate, or only above the horizon, and write the number 30 next to it.

In the same way we find the center W' of the azimuthal circle for azimuth 60^o: draw angle VQW'=60^o where W' is on the perpendicular through V. Then draw the azimuthal circle for azimuth 60^o (purple in Figure 3a) and put the number 60 next to it.

Compare the work with the picture of the astrolabe plate on the second plate of this workshop. Circles for azimuths close to 90^o have a very large radius, which presents additional difficulties for drawing. Also compare your work with Taqi al-Din's Figure 3b (rotated 90 degrees). The red lines are Taqi al-Din's lines of construction, which can be erased when the work is finished.

Questions to think about:

a. Explain why the construction of the nadir is correct.

b. Can you devise a shortcut to find the position of the centre V of the prime vertical, without point U?

c. Why is the construction of the azimuthal circles correct? Hint: you can use the fact that the stereographic projection is conformal, that is to say, that it preserves angles. This property was probably known to some of the Islamic astrolabe makers but a proof has not (yet?) been found in Arabic texts. The oldest known proof seems to be by the English mathematician Thomas Harriott (1560-1621).

Comparison with Taqi al-Din's tables.

In your figure, measure UE, QV and VE and compare it with the following information in Table A (between the first and second group, just right of the red arguments from 31 to 60):

"The southern position of the first vertical [i.e., the nadir] is 108 15, the radius of it is 70 46, and the distance of its centre from the centre of the astrolabe is 37 30." (You can read the abjad numbers in red).

Taqi al-Din constructs the other azimuthal circles by the following method, which has not yet been found in other sources, so it may be his own discovery.

Table B is Taqi al-Din's table for almucantars for a locality on earth on the equator $\phi = 0^o$. The table presents for every degree of altitude **a** between 1 and 90 the radius **r** of the almucantar and the distance **d** of its centre to the centre of the astrolabe. The celestial equatior is assumed $R = 60$. Compare also the figure (taken from Morrison p. 66).

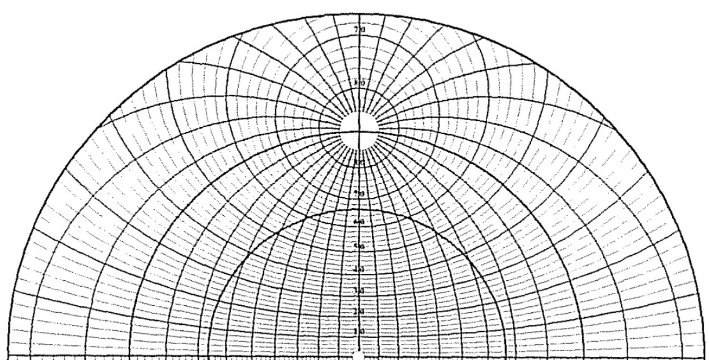

Plate for localities on the equator ($\phi = 0^o$). The horizon is a straight line and the zenith is located on the celestial equator.

For the azimuthal circles,Taqi al-Din prescribes the following procedure:

First compute the scaling factor c: radius of prime vertical divided by the radius of the equator 60. For $\phi = 32^o$, c= (70;46)/60.

For $\phi = 32^o$, and the circle for azimuth **z** (which Taqi al-Din measures from the East or West point of the horizon) the distance between its centre to point V is c times the radius of the almucantar for 90^o-**z** in the table for $\phi = 0^o$. The radius of the azimuthal circle for azimuth **z** is c times the distance fo the centre of the almucantar 90^o-**z** for $\phi = 0$.

Now measure $VW, WQ, VW', W'Q$ and compare them with the values which you get in this way from Table B.

Problems to think about:

Show by modern trigonometrical formulas that the procedure is mathematically correct, by computing the radius and distance to the centre for almucantars with altitude 90^o-$\mathbf{z}$ for $\phi = 0^o$, and then the radius and distance to the centre of the azimuthal circles for azimuth $\mathbf{z}$ for $\phi > 0$ (suppose $\phi < (90 - \varepsilon)$.

Can you think of a simple geometrical way to show that this procedure is correct? (Taqi al-Din describes the procedure but does not give a motivation in his astrolabe text)

References: Taqi al-Din's astrolabe treatise is still unpublished, but it has been preserved in several Arabic manuscripts.

The geometrical constructions in this workshop have also been inspired in part by Abū al-Rayḥān al-Bīrūnī, *Istī^{c}āb al-wujūh al-mumkina li-ṣan^{c}at al-usṭurlāb* (Full Discussion of all possible ways to construct the astrolabe), see work no. 3 on the website www.albiruni.nl, and by several works by his teacher Abū Naṣr ibn ^{c}Irāq (mainly N4 on the website www.albiruni.nl).

Older numerical tables for the astrolabe are to be found in al-Farghāni's work *Kitāb al-kāmil fī ṣan^{c}at al-asṭurlāb al-shamālī wal-janūbī wa-^{c}ilaluhumā bi-l-handasa wa'l-ḥisāb* (*Complete book on the construction of the northern and southern astrolabe and the reasons for them by geometry and by computation*). The work has been edited in Arabic and translated into English by R. Lorch (ed.). Al-Farghānī on the Astrolabe: Arabic text Edited with Translation and Commentary. Stuttgart: Franz Steiner, 2005. Series: Boethius, vol. 52.. The tables can be found in London, British Library, Ms. Or 14270, available online on Qatar Digital Library, https://www.qdl.qa (click on Explore the Archive, and search with "Al-Farghani astrolabe" , then you will find this manuscript and much else). Note that Taqi al-Din probably did not know al-Farghani's work.

The mathematical background is explained in, e.g., James E. Morrison, *The Astrolabe*, Rehoboth Beach: Janus, 2007, This book (and many other sources) can be downloaded from archive.org via the link on www.jphogendijk.nl/astrolabe.html

See also the websites www.astrolabes.org and www.astrolabeproject.com if you want to actually draw an astrolabe yourself.

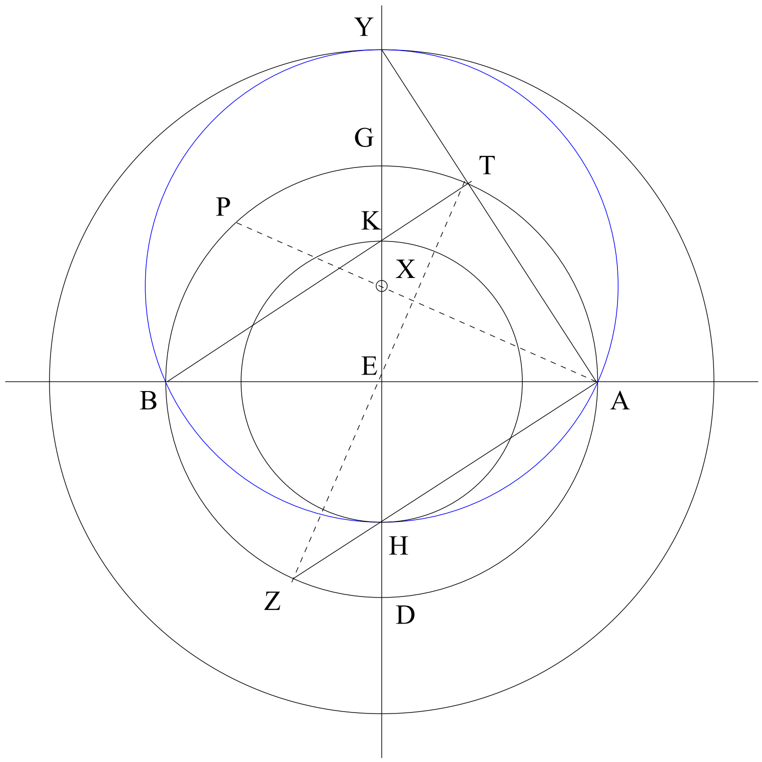

Figure 1a

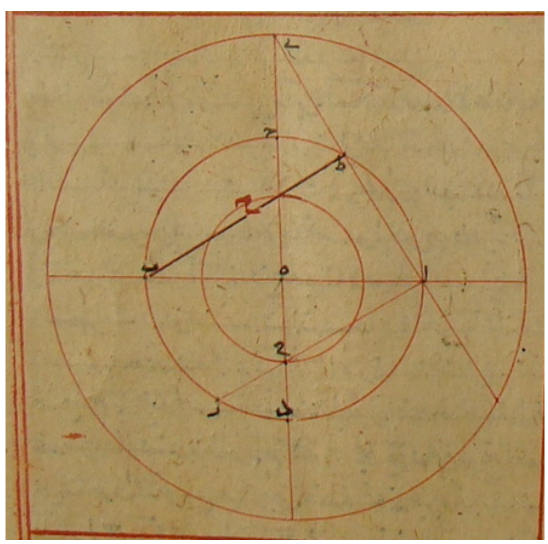

Figure 1b

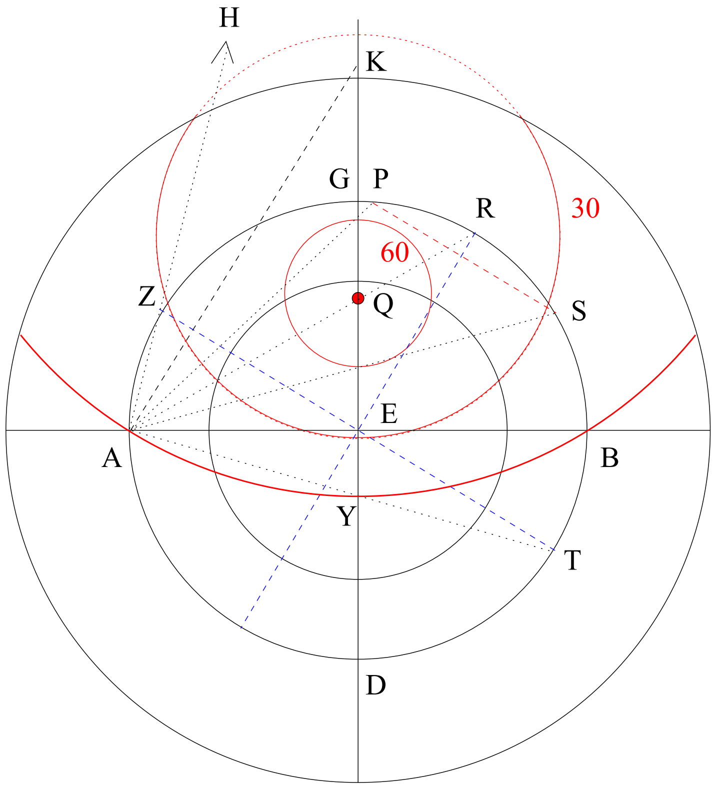

Figure 2a

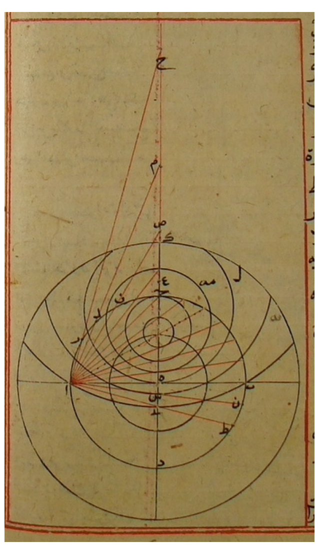

Figure 2b

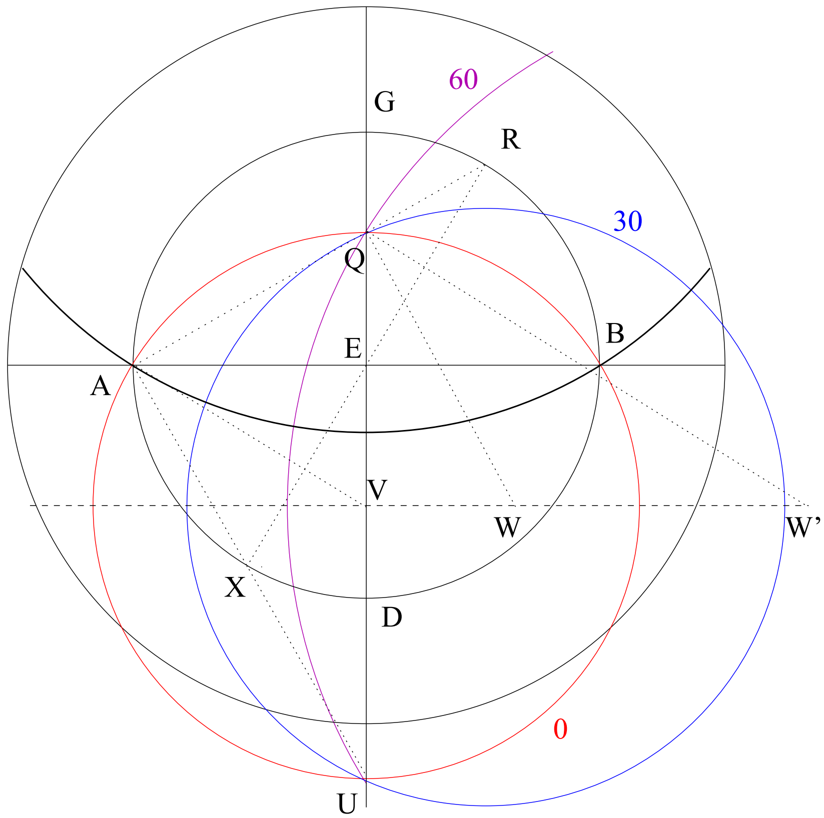

Figure 3a

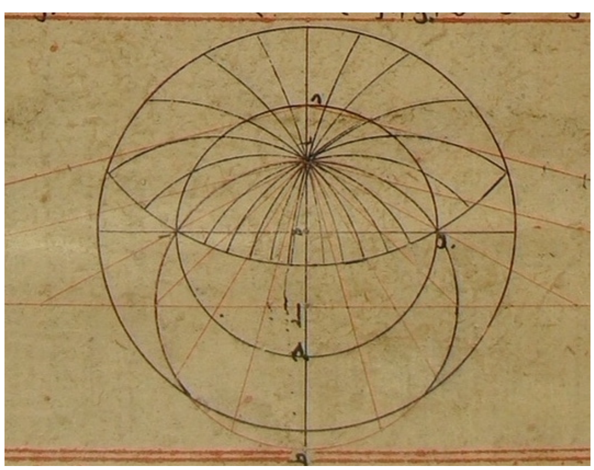

Figure 3b