THE CONTRIBUTIONS BY ABŪ NAṢR IBN ʿIRĀQ AND AL-ṢAGHĀNĪ TO THE THEORY OF SEASONAL HOUR LINES ON ASTROLABES AND SUNDIALS

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1. *Introduction*

In the civil time-reckonings of Hellenistic Greece, the Roman empire, the Islamic world, and medieval Europe, the period between sunrise and sunset was divided into twelve equal day hours. These hours are called seasonal hours because the length of the hour depends on the season of the year. Similarly, the period between sunset and sunrise was divided into twelve seasonal night hours. Many ancient and medieval sundials and astrolabe plates display lines or curves indicating the ends of the seasonal hours. These hour lines are always represented as straight lines on plane sundials, and as arcs of circles on astrolabe plates. Whether these representations are exact depends on the geographical latitude of the locality for which the astrolabe plate or horizontal sundial was designed. For localities on the equator, the hour lines on an astrolabe plate or horizontal sundial are straight lines. For other localities, only the line indicating the end of the sixth seasonal hour (noon) is a straight line, and the other hour lines are not straight lines or circles, but more complicated curves.

The European history of the subject was summarized by Drecker [5, pp. 12-20]. Clavius seems to have been the first European mathematician who proved that the hour lines on a horizontal sundial are not straight lines, except in the above-mentioned special cases. His 1593 proof was unknown to many later mathematicians, and in 1817 Delambre still stated that the hour lines on a horizontal sundial are straight lines. In 1834, T. S. Davies showed that the hour lines on a horizontal sundial are algebraic curves whose degrees depend on the ordinal number of the seasonal hour (see [5, p. 18]).

Of course, hour lines had been studied long before the European Renaissance. Ptolemy (ca. AD 150) may have known that the hour lines on a horizontal sundial are in general not straight lines [5, p. 12],

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but no ancient Greek proof of this fact has been found. Thābit ibn Qurra (AD 836-901) explicitly says that at least some hour lines on a horizontal sundial are not straight [7, p. 7]. In Book 2 of the *Treatise on Shadow Instruments*, Thābit's grandson Ibrāhīm ibn Sinān (AD 907-946) attempted to prove that the hour lines on a horizontal sundial cannot be straight lines by showing that the corresponding hour curves on the celestial sphere cannot be circles. Unfortunately, the only surviving manuscript of the text breaks off in the middle of the proof. In his commentary to the text, Paul Luckey (1884-1949) argued that Ibrāhīm's proof was insufficient because he only showed that *some* hour curves on the sphere are not circles [11, pp. 63-74]. Luckey's reconstruction is confirmed by a hitherto unpublished treatise on hour lines, which was written by Ibn al-Haytham (ca. AD 965-1041) towards the end of his life, and in which Ibrāhīm's proof is mentioned.[1] Ibn al-Haytham proves that the hour lines on a horizontal sundial cannot be straight lines (except in the above-mentioned special cases), but that the difference is so small that it can often be ignored for practical purposes.[2]

The present paper is devoted to two texts on hour lines which were written in the period between Ibrāhīm ibn Sinān's *Treatise on Shadow Instruments* and Ibn al-Haytham's treatise on hour lines. Section 2 is a brief introduction to the hour line problem in its different forms.

Section 3 focuses on the important contribution by Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq, a mathematician and astronomer who belonged to an ancient Khwārizmian family.[3] He spent most of his life in Khwārizm (modern Turkmenistan and Uzbekistan) and probably died in Ghazna (Afghanistan) around AD 1030. His date of birth is unknown, but his scientific career began well before AD 1000. Abū Naṣr was an expert in spherical trigonometry. He made the best revision of the *Spherics* of Menelaus of Alexandria (ca. AD 100) that has come down to us [10], and he was the most important mathematics teacher of al-Bīrūnī (AD 973-1048).[4]

Probably between AD 990 and 1000, Abū Naṣr wrote a letter to al-Bīrūnī in response to questions which al-Bīrūnī had asked, mainly on

[1] See [11, p. x]. The treatise is listed in [15, vol. 5, p. 368 no. 19]. The last extant theorem of Ibrāhīm's proof was also studied by al-Sijzī in his *Taʿlīqāt Handasiyya* (see, e.g., MS. Dublin, Chester Beatty 3045, 83b:6–14).

[2] A more precise description of his results will be given in Section 3 below.

[3] On the life and works of Abū Naṣr, see, e.g., [15, vol. 5, pp. 338-341, vol. 6, pp. 242-245], [10, pp. 109-116].

[4] On al-Bīrūnī, see, e.g., [8, vol. 2, pp. 148-158].

hour lines on astrolabe plates. The letter was printed in Hyderabad in [1] and translated into Spanish by Julio Samsó in his excellent book [14]. Samsó points out that Abū Naṣr's letter contains many obscure passages. In Section 3 of this paper, I extend Samsó's mathematical commentary by a detailed discussion of the connections between the propositions in Abū Naṣr's letter and al-Bīrūnī's questions. I show that Abū Naṣr's letter contains a proof of the fact that the hour curves on the celestial sphere are not circles. It follows that the hour lines on a plane sundial cannot be straight lines and that the hour curves on the astrolabe plates cannot be circles, except in the above-mentioned special cases. Abū Naṣr's proof is the oldest correct proof that has yet come down to us. Abū Naṣr left most of the details to the reader, and thus he treated his pupil al-Bīrūnī in the same way in which a modern mathematics professor would treat a very good student. Section 3 of this paper also includes an English translation of the part of Abū Naṣr's letter dealing with hour lines.

To place Abū Naṣr's letter in a historical context, I have included in Section 4 a contribution by the mathematician, astronomer and astrolabe maker Abū Ḥāmid Aḥmad ibn Muḥammad ibn al-Ḥusayn al-Ṣaghānī, who worked in the late tenth century AD in Baghdād and died in AD 990.[5] Al-Ṣaghānī wrote a treatise on hour lines of which only the first chapter is extant. This first chapter treats the circular arcs which represent the hour lines on an astrolabe plate. Al-Ṣaghānī says that many people in his time believed that these arcs pass through the projections of the North and South points of the horizon. He then proves that on astrolabe plates for the temperate latitudes, the circular arcs for the ends of the first, second and third seasonal hour cannot *all* pass through the projections of the North and South points. Abū Naṣr proves in his letter that none of these arcs passes through the North and South points. This is an improvement over al-Ṣaghānī, but Abū Naṣr's proof is based on a difficult theorem in plane geometry which he does not bother to prove (or even enunciate), while al-Ṣaghānī's explanation is very clear. Section 4 includes an edition, translation and commentary of the first chapter of al-Ṣaghānī's treatise. I have no further information on the remainder of his text, which is now lost.

Al-Ṣaghānī's treatise may have been practically oriented, but the work by Ibrāhīm ibn Sinān, Abū Naṣr, and Ibn al-Haytham on the theory of the hour lines was clearly motivated by an interest in mathematics for its own sake. The interaction between theory and practice is one of

[5] On al-Ṣaghānī, see, e.g., [15, vol. 5, p. 311; vol. 6, pp. 217-218].

the reasons why the medieval Islamic history of the hour line problem deserves our attention.

2. *The hour line problem*

Figure 1 shows the plane of a horizontal sundial with gnomon at point G perpendicular to the plane of the paper for a locality in the temperate northern latitudes.[6] If we ignore the change in solar declination during a single day, the tip of the shadow describes a straight line during the days of vernal and autumnal equinox, and a hyperbola during all other days. For any seasonal day hour, the hour line can be drawn by joining the points which mark the end of this hour on the hyperbolas and on the straight line. Since the sun is due south at noon, the line for the end of the sixth hour is the North-South line, which is a straight line. The other seasonal hour lines also look like straight lines (see the computer drawing Figure 1), and the question is: are they really straight lines?

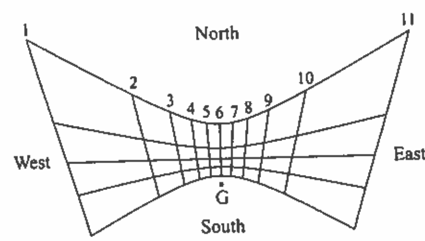

Figure 1

To construct the hour lines on the plate of an astrolabe (Figure 2), consider the stereographic projections of the declination circles between the tropics of Cancer and Capricorn.[7] The points indicating the ends of the seasonal hours on these circles can be obtained by dividing their arcs between the projection of the meridian and the projections of the Eastern and Western horizon into six equal parts.[8] The curves joining these points look like circles, as in the computer drawing Figure 2, and the question is: are they actually circles?

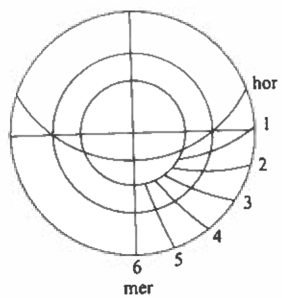

Figure 2

[6] Figures 1, 2 and 3 have been drawn for a geographical latitude of 38 degrees.

[7] The pole of projection is the celestial South pole. The standard source on the theory of the astrolabe is [12].

[8] Most astrolabe plates display only curves for the night hours, but these curves were also used for the day hours using obvious symmetries.

The following argument shows that these two problems are equivalent. The hour lines on a horizontal sundial and an astrolabe plate are gnomonic and stereographic[9] projections of hour curves on the celestial sphere. These hour curves can be constructed as follows. The horizon and meridian divide each declination circle into four sections.[10] Divide each section into six equal parts, and then join corresponding division points on the different declination circles.

Figure 3 displays the hour curves in the quadrant of the celestial sphere for the morning hours.[11] The hour lines on a horizontal sundial are straight lines if and only if the hour curves on the celestial sphere are great circles. The hour lines on an astrolabe plate are circles if and only if the hour curves on the celestial sphere are (great or small) circles. Thus the problems for the horizontal sundial and the astrolabe are equivalent if the following theorem is assumed: if an hour curve on the celestial sphere is a circle, it must be a great circle. This theorem was well known in the medieval Islamic tradition, and one medieval Islamic proof will be presented in Section 3. There I will also discuss a medieval Islamic quotation suggesting that the problems for the horizontal sundial and the astrolabe plate are similar.

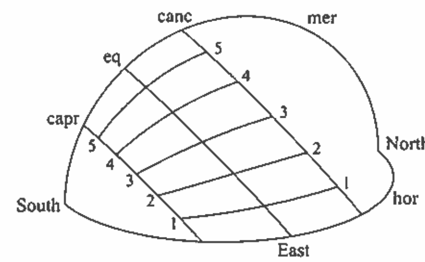

Figure 3

In the rest of this paper, I will use the term "hour line" or "hour curve" only for mathematically exact hour lines or curves for the end of the first through fifth and seventh through eleventh seasonal (day or night) hours, for localities which are neither on the equator,[12] nor in the arctic and antarctic regions. I will no longer use "hour line" and "hour

[9] For an introduction to gnomonic and stereographic projections see [3], Ch. 6.9.

[10] We consider only declinations between $-\varepsilon$ and ε, with ε the maximum inclination of the ecliptic, and we assume that the locality is not on the equator and not in the arctic and antarctic regions.

[11] The afternoon quadrant is symmetrical to the morning quadrant with respect to the meridian plane, and the night hour curves are symmetric to the corresponding day hour curves with respect to the centre of the celestial sphere.

[12] For localities on the equator, the seasonal hours coincide with the equinoctial hours, and the hour lines on the astrolabe plate and the horizontal sundial are indeed straight lines.

curve" for the mathematically uninteresting cases of the meridian (end of the 6th hour) and the horizon (end of the 12th hour). The ancient and medieval instrument makers drew the hour lines on a horizontal sundial as straight lines, and the hour lines on an astrolabe plate (except the meridian line) as circles, as follows. For any seasonal hour, find the points defining the end of this hour on the projection of the equator and the projections of the tropics of Cancer and Capricorn. On a plane sundial, these three points lie on one straight line, which is drawn as (an approximation of) the hour line. On an astrolabe plate, draw the circle through these three points. I will use the term "hour circle" for such a circle approximating the hour line on an astrolabe plate.

3. *The contributions by Abū Naṣr ibn ʿIrāq*

In or before AD 1016, Abū Naṣr ibn ʿIrāq wrote a treatise on the practical construction of the astrolabe dedicated to Abū ʿAbdallāh Muḥammad ibn ʿAlī al-Maʾmūnī, who ruled Khwārizm from 1016 until the country was conquered by Maḥmūd of Ghazna in 1017.[13] In this treatise, Abū Naṣr discusses the above-mentioned construction of the hour circles on an astrolabe plate, but he says that this construction does not produce the exact hour lines, and he suggests that the hour line problems for the astrolabe and the sundial are related. Here is the relevant passage:

"Construction of the lines of the seasonal hours. The circles which we draw on the astrolabe for the beginnings of the (seasonal) hours are drawn as follows (cf. Fig. 2 above). Of each of the three orbits (i.e. the equator and the tropics of Cancer and Capricorn) drawn on the astrolabe, we divide each part under the horizon until the meridian into six equal parts. Then we look for the center of the circle which passes through the three endpoints of the first sixth parts, and (the method for) drawing the circle through these three points is mentioned in the Book of *Elements* (of Euclid).[14] Similarly for the endpoints of the second sixth parts, and the third, until the sixth (sixth parts), on both sides (of the meridian).

[13] In [14, p. 46], Samsó points out that the treatise must have been written before al-Maʾmūnī's accession to the throne because Abū Naṣr does not yet call him Khwārizm-Shāh (king of Khwārizm).

[14] A circle can be constructed through three given points on the bases of *Elements* IV:5, see [6, vol. 2, pp. 88–89].

This (method) does not lead to the exact result, except on the three orbits themselves. I have proved this in my book *On Azimuths*, and in my *Answer to Abū 'l-Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī concerning what he asked about these circles and similar problems*, using the *Spherics* (of Menelaus). But it is the limit (i.e., the most accurate) of what is possible in this for the astrolabe, just like (the method) which is also used in sundials, because it is like this" (Arabic text in [1, no. 15, pp. 16–17], Spanish translation in [14, p. 85]).

The book *On Azimuths* is now lost, but the second text which Abū Naṣr mentions is extant, under the title *Letter by Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq, Associate of the Commander of the Faithful, to Abū 'l-Rayḥān Aḥmad ibn Muḥammad al-Bīrūnī on the Circles which Define the Seasonal Hours and on Something Related to the Construction of the Astrolabe.* I will abbreviate this long title to *Letter on the Seasonal Hours*. The *Letter on the Seasonal Hours* is mentioned in [15, vol. 5, p. 244 no. 8], and it has come down to us in the Arabic manuscript Patna, Khuda Bakhs Library, Bankipore 2468, ff. 96b–98b, which was printed in [1] and translated into Spanish in [14, pp. 53–58]. I have also used the Arabic manuscript Oxford, Bodleian Library, Marsh 713, ff. 251a–253b, which was not available to Samsó in [14]. According to the passage quoted above, the most important part of the *Letter on the Seasonal Hours* was also found in the lost book *On Azimuths*. The *Book on Azimuths* must have been written before AD 998, because Abū Naṣr says that this work was cited by Abū 'l-Wafāʾ al-Būzjānī, who died in AD 997/8 [14, pp. 17–18, 40–41]. Thus the *Letter on the Seasonal Hours* probably dates back to the period AD 993–998, when al-Bīrūnī was in his early twenties.

A translation of the first part of the *Letter on the Seasonal Hours* is to be found at the end of this section. I begin with a commentary.

First, Abū Naṣr mentions several questions by al-Bīrūnī, most of which relate to hour lines on the astrolabe. Abū Naṣr then presents five propositions related to seasonal hour lines, and four other propositions on the astrolabe which do not concern us here. The five propositions are very abstractly worded, in the style of the *Spherics* of Menelaus [10], and Abū Naṣr does not bother to explain why they entail the answers to al-Bīrūnī's questions. I will therefore discuss the connections between these five propositions and the theory of hour lines and hour circles, in notations adapted to those of Abū Naṣr, so that the reader can see how the propositions were to be used.

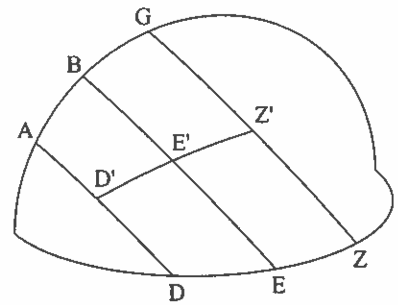

Figure 4

Propositions 1 and 2 are preliminary properties on hour circles. Proposition 1 can be used to show that the hour circles are projections of great circles on the celestial sphere. Figure 4 represents the celestial sphere in notations adapted to Figure 11 below: ABG is the meridian, DEZ the horizon, BE the celestial equator, and AD and GZ are the tropics of Cancer and Capricorn. In Proposition 1, Abū Naṣr shows that arc GZ − arc BE = arc BE − arc AD.[15] Suppose that points Z', E', D' define the end of the k-th seasonal hour on arcs GZ, BE, AD; this is to say that arc $GZ' = \frac{n}{6}$arc GZ, arc $BE' = \frac{n}{6}$arc BE, arc $AD' = \frac{n}{6}$arc AD for $n = |6 - k|$, so arc GZ' − arc BE' = arc BE' − arc AD'. Let the great circle through Z' and E' intersect arc AD at point D''. According to Proposition 1, arc GZ' − arc BE' = arc BE' − arc AD''. Thus $D'' = D'$, so the circle through Z', E' and D' is a great circle, whose stereographic projection is an hour circle.

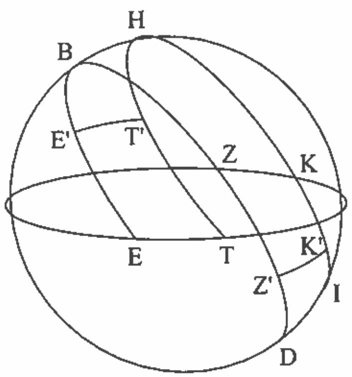

Figure 5

Proposition 2 can be used to show that the two hour circles for the n-th seasonal day and night hour belong to the projection of the same great circle on the celestial sphere. In Figure 5 (notations adapted to Figure 12), $BEDZ$ is the celestial equator, $ETKZ$ the horizon, $BHID$ the meridian, and $HTIK$ the tropic of Cancer or Capricorn. We have arc BE = arc ZD = 90°, and Proposition 2 shows that arc HT − arc BE = arc ZD − arc IK.[16] Let the great circle $E'T'$, whose projection includes the hour circle for the end of the k-th seasonal day hour,

[15] This interpretation differs from Samsó's interpretation $GZ = BE = AD$ in [14, pp. 54, 106].

[16] My interpretation differs from Samsó's interpretation $HT = BE = ZD = IK$ in [14, pp. 55, 106].

intersect arcs DZ at Z' and arc IK at K', and let $n = |6 - k|$. Then, by Proposition 2, arc HT' − arc BE' = arc $Z'D$ − arc IK'. But arc $HT' = \frac{n}{6}$arc HT, arc $BE' = \frac{n}{6}$arc BE, arc $Z'D = \frac{n}{6}$arc ZD because $BE'ZD'$ is the celestial equator, hence arc $IK' = \frac{n}{6}$arc IK. Thus the projection of the great circle $E'T'$ also includes the hour circle for the end of the k-th seasonal night hour.

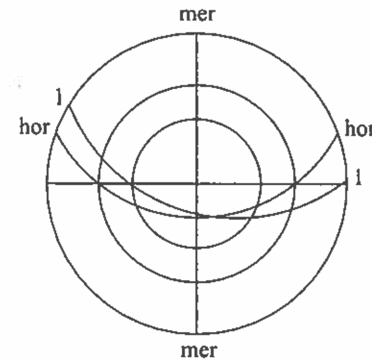

Figure 6

For a discussion of the easy proof of Propositions 1 and 2, I refer to my footnotes to the translation below. Figure 6 shows the projections on the astrolabe plate of the horizon and of the great circle which includes the hour circles for the end of the first day and night hour.

Proposition 3 implies Abū Naṣr's (negative) answer to al-Bīrūnī's question whether "many" hour circles, including the meridian and the horizon, can intersect at one point. I first explain Abū Naṣr's proof of Proposition 3 for the case of an hour circle, the meridian and the horizon. Historically, this is the most interesting case because many tenth-century mathematicians seem to have believed that the hour circles pass through the projections of the North and South point of the horizon (see Section 4 below).

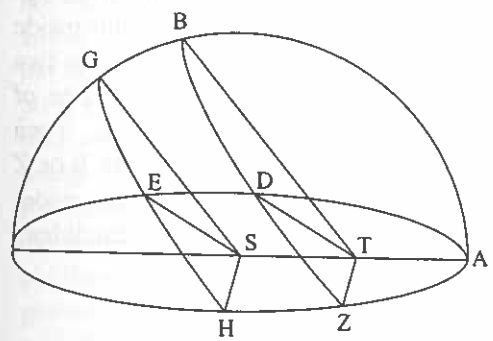

Figure 7

Figure 7 (notations adapted to Figure 13) shows part of a quadrant of the celestial sphere, with the horizon AH, the meridian AG and the celestial equator GH, whence arc GH = 90°. Let E be any point of intersection of an hour curve with the equator (thus GE and EH are non-zero multiples of 15 degrees). Let BZ be the tropic of Cancer or Capricorn, with B on the meridian and Z on the horizon as in Figure 7, and suppose that the great circle arc EA intersects arc BZ at point D. Let S be the center of the celestial sphere. Line AS meets the plane of the tropic of Cancer at a point T. Draw straight lines SH, SE, SG, TZ, TD, TB. Then $SH \parallel TZ, SE \parallel TD, SG \parallel TB$, so arc GE : arc $EH = \angle GSE : \angle ESH = \angle BTD : \angle DTZ$. For a locality with northern nonzero latitude, not in the arctic regions, T is not the center of circle BDZ. Abū Naṣr concludes $\angle BTD : \angle DTZ \neq$ arc BD : arc DZ. We will return to this step below. Thus arc GE : arc $EH \neq$ arc BD : arc DZ.

Now suppose that any hour circle on the astrolabe passes through the North and South points of the horizon. This hour circle is the projection of a great circle on the celestial sphere through a division point E of the equator. This circle must divide the tropic of Cancer at point D such that arc GE : arc EH = arc BD : arc DZ, and it must pass through point A, which is the North or South point. This yields a contradiction.

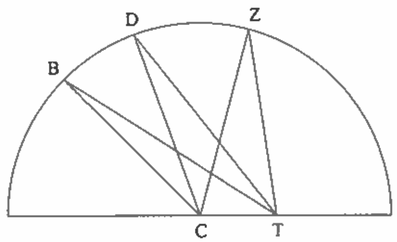

Figure 8

Abū Naṣr states his Proposition 3 for a more general case: BZ is an arbitrary circle parallel to the equator, not necessarily the tropic of Cancer or Capricorn, the numerical values of arcs GE, EH are unspecified, and all he assumes about arcs AG, AE, AH is that they are great circle arcs through A on the same side of the meridian plane, and that point A is not on the celestial axis (that is to say that the locality is not on the equator). His proof appears to be based on the following theorem: Let T be a point inside a circle, distinct from its center C. Divide the circumference in two halves by diameter TC, and let B, D, Z be three points on one of the halves such that $|BT| > |DT| > |ZT|$, as in Figure 8. Then arc BD : arc $DZ > \angle BTD : \angle DTZ$. One or both of the points B or Z can be on the diameter TC. Abū Naṣr evidently assumed that his reader (al-Bīrūnī) knew this theorem; a proof of it by elementary Euclidean geometry is possible but not trivial.[17]

[17] The theorem has the following interpretation in the context of the Ptolemaic theory of solar motion: suppose that T is (the center of) the earth, and C the center of the solar orbit. Then the mean motion of the sun is measured by arc BD and arc DZ, and its apparent motion by $\angle BTD$ and $\angle TDZ$. Since the mean solar motion is a linear function of time, the theorem $\angle BTD$: arc $BD < \angle DTZ$: arc DZ implies that the apparent solar velocity increases monotonously as the sun moves on its orbit from apogee to perigee.

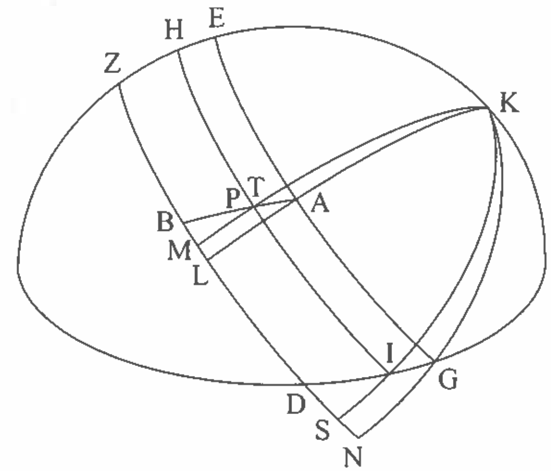

Figure 9a

Proposition 4 is the most interesting part of the *Letter on the Seasonal Hours*. In this proposition, Abū Naṣr shows that for localities not on the equator and not in the arctic regions, the hour curves on the celestial sphere cannot be circles. As a consequence, the hour lines on an astrolabe plate cannot be circles. This proposition implies the answer to al-Bīrūnī's question as to whether the hour circles are exact if the sun is not on the equator or on one of the two tropics. Figure 9a represents the celestial sphere in notations adapted to Figure 14: K is the celestial North pole, KEZ the meridian, ZD the celestial equator, EG the tropic of Cancer, and DG the horizon.[18] Suppose B and A are the division points for the end of the same seasonal (day or night) hours on the equator and the tropic of Cancer, respectively. Then arc ZB : arc BD = arc EA : arc $AG = n : (6 - n)$ for some integer n between 0 and 6. Now let HI be another circle parallel to the equator, not one of the tropics. Let the great circle through AB intersect HI at point T. In Proposition 4, Abū Naṣr proves arc HT : arc $TI \neq$ arc ZB : arc $BD = n : (6 - n)$. The seasonal hour curve through A and B intersects HI at point P such that arc HP : arc $PI = n : (6 - n)$. It follows that $P \neq T$, so the great circle AB has only three points in common with the hour curve, namely

[18] Alternatively, K can be taken as the celestial South pole, EG the tropic of Capricorn, and so on.

A, B and the point on the tropic of Capricorn. Thus, the corresponding hour circle and hour line on the astrolabe plate have only three common points as well.

Abū Naṣr's proof of Proposition 4 can be summarized as follows in modern terms:

- Through points T and I draw great circles KT and KI to meet the equator at points M and S. Then $\sin DN : \sin DS = \sin BL : \sin BM = \tan \varepsilon : \tan \delta$, where ε and δ are the declinations of circles EG and HI, respectively. In my commentary, I use the modern sine function, but the reader should bear in mind that Abū Naṣr defined the sine of any arc as a segment in a circle with fixed radius $R = 60$. Of course, the R cancels out in ratios.
- Since arc EA : arc EG = arc ZB : arc $ZD = n : 6$, also arc BL : arc $DN = n : 6$, so $BL < DN$.
- Since $BL < DN$ and $\sin DN : \sin DS = \sin BL : \sin BM$, hence $DN : DS \neq BL : BM$. Abū Naṣr states this conclusion without explanation. He and al-Bīrūnī must have known a mathematical equivalent of the following theorem: for $0 < \alpha < \beta < 90°$ and $0 < x < 1$, $\sin\beta : \sin\alpha \neq \sin x\beta : \sin x\alpha$. To apply this theorem, we take $\alpha = DS, \beta = DN, x = n/6$, so $x\beta = BL$. If we drop great circle arc PQ perpendicular to the equator, $BQ = x\alpha$. Then $DN : DS = BL : BQ$, and $\sin DN : \sin DS \neq \sin BL : \sin BQ$, so $BQ \neq BM$.

Thus, Abū Naṣr's proof is correct, but it uses a theorem which he and al-Bīrūnī must have discussed elsewhere. It would be interesting to find out more about the sources because this theorem may have inspired Ibn al-Haytham as well.

Here is the essence of Ibn al-Haytham's contributions in his unpublished treatise on hour lines [15, vol. 5, p. 368 no. 19]. Ibn al-Haytham first proves, by means of elementary Euclidean geometry, a theorem which implies the theorem used by Abū Naṣr. Ibn al-Haytham's theorem is as follows in modern notation: For $0 < x < 1$ and $0 < \alpha < \beta < 90°$, we have $x < \sin x\alpha / \sin\alpha < \sin x\beta / \sin\beta$. From the second inequality we conclude in Figures 9ab, with x, α, β as above: $\sin BQ : \sin DS < \sin BL : \sin DN = \sin BM : \sin DS$ as above, so $BQ < BM$. To utilize the first inequality, define the great circle arc BA' (with A' on the tropic EG) such that $\tan \angle A'BL : \tan \angle GDN = 6 : n$. Figure 9b displays the part of Figure 9a near the hour curve through A and B.

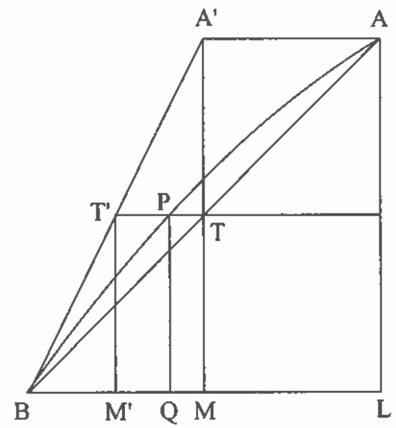

Figure 9b

Let arc BA' intersect circle HI at point T' and drop the perpendicular arc $T'M'$ onto arc ZD. Since arc IS = arc $T'M'$, we have $\sin BM' : \sin DS = \tan \angle GDN : \tan \angle A'BD = n/6 = x < \sin BQ : \sin DS$, so $BM' < BQ$. Thus, the hour curve is entirely between the great circle arcs BA and BA'. The angle between these arcs is so small that the hour curve can be identified with arc BA for most practical purposes.

Ibn al-Haytham does not work with the curves and great circles on the sphere but with their gnomonic projections on the plane of the sundial. He computes the angle between the projections of BA and BA' only for a specific latitude of approximately 33°.

In Proposition 5, Abū Naṣr discusses the construction of the centers of the hour circles on the astrolabe. As has been mentioned above, the craftsmen constructed the hour circles from their intersections with the equator and the tropics of Cancer and Capricorn. If the hour circle resembles a straight line (see Fig. 2), this construction is not very accurate, so al-Bīrūnī asked if the centers of the hour circles can also be found in another way.

Abū Naṣr argues as follows. Figure 15 below is the celestial sphere, and Figure 10 its projection on the astrolabe plate. The projection of any point P on the celestial sphere in Figure 15 is indicated by P' in Figure 10. Suppose that B is the celestial north pole, BZG the meridian, GD the celestial equator, ZH the tropic of Cancer, and let points E, H define the end of the k-th hour after sunrise. Draw the great circle through B and H intersecting the equator at D. Then arc $GE = 90 - k \cdot 15°$ and

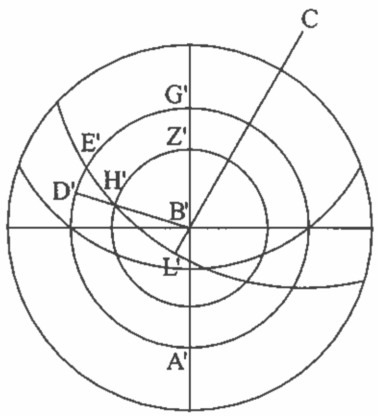

Figure 10

arc $ZH = |6-k| \cdot (15° + d)$, so arc $DE = |6-k| \cdot d$, where d is the difference between the lengths (in degrees) of one seasonal hour in the beginning of summer and one seasonal hour in the beginning of spring. We have $\sin 6d = \tan\phi \tan\varepsilon$, where ϕ is the geographical latitude. Abū Naṣr drops the perpendicular arc BL onto the hour circle. Since arc $DH = \varepsilon$ and $\angle D = 90°$, three elements in the spherical triangle EDH in Figure 15 are known, so angle H can be found. Since arc $BH = 90° - \varepsilon$ is known and $\angle L$ is a right angle, three elements in triangle BLH in Figure 15 are known, so we can also find $\angle HBL$ and arc BL. Since $\angle HBL$ is known, $\angle ABL = 180° - \angle HBL - \angle DBG$ is known.

Abū Naṣr knew that the center of the projection of the hour circle in Figure 10 is on $B'L'$, which is determined by $\angle A'B'L' = \angle ABL$, and since the distance arc BL between the pole and the circle is now known, its stereographic projection can be constructed in the same way as the projection of the horizon. Abū Naṣr and al-Bīrūnī could compute the centers of the hour lines in this way because they possessed general methods for solving spherical right triangles [4]. In this case it is unlikely that Abū Naṣr had actually worked out the details himself.[19]

[19] If he had done the computation, he would have found the following simplification. Since the arcs BE and EL are both quadrants, $\angle EBL = 90°$. Therefore $\angle ABL = 180° - \angle GBE - \angle EBL = 15k°$. If C in Figure 10 is the center of the circle through E', H' and L', and R is the radius of the projection of the equator on the astrolabe, we have $B'C = R/\tan\angle BEL = R\tan\angle LED = R\tan\varepsilon/\sin|6-k|\cdot d$.

In the following translation of the first five propositions of Abū Naṣr's letter on the hour lines, the notations **B** and **O** refer to the Bankipore and Oxford manuscripts mentioned above. Manuscript **B** was printed in [1, no. 1], and the page numbers in the following translation refer to this edition. Although **O** is a bad manuscript, it can be used to add some words and passages which were left out in **B** by scribal error. A notation such as GE [HE] means that the correct expression GE was erroneously printed in [1] as HE. The notation < ZM > means that the word ZM is missing in [1].

Translation

Letter by Abū Naṣr Manṣūr ibn ʿAlī ibn ʿIrāq, Associate of the Commander of the Faithful, to Abū 'l-Rayḥān Aḥmad ibn Muḥammad al-Bīrūnī on the Circles which Define the Seasonal Hours and on Something Related to the Construction of the Astrolabe. Page 1

In the name of God, the Merciful, the Compassionate. You asked, may God support you, about the circles drawn on the plane of the astrolabe through the beginnings of the seasonal hours,[20] and you said: is it correct to work with them for the other (declination) circles which are not drawn on the plane of the astrolabe[21] or not? And what is the proof of whichever of these statements which is correct?[22] What is the way to find the centers of these circles, other than the usual method for it by the craftsmen?[23] And you said: can many of these circles intersect at a single point or not?[24] And you reported, on the authority of Abū Muḥammad al-Sayfī[25] a method for finding the centers of the azimuthal circles and for knowing the magnitudes of their diameters, which he published without establishing a proof. You were amazed by the easiness Page 2

[20] See Props. 1, 2 below.

[21] The equator and the two tropics are the only declination circles which were engraved on astrolabe plates.

[22] See Prop. 4 below.

[23] See Prop. 5 below.

[24] See Prop. 3 below.

[25] On al-Sayfī see [15, vol. 6, p. 233]. In the *Answers to Geometrical Questions by People of Khorāsān*, al-Sijzī says that he attended a meeting at which "Shaykh Abū Muḥammad al-Sayfī" was present, so al-Sayfī must have flourished in Iran in the late tenth century AD. Manuscript: Istanbul, Reşit 1191, 113a:20, see: Al-Sijzī, *Collection of Geometrical Works*, ed. F. Sezgin, Frankfurt, Institut für Geschichte der Arabisch-Islamischen Wissenschaften, 2000, Series C, vol. 64, p. 161.

of this method, so you asked for a proof of what he mentioned.

So I was obliged to reply to you with what you asked. Here I am explaining this to you in an organized way, and I am indicating (it) at a higher level than others before me, so that it is most complete in use, and most correct with respect to (my) relation (to my predecessors).[26] With God are Power and Strength.

Page 3 Prop. 1. If there are on the sphere parallel circles and two great circles, of which one or both are inclined (non-perpendicular) to the parallel circles, then the (arcs) which they (the two great circles) cut off from any two parallel circles at equal distance from the parallel circle which is a great circle, have equal differences with the arc which they cut off from the (i.e., this) great circle.

Example (Figure 11): One or both of the great circles ABG, DEZ are inclined to the parallel circles AD, BE, GZ; of these (parallel circles) BE is a great circle and the distance of AD to it is equal to the distance of GZ to it. I say that AD, GZ have equal differences[27] with BE.

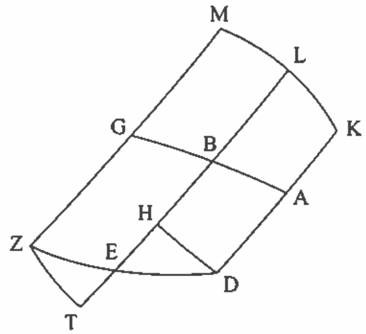

Figure 11

Proof: We draw through points D, Z two (great) circles $DH[DG]$, ZT perpendicular to the parallel circles. Since DH $[DG]$ is equal to ZT and angles H, T are equal, and the opposite angles E are equal, triangle DEH is equal to triangle ZET, and (arcs) EH, ET are equal.[28] So if

[26] I owe this translation to Samsó [14, p. 105]. I take this to mean that the propositions in Abū Naṣr's treatise are his original work, and also that he assumes that al-Bīrūnī knew the literature on hour lines by Abū Naṣr's predecessors.

[27] My interpretation of the Arabic: AD, GZ *yatakāfaʿa ʿinda* BE as "AD, GZ have equal differences with BE", meaning $|BE - AD| = |GZ - BE|$, eliminates the following mathematical problem in Samsó's interpretation $AD = BE = GZ$ [14, p. 106]. If $AD = BE = GZ$, great circle DEZ is perpendicular to the parallel circles, in contradiction with the hypothesis.

[28] Here Abū Naṣr uses Menelaus, *Spherics* I:17 [10, pp. 137-138]. In the statement and proof of this theorem, Menelaus assumes that the sum of arcs ZE, ED is not a semicircle. Among the points of intersection of the great circle through E and the two equidistant parallel circles, points Z and D are closest to E because ZE and ED are the arcs "cut off" by the parallel circles. Therefore arc $ZD < 180°$, so $ZE + ED$ is not a semicircle, unless ZED is tangent to the small circles at Z and D. Then also arc HE = arc $ET = 90°$.

ABG is perpendicular to the parallel circles, it is now clear that AD, GZ have equal differences with BE. If this is not the case, we draw (great circle) KLM perpendicular to the parallel circles. Then AK, GM have equal differences with BL, and similarly, DK, < ZM >[29] have equal differences with EL, so AD, GZ have equal differences with BE, and that is what we wanted to prove.

Prop. 2. If there are on the sphere parallel circles and two great circles, of which one or both are inclined (non-perpendicular) to the parallel circles, then the (arcs) which they cut off on the opposite sides from each small circle among the parallel circles have equal differences with the (arcs) which they cut off on the opposite sides from the great circle among the parallel circles. Page 4

Example (Figure 12): One or both of the great circles $ABGD$, $AEGZ$ are inclined to the parallel circles, (of which) $BEDZ$ is the great circle and circle $HTIK$ is one of the small circles. I say that IK, HT have equal differences with BE.

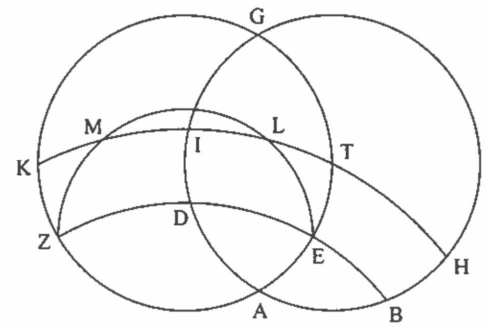

Figure 12

Proof: We draw circle $ELMZ$ perpendicular to the parallel circles. Then ZK, ET are equal <as has been proved in the previous proposition, and therefore TL, KM are equal>.[30] So if circle $ABGD$ is perpendicular to the parallel circles, TH, KI have equal differences with BE. If this is not the case, we argue as we have argued in the preceding proposition, and it will be clear that in a similar way, TH, KI have equal differences with BE.[31] That is what we wanted to prove.

[29] The word ZM is extant in MS. O 251b:11.

[30] I have supplied this passage from MS. O 251b:17–18. Arabic text: *kamā tabayyana fī 'l-shakl al-mutaqaddam wa-li-dhālika* TL KM *mutasāwiyatān* . Abū Naṣr applies Menelaus, *Spherics* I:17 to triangles ETP and ZKQ, where TP and KQ are great circle arcs perpendicular to the great circle $BEDZ$. If the great circle $AEGZ$ is tangent to the small circle $HTIK$ we cannot apply *Spherics* I:17, but the conclusions remain valid.

[31] This translation eliminates the problem which Samsó mentioned in [14, p. 107, note 14] with the interpretation $TH = KI = BE$.

Prop. 3. If there are on the sphere parallel circles and great circles intersecting at one point, and not all perpendicular to the parallel circles, then the ratios of the arcs of the great circle of the parallel circles, which (arcs) are between them (the great circles intersecting at one point) and which (arcs) are on one side of their pole,[32] are not equal to the ratios of the arcs of each of the small (circles) which fall between them (the great circles intersecting at one point).

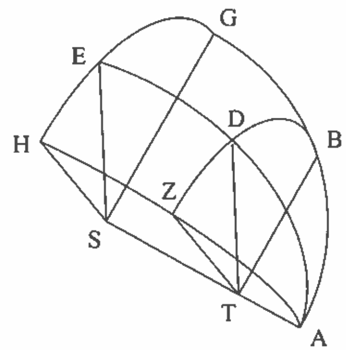

Figure 13

Example (Figure 13): The great circles ABG, ADE, AZH (cut off arcs) from two circles GEH, BDZ $[MDZ]$ in the way we have mentioned, and
Page 5 GEH is the great circle. I say that the ratio of GE to EH is not equal to the ratio of BD to DZ.

Proof: We draw the line common to the three (great) circles to the center of the sphere, and let it be AS. Let it intersect the plane of circle BDZ at T. We draw the straight lines SG, SE, SH, TB, TD, TZ. Since points $T, B, S, G[H]$ are in the plane of circle ABG, and intersect the parallel circles GEH, BDZ at lines SG, TB, therefore lines SG, TB are parallel, and similarly, lines TD, SE are parallel and lines TZ, $SH[SG]$ are parallel. Therefore angles BTD, GSE are equal, and angles DTZ, ESH are equal. But point S is the center of circle $\langle GEH \rangle$,[33] and point T is not the center of circle BDZ, and none of the lines BT, DT, ZT is on the other side with respect to its pole.[34] So the ratio of GE to EH is equal to the ratio of angle GSE to angle ESH, and the ratio of BD to DZ is not equal[35] to the ratio of angle BTD to angle DTZ. Thus, the ratio of GE to EH is not equal to the ratio of BD to DZ, and that is what we wanted to prove.

[32] The text is unclear. Abū Naṣr means that the parallel circles are intersected by 180 degree arcs of great circles, beginning at a single point A, and that these great circle arcs are on the same side of the great circle through A and the pole of the parallel circles.

[33] Samsó's emendation GEH [14, p. 107] is confirmed by MS. O 252a:6 HEH.

[34] I interpret this statement to mean that arcs BT, DT, ZT are on one side of the diameter through T of circle BDZ. This diameter is in the plane through A, the center of the sphere, and the pole of the parallel circles.

[35] Here Abū Naṣr assumes the theorem of Figure 8 above.

If ADE is perpendicular to the parallel circles, and angles EAG, Page 6
EAH are equal, then GE, EH are equal, and similarly BD, DZ, that is, the two angles E are equal, and similarly the two angles D, and triangle AEH is equal to triangle $AEG[AEH]$, and triangle ADZ is equal to triangle ADB.[36]

Prop. 4. If there are on the sphere parallel circles, and two great circles inclined to them cut (from) the great circle of the parallel circles and (from) one of the small circles (arcs) between the two (inclined great circles) and one of the circles perpendicular to the parallel circles, (the arcs being) on the same side of it (the perpendicular circle) and according to the same ratio, then they do not cut off from the other parallel circles which are not equal to that small circle (arcs) according to that ratio.

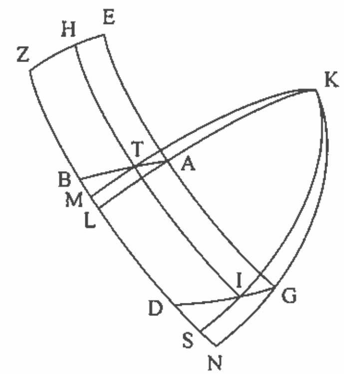

Figure 14

Example (Figure 14): the great circles AB, GD are inclined to great circle BD and circle AG parallel to it, and they cut (from) them (arcs) between them and circle EZ perpendicular to the parallel circles according to the same ratio, and the two inclined circles are on the same side of the two[37] perpendicular circles. I say that they do not divide circle HTI,[38] which belongs to the small circles, according to that ratio.

Proof: We draw through the pole of the parallel circles and points A, T, G, I great circle arcs KAL, KTM, $KGN[KHN]$ and $KIS[KSN]$. Since the angles L, M in triangles Page 7
LBA, MBT are equal and angle B is common, the ratio of the sine of LB to the sine of MB is equal to the ratio of the sine of AL to the sine of TM compounded with the ratio of the sine of angle LAB to the sine of angle MTB.[39] Again, similarly in triangles NDG, SDI, the ratio of

[36] The last paragraph shows that the theorem is not valid if the intersecting great circles are perpendicular to the parallel circles.

[37] Abū Naṣr refers to arc EZ and the arc diametrically opposite it.

[38] The emendation of the word *khaṭay* in [1] to HTI solves the problem mentioned in [14, p. 108 fn. 30]

[39] This can be seen by three applications of the sine theorem, where R is the radius of the circle which Abū Naṣr uses to define his sine function: $\sin TM : \sin AL = \sin BT : \sin AB$, $\sin \angle LAB : R = \sin LB : \sin AB$, and $\sin \angle MTB : R = \sin MB : \sin BT$.

the sine of ND to the sine of DS is equal to the ratio of the sine of NG to the sine of SI compounded with the ratio of the sine of angle NGD to the sine of angle SID. But $NG[BG]$ is equal to AL and SI is equal to MT, so the ratio of the sine of $NG[BG]$ to the sine of SI is equal to the ratio of the sine of AL to the sine of MT. Again, the ratio of the sine of angle $NGD[BHD]$ to the sine of angle $SID[SD]$ is equal to the ratio of the sine of angle LAB to the sine of angle MTB, since KT is equal to KI and KA is equal to KG.[40] So the ratio of the sine of $ND[BD]$ to the sine of DS is equal to the ratio of the sine of LB to the sine of BM.[41]

But the ratio of $NZ[BZ]$ to ZL is equal to the ratio of DZ to $ZB[DB]$, so the ratio of $ND[BD]$, the remainder, to LB, the remainder, is equal to the ratio of DZ to ZB. Therefore[42] ND is greater than BL.

But the sines are proportional as we have shown.[43] So the ratio of $ND[BD]$ to DS is not equal to the ratio of LB to $BM[LM]$.[44] *Alternando*,[45] the ratio of $ND[BD]$ to LB is not equal to the ratio of DS to $BM[LM]$. So the ratio of DS to $BM[LM]$ is not equal to the ratio of DZ to ZB, so the ratio of SZ to $ZM[DM]$ is not equal to the
Page 8 ratio of DZ to ZB, so the ratio of $IH[BH]$ to $HT[GT]$ is not equal to the ratio of DZ to ZB,[46] and that is what we wanted to prove. In

40 We have $\sin \angle NGD : \sin \angle SID = \sin \angle KGI : \sin \angle KIG = \sin KI : \sin KG$, and similarly $\sin \angle LAB : \sin \angle MTB = \sin KT : \sin KA$.

41 This part of the proof can be simplified if we apply the theorem of Menelaus in the form: $\sin LB : \sin BM = (\sin LA : \sin AK) \cdot (\sin KT : \sin TM)$ and similarly $\sin ND : \sin DS = (\sin NG : \sin GK) \cdot (\sin KI : \sin IS)$. If δ_1 and δ_2 are the declinations of circles AG and ITH with respect to the equator BD, we obtain $\sin LB : \sin BM = \sin ND : \sin DS = \tan \delta_1 : \tan \delta_2$.

42 Samsó [14, p. 109] translates this paragraph as follows: "But the ratio of BZ to ZL is equal to the ratio of DZ to $NZ[DB]$, and the ratio of BD, the difference, to $LN[BL]$, the difference, is equal to the ratio of DZ to $ZN[ZB]$, and ND is greater than BL." My interpretation explains the word "therefore" *fa-* in the text. The corresponding passage in manuscript O 252b:5–6 is corrupted. In almost all other instances in Prop. 4, I follow Samsó's corrections. Note that Samsó's point H corresponds to my point G and vice versa. The labels H and G are often indistinguishable in Arabic geometrical texts.

43 Abū Naṣr means $\sin ND / \sin DS = \sin LB / \sin BM$.

44 Here Abū Naṣr assumes a theorem on sines, see the commentary above.

45 The word *alternando* indicates the following operation in Heath's translation of Book V of Euclid's *Elements* [6, vol. 2]: $a : b = c : d \rightarrow a : c = b : d$.

46 Since $DS : DN < BM : BL$, we have $ND : DS > BL : BM$, so $DZ : ZB = ND : LB > DS : BM$, hence $DZ \cdot BM > DS \cdot ZB$. By addition of $DZ \cdot ZB$,

what we have answered about the properties of these circles is sufficient explanation, in accordance with your familiarity with this science.

Prop. 5. This is how the centers of these circles can be found in a way different from the usual way of the craftsmen:

(Figure 15) Let ABG be one of the circles perpendicular to the parallel circles, let GD be their great circle, and let ZH be one of the parallel circles at a known distance of GD. Let circle HE be the circle of which (i.e., of whose projection) we want to find the center, and let ZH and $GE[GD]$[47] be assumed. We draw through the pole of the parallel circles and through point $H[G]$ the great circle BHD. We also drop from B onto circle EH the perpendicular (arc) BL. Then, since each of DE, $DH[ZH]$ is known and angle D is known,[48] triangle DHE is known in shape,[49] <so angle H is known. But angle L is assumed (to be a right angle), and BH is known>,[50] so the sides and the shape of triangle BLH are known. Angle B is assumed, so by subtraction[51] angle ABL is known. So on line BL which is known in position in the plane of the astrolabe,[52] we look for the center of the circle whose

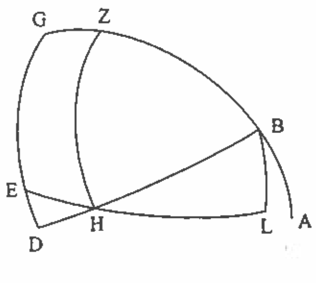

Figure 15

$DZ \cdot ZM > SZ \cdot ZB$, so $DZ : ZB > SZ : ZM = IH : HT$. Most of the corrections to the text of [1] are supported by MS. O.

47 I have emended the text following MS. O, 252b:13. The emendation is required by the mathematical context; point D is defined by the next sentence.

48 DE is the difference between the known arcs ZH and GE, DH is the given distance between the parallel circles DE and ZH. Samsó reads: "since all (points) D, E, L, H are known" [14, p. 110].

49 Abū Naṣr probably means that the angles of the triangle are known. The notion "known in shape" is strange for spherical triangles, because the size of such a triangle is determined by its angles.

50 This passage *fa-zāwiyat H maʿlūma wa-zāwiyat L mafrūḍa wa-BH maʿlūm* in MS. O is missing from the text in [1].

51 The "assumed" angle B is probably $\angle GBD$, which is measured by the given arc ZH. For seasonal hour lines, the computation of $\angle ABL$ can be simplified: if EH is the seasonal hour circle for $k < 6$ hours before or after sunrise or sunset, $\angle LBA = 15k°$, see the commentary above.

52 Abū Naṣr identifies arc BL on the sphere with its stereographic projection on the astrolabe plate.

distance from a (i.e., the) pole is the known arc BL. Thus, we find it.[53] The centers of the other circles are found the same way. That is what we wanted to find.

What you said on the authority of al-Sayfī is correct, and the proof is this, which I will now mention: ...[54]

4. *The contribution by al-Ṣaghānī*

Chapter 1 of al-Ṣaghānī's treatise on hour lines is entitled "The first Chapter of what Abū Ḥāmid Aḥmad ibn Muḥammad ibn al-Ḥusayn al-Ṣaghānī said on the hours which are constructed on the plates of the astrolabe."

Al-Ṣaghānī gives two reasons why many people of his time erroneously believed that the hour circles pass through the North and South points of the horizon: 1. In the astrological work *Tetrabiblos* III:10, Ptolemy assumed that the great circles through the North and South points of the horizon are approximately hour lines, so he had given a wrong example; 2. On an astrolabe plate for latitude $\phi = 90° - \varepsilon$, where $\varepsilon \approx 23°35'$ is the obliquity of the ecliptic, the night hour circles pass through the North point. According to al-Ṣaghānī this is incorrect. He then proves that the hour circles will only pass through the projection of the North point at localities on the equator (and not in the arctic regions). So for any locality with nonzero geographical latitude, not all hour circles pass through the projection of the North point. In his proof, al-Ṣaghānī tacitly assumes that the locality is not on the polar circle or in the arctic regions. Finally, he gives an incorrect discussion of the hour circles for localities on the arctic circle.

I now explain the details of al-Ṣaghānī's most important proof with reference to his second figure.

In Figure 16, N is the center of the celestial sphere, A and D are the North and South points of the horizon,[55] $ABGD$ is the plane of the local meridian, $GLKTH$ the plane of the celestial equator and $BFOSZ$ a

[53] The method is mathematically identical to finding the center of the stereographic projection of the horizon on the astrolabe plate.

[54] In the remaining Propositions 6-9, Abū Naṣr proves a simple method for finding the centers and radii of the projections of the azimuthal circles on the astrolabe plate. His explanation is as concise as in Propositions 1-5.

[55] We can take either A or D as the North point of the horizon, and thus obtain a proof for day or night hours, respectively.

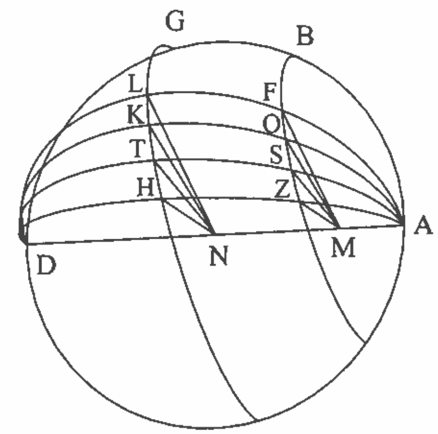

Figure 16

plane parallel to the equator. Line AD intersects the plane of $BFOSZ$ at point M. Take three successive equal arcs HT, TK, KL on the celestial equator. The four great circles through A, D and the four points H, T, K and L intersect circle $BFOSZ$ at points F, O, S, Z as in Figure 16. Al-Ṣaghānī supposes that $AZHD$ is the horizon. Draw lines NH, NT, NK, NL and MZ, MS, MO, MF. Because HT, TK, KL are equal arcs of a circle with center N, we have $\angle HNT = \angle TNK = \angle KNL$. Because the planes of circles $GLKTH$ and $BFOSZ$ are parallel, $HN \parallel ZM, TN \parallel SM, KN \parallel OM, LN \parallel FM$. Thus, point M is inside circle $BFOSZ$, in such a way that $\angle ZMS = \angle SMO = \angle OMF$.

Al-Ṣaghānī must have known that the hour circles are projections of great circles on the celestial sphere, see Section 2 above. Now suppose that all hour circles pass through the North and South points A and D of the horizon. Al-Ṣaghānī does not specify the numerical value of his arcs GL, HT, TK, KL and the position of circle $BFSOZ$, but he evidently was thinking about the case where $HT = TK = KL = 15°, GL = 30°$, and $BFOSZ$ is the tropic of Cancer or Capricorn; al-Ṣaghānī tacitly assumes that the tropics meet the horizon. The projections of great circles $AFLD, AOKD, ASTD$ on the astrolabe must be hour circles, so arcs ZS, SO, OF must also be 15 degrees. Al-Ṣaghānī concludes that M is a point inside[56] circle $BFOSZ$ such that $\angle ZMS = \angle SMO = \angle OMF$ and arc SZ = arc SO = arc OF. In his first proposition, al-

[56] This is true if the locality is not on the arctic or antarctic circle.

Ṣaghānī considers this situation in a different notation, and he proves that, in the notation of Figure 16, point M must be the center of circle $BFOSZ$. Thus, AD must be the celestial axis, which is only true at localities on the terrestrial equator. Al-Ṣaghānī concludes that the three arcs ZS, SO, OF cannot be equal at a locality which is not on the terrestrial equator. Thus, if AZD is the Eastern horizon, the four great circles $AFLD$, $AOKD$, $AST\,D$ cannot all be identical to the great circles whose projections are the hour circles for the end of the first, second and third seasonal hour. Thus, these hour circles cannot all pass through the North and South points of the horizon. Al-Ṣaghānī's proof does not exclude the possibility that any individual hour circle (for example, for the end of the first seasonal hour) passes through the North and South points of the horizon. In Section 3, we saw that A and D do not lie on any of the hour circles.

Chapter 1 of al-Ṣaghānī's text has come down to us in the manuscript Oxford, Bodleian Library, Thurston 3, f. 119b, in a collection of writings on the astrolabe. A defective copy of this manuscript is Oxford, Bodleian Library, Marsh 713, 238b–239b, with figures on f. 237b. I have consulted this copy but I have not listed the numerous scribal errors. Words which I have restored to the text are in angular brackets. My explanatory additions to the translation are in parentheses.

Arabic text

الفصل الأول من كلام أبي حامد أحمد بن محمد بن الحسين الصغاني في الساعات المعمولة على صفائح الأصطرلاب وهو البرهان على أنّ الدوائر العظام المارّة بالفصلين المشتركين بين أفق البلدة ونصف نهارها ليست هي دوائر الساعات المعوجة على ما ظنّ كثير من الناس.

وأظن أنّ الذي هداهم إلى الرأي ما يوجد في كتاب الأربع مقالات لبطلميوس في باب التسييرات وما يرى على سطح الأصطرلاب لعرض تمام الميل وهذا ظنّ فاسد.

فليكن لبيانه دائرة ابجد وقد خرج من ز فيها زه زد زج زب وصارت قسي هد دج جب وزوايا هزد دزج جزب < متساوية > · ف ز مركزها إذ نخرج زد إلى اب ونصل اه اج فلتساوي هد دج تتساوى زاويتا ا في مثلثي هاز جاز وكذا زاويتا هزا جزا و از مشترك · ف ها ك جا وكذا قوساهما و هد ك دج ف دا

قطر وكذا يبين أنّ زج قطر ف ز مركزها وهو المطلوب· وهذا الحكم واجب إن فرضت القسي والزوايا أكثر من ثلاثة ·

ليكن ابجد نصف نهار للدنيا و حج نصف المعدل و بز مواز له ونفرض هط طك كل متساوية وكذا زس سع عف متساوية ونرسم عظام ازحد اسطد اعكد افلد· أقول ف ا د قطبا المعدل ·

برهانه ليكن ن مركز الكرة فمن البين أنّ سطوح تلك العظام تمرّ ب ن فليكن ان الفصل المشترك بينها وانّ ان يمرّ بسطح زب فليكن الفصل بينهما م · ونصل زم سم عم فم حن طن كن لن· فلتوازي السطحين تتوازى تلك الخطوط ولتساوي زوايا هنط طنك كنل لتساوي هط طك كل وكون ن مركز الكرة تتساوى زمس سمع عمف· و زس سع عف متساوية ف م مركز مدار زف و ن مركز الكرة ف ان محورها ف ا د قطبا المعدل ·

فمن بعد ما بينا هذا نفرض ازحد أفق بلد نصف نهارها ابجد ونفرض حج بزمن المعدل والمدار منقسمة بأقسام متساوية وتمرّ بها الدوائر العظام كما في هذا الشكل لأنّه كذا يدعى· فيلزم أن يكون ا د قطبي المعدل وهذا خطأ لأنّه يدعى أن هذا عام لسائر العروض فيصير بحيب هذا الدعوى لمعدل النهار أقطاب كثيرة وهذا محال· فليست هذه هي الدوائر التي سطحت على سطح الأصطرلاب لما قسمت ما تحت الأفق من دائرة معدل النهار ومداري الجدي والسرطان باثني عشر قسمًا إلا في البلدان التي لا عرض لها وذلك ما أردنا أن نبين ·

وإن يجعل مدعي حجته في هذا الدعوى ما عدّ من عمل الساعات في عرض تمام الميل فيكون الخطأ أعظم· فنفرض ابجدهز نصف النهار بتمام الميل و زج من المعدل و احب هد نصف مداري الجدي والسرطان فأفق ذلك العرض عاس مداري احب هد على ا د · فليكن نصف الأفق اكد و نجعل كط سدس كج و اح سدس اب وقد جرت عادة الأصطرلابيين أنهم يعملون دائرة تمرّ بنقط ح ط د فإن كانت تلك الدائرة عظيمة تبين أنها تمرّ بنقطة ا فيلزم بحسب ما بيّنّا قبل أن يكون ا د قطبي المعدل فالقوس التي تمرّ بنقطة ح ليست من الدوائر العظام وذلك ما أردنا أن نبين ·

Translation

The first Chapter of what Abū Ḥāmid Aḥmad ibn Muḥammad ibn al-Ḥusayn al-Ṣaghānī said on the hours which are constructed on the plates of the astrolabe. It is the proof of (the fact) that the great circles through the two intersections of the horizon of the locality and its meridian are not the circles of the seasonal hours, as many people think.

I think that they adopted this view because of what is found in the *Tetrabiblos* by Ptolemy in the Chapter on progressions,[57] and what is seen on the plane (i.e., plate) of the astrolabe for latitude (equal to) the complement of the (maximal ecliptic) inclination,[58] but this view is false.

(Prop. 1, Figure 17) For the proof of this, let there be a circle $ABGD$. From point Z inside it, ZE, ZD, ZG, ZB have been drawn so that arcs ED, DG, GB and angles EZD, DZG, GZB are <equal>. Then Z is the center of it. For we extend ZD to AB, and we join AE, AG. Then, since ED, DG are equal (arcs), angles A in triangles EAZ, GAZ are equal, and similarly angles EZA, GZA (are equal), and AZ is common. So (segment) EA is equal to (segment) GA, and also their arcs (are equal). But (arc) ED is equal to (arc) GD, so DA is a diameter. Similarly, ZG is proved to be a diameter. Thus, Z is its center, and that is what was required. This theorem is (also) valid if the arcs and angles are supposed to be more than three.

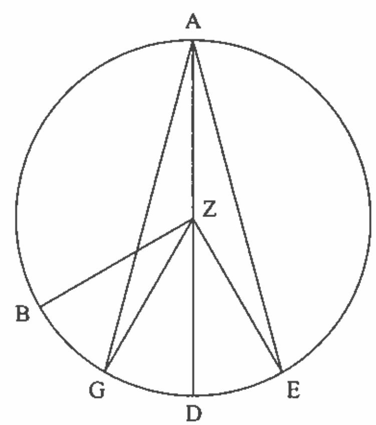

Figure 17

[57] The "I" in this sentence is probably al-Ṣaghānī. The theory of astrological progressions, explained by Ptolemy in *Tetrabiblos* III:10 [13, pp. 270–307], is based on great circles through the North point N and the South point S of the local horizon. For any point P on the ecliptic, Ptolemy has to compute the intersection P' of great circle NPS with the celestial equator. In his computation, Ptolemy assumes that the great circle NPS is approximately identical to the seasonal hour curve through P. For further details, see [9].

[58] In the early ninth century, the astronomers of Caliph al-Ma'mūn found $\varepsilon = 23°35'$ for the obliquity of the ecliptic, and al-Ṣaghānī also observed this value himself in Baghdād in AH 374 [2, p. 69]. At localities with northern latitude $90° - \varepsilon$, the tropic of Cancer is tangent to the local horizon at the North point, so all hour circles on the astrolabe do indeed pass through the North point and the South point, even though al-Ṣaghānī attempts to refute this view in his last proposition.

(Prop. 2, Figure 16) Let $ABGD$ be a meridian of the universe, GH half[59] of the equator, and BZ parallel to it. We assume HT, TK, KL equal and, similarly, ZS, SO, OF equal, and we draw great circles $AZHD$, $ASTD$, $AOKD$, $AFLD$. I say: then A, D are the poles of the equator.

Proof of this: Let N be the center of the sphere. It is clear that the planes of these great circles pass through N, so let their intersection be AN,[60] and that AN passes through plane ZB, so let their common (point) be M. We join ZM, SM, OM, FM, HN, TN, KN, LN. Since the two planes (of circles GH and BZ) are parallel, those lines are parallel. Since angles HNT, TNK, KNL are equal — because HT, TK, KL are equal and N is the center of the sphere — angles ZMS, SMO, OMF are equal, but ZS, SO, OF are equal, so (by Prop. 1) M is the center of orbit ZF. But N is the center of the sphere, so AN is its axis, so A, D are the two poles of the equator.

(Prop. 3, Figure 16) After we have proved this, we assume that $AZHD$ is the horizon of a locality whose meridian is $ABGD$, and we assume that (arcs) GH, BZ of the equator and the orbit (i.e., circle parallel to the equator) have been divided into equal parts, and that great circles pass through them as in this figure, because it was alleged to be this way. Then it is necessary that A and D are the two poles of the equator, but this is false since this was alleged to be (a) general (truth) for the other (nonzero) latitudes. So this assertion has as its consequence that the equator has many (different) poles, but this is impossible. So these are not the circles which have been projected on the plane (i.e., plate) of the astrolabe if the parts of the equator and the tropics of Capricorn and Cancer under the horizon are divided into twelve equal parts, except at localities which have no (i.e., zero) latitude. That is what we wanted to prove.

(Prop. 4, Figure 18) If someone who makes this claim bases his argument on what was considered (to be true) in the construction of the hours (i.e., hour lines) for a latitude of the complement of the (maximum ecliptical) inclination, then the mistake is even greater. We assume that

[59] The word "half" *nisf* in the text is strange because GH is supposed to be one quadrant of the equator in Proposition 3 below.

[60] The planes of the four great circles through A and D have a common intersection AD by hypothesis.

$ABGDEZ$ is the meridian for (a locality whose latitude is) the complement of the inclination, ZG is (part) of the equator, and AHB, ED are half of the tropics of Capricorn and Cancer. Then the horizon for that latitude is tangent to the two orbits (parallel circles) AHB, ED, at A, D. Let half of the horizon be AKD, and we make KT one-sixth of KG and AH one-sixth of AB. The usual method of the makers of astrolabes is that they construct a circle through points H, T, D. If this circle were a great circle, one could prove that it passes through point A, so it would be necessary, as we have proved before, that A, D are the two poles of the equator.[61] Thus, the arc which passes through point H is not a great circle.[62] That is what we wanted to prove.

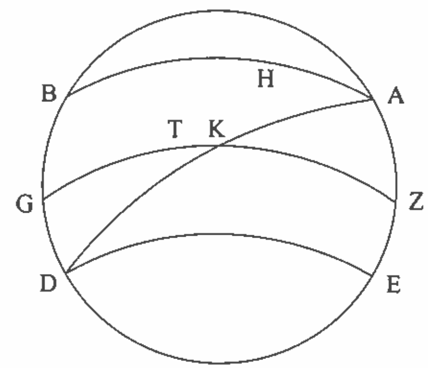

Figure 18

Acknowledgements

I am grateful to Prof. Dr. F. Sezgin for making manuscripts available to me and for the suggestion that I write this article, and to Prof. Dr. Karen Parshall and Dr. Benno van Dalen for their comments on a preliminary version. This article was written during my stay at the Department of Mathematics of the University of Virginia, Charlottesville, Va., USA.

[61] This argument is false. Proposition 1 is not valid for points on the circumference of the circle because in Figure 17 $\angle EAD = \angle DAG = \angle GAB$ even though A is not the center of the circle. Thus, if point M in Proposition 1 is on circle $FOSZ$, it does not follow that point M is the center of that circle, and points A and D need not be the celestial poles.

[62] As a matter of fact, the arc through D, T, H is part of a great circle through A; the proof in Abū Naṣr's Proposition 1 above is valid in this case.

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