Key to the workshop on constructing the astrolabe according to traditional geometric methods and the simplified methodology of Taqi al-Din

Part 1

a. We can look at Figure 1a in two different ways. On one hand (1) it is the plane of projection, which is the plane of the celestial equator. On the other hand (2) one can also see it as a cross-section of the celestial sphere, with A the pole of projection, B the north celestial pole, and GD a cross-section of the projection plane, and TZ a cross-section of the ecliptic. Then T is the winter solstice, Z the summer solstice, and $\angle TEG = \angle ZED = \varepsilon$, the obliquity of the ecliptic. If the figure is viewed as a cross-section, Y is the projection of T and H the projection of Z. Therefore EY and EK are the radii of the tropics of Capricorn and Cancer respectively.

b. There are many different ways to prove this. Here is one example. Because $\angle TEG = \angle ZED = \varepsilon$, TEZ is a diameter of the circle $AGBD$.

We have $\angle ATE = \angle AYE + \angle TEY$, therefore $\angle AYE = \angle ATE - \angle TEY = \frac{1}{2}$ arc $AZ-$ arc $GT = \frac{1}{2}(90+\varepsilon) - \epsilon = \frac{1}{2}(90-\varepsilon)$. Because X is the centre of the circle through Y, A and H, we have $YX = XA = XH$, therefore $\angle YAX = \angle AYX = \angle TYE = \frac{1}{2}(90-\varepsilon)$ But $\angle TAB = \frac{1}{2}$ arc $BT = \frac{1}{2}(90+\varepsilon)$. We conclude $\angle PAB = \varepsilon$. Therefore arc $PB = 2\varepsilon$. Since AB and GD are perpendicular, while $\angle BAP = \angle GET = \varepsilon$, AP is perpendicular to TZ. This last theorem will be useful to find similar shortcuts in the future.

c. Put $AE = R = 60$. Since $\angle BAT = \frac{1}{2}(90+\varepsilon)$, therefore $EY = R\tan\frac{1}{2}(90+\varepsilon) = 60\tan\frac{1}{2}113.5^o = 91.515209\ldots$. Since $\angle BAZ = \frac{1}{2}(90-\varepsilon)$, $EH = R\tan\frac{1}{2}(90-\varepsilon) = 60\tan\frac{1}{2}66.5^o = 39.337724\ldots$ Using these two values one can find the radius of the ecliptic $\frac{1}{2}(91.515209\ldots + 39.337724\ldots) = 65.42647\ldots$ and the distance of its centre to the centre of the astrolabe as $\frac{1}{2}(91.515209\ldots - 39.337724\ldots) = 26.08874\ldots$.

d. Taqi al-Din (manuscript ff. 61b, 62a in the table headers) gives rounded values of the four numbers in two sexagesimal places. Rounded to three sexagesimal places (wolframalpha.com) the values are 91;30,54,45 and 39;20,15,48 and 65,25,35,17 and 26;5,19,28.

Part 2

a. See a, part 1. We can consider Figure 2a in two ways: (1) as the plane of projection, which is the plane of the celestial equator, but also (2) as the local meridian plane with A the celestial south pole, B the celestial north pole,R the zenith, GD the cross-section of the celestial equator, ZT the cross-section of the local horizon. Then arc $AZ = BT = \phi$, the geographical latitude. Since R is the zenith, $BR = 90^o$, therefore also arc $RG = \phi$. Then Q is the projection of the zenith, and H and Y are the projections of the south point and north point of the horizon.

Similarly, if PS is the cross-section of an almucantar of altitude a we have arc $ZP =$ arc $TS = a$, therefore arc $SR =$ arc $RP = 90^o - a$. The construction now follows.

b. For the shortcut see part 1, key to b, where we change ε to $90^o - \phi$. Note that we have $\angle KAE = 90^o - \phi$; this gives a shortcut for the computation of KE.

Part 2: Comparison with Taqi al-Din's methodology

c. Table A contains the following relevant data for $\phi = 32^o, R = 60$:

Distance from the zenith to the centre of the astrolabe 33;16, radius 0;0 (the zenith is a point)

Almucantar for altitude 60^o: distance to the centre of the astrolabe 36;27, radius 21;29

Amucantar for altitude 30^o: distance to the centre of the astrolabe 49;24, radius 50;26

We can round these values to obtain the "ideal" measurements in millimeters in our figure: zenith 33 mm, almucantar for altitude 60^o distance 36 mm, radius 21 mm, and so on.

d. For an almucantar of altitude a, the intersections with GD have the following distances to the centre: $R \tan \frac{1}{2}((90-\phi)+(90-a))$ and $R \tan \frac{1}{2}((90-\phi)-(90-a))$. I If $a < \phi$ we have $90 - a > 90 - \phi$ so that $R \tan \frac{1}{2}((90-\phi)-(90-a)) < 0$, this means that the almucantar intersects the meridian GD north of (i.e. below) the pole E.

We conclude that the radius of the almucantar is $\frac{R}{2}(\tan(90 - \frac{1}{2}(a + \phi)) - \tan\ \frac{1}{2}(a - \phi))$ and the distance of the centre of the almucantar to the centre of the astrolabe is $\frac{R}{2}(\tan(90-\frac{1}{2}(a+\phi))+\tan\frac{1}{2}(a-\phi))$, modulo typing errors.

Taqi al-Din computes all these radii and distances on the basis of his "fundamental table" in the beginning of the manuscript, which presents values for $R \tan x$ for intervals of 5 minutes of arc, from $0^o5'$ to $89^o55'$.

Part 3: Azimuthal Circles

The relevant entries in table B (for the equator $\phi = 0^o$) are as follows:

Almucantar for altitude 30^o: distance of the centre from the centre of the astrolabe 120;0. Radius of the almucantar 103;55.

Almucantar for altitude 60^o: distance of the centre from the centre of the astrolabe 66;16. radius of the almucantar 34;38.

The computations for the azimuthal circles for $\phi = 32^o$ are as follows:

constant $c = (70 + \frac{46}{60})/60 = 1.179444\ldots$

For the azimuthal circle for 30^o, the distance of the centre to V is $c \cdot (34 + \frac{38}{60}) \approx 40.8$. radius of the azimuthal circle $c \cdot (66 + \frac{16}{60}) \approx 78.2$

For the azimuthal circle for 60^o, the distance of the centre to V is $c \cdot (103 + \frac{55}{60}) \approx 122.6$. radius of the azimuthal circle $c \cdot 120 = 2 \cdot (70 + \frac{46}{60})$, the same as the distance between zenith and nadir (which together with the centre of the azimuthal circle form an equilateral triangle).

General: The table for $\phi = 0^o$ contains the following entries. Put $R = 60$.

The almucantar with altitude a intersects the meridian in $R \tan \frac{a}{2}$ and $R \tan(90^o - \frac{a}{2})$. Thus we obtain

the centre of the almucantar $\frac{R}{2}(\tan(90^o - \frac{a}{2}) + \tan \frac{a}{2}) = \frac{R}{\sin a}$, thus for $a = 30^o$ we obtain exactly 120.

the radius of the almucantar $\frac{R}{2}(\tan(90^o - \frac{a}{2}) - \tan \frac{a}{2}) = R \tan(90^o - a)$.

With these formulas one can show that Taqi al-Din correctly used the table for the azimuthal circles:

For example, in the right triangle QVW, if we put $\angle VQW = 90 - a$ and $QV = R_1$, then $VW = R_1 \tan(90^o - a)$ and $QW = \frac{R_1}{\sin a}$. Therefore we need to multiply the numbers in the table for $\phi = 0^o$ with a factor $\frac{R_1}{R}$.