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A MEDIEVAL ARABIC TREATISE ON MENSURATION BY QĀḌĪ ABŪ BAKR

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1. Introduction and summary

The Istanbul manuscript Fatih 3439 contains on f. 182b–189b an Arabic treatise on mensuration by the qāḍī (judge) Abū Bakr. The treatise has been listed in Krause[1] and Sezgin[2], but it has not been investigated until the present. Even the correct title of the treatise, *Book of the Levels in the Explanation of the Measurements* (Kitāb al-ṭabaqāt fī sharḥ al-misāḥāt), has not been generally known, because it appears only in the middle of the preface. We know nothing about the identity of the author other than that he wrote the treatise before the end of the twelfth century A. D, when the extant manuscript was copied[3].

In 1968 H. Busard[4] published a Latin translation by Gerard of Cremona of an Arabic text, called the *Liber Mensurationum*, and also written by "Abū Bekr". Sezgin[5] raised the question whether there is any connection between the two texts. We will answer this question in this paper.

Section 2 of this paper is a survey of the mathematical part of the *Book of the Levels in the Explanation of the Measurements.* In Section 3 I compare this text with the *Liber Mensurationum*, and

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[1] *Stambuler Handschriften*, p. 515. Complete references can be found in the bibliography.

[2] GAS V, p. 386.

[3] The date and place are found on f. 182a, at the end of the preceding treatise, dealing with magic squares; see the end of Section 3 of this paper.

[4] In l'*Algèbre au moyen âge.*

[5] GAS V, p. 390.

I argue that the two texts are unrelated. It is even likely that the authors were different.

Section 4 of this paper includes a translation of the first mathematical section, on the definition of a point. In Section 5 I have rendered the Arabic text of the beginning of the treatise (preface, table of contents and the part translated in Section 4) as well as I could. The manuscript is extremely difficult to read, and there are a number of passages that I have not been able to decipher. Because of the verbosity of the author, the general meaning of the preface is clear, and almost all of the mathematical argument can be reconstructed in detail.

At the present state of research a complete edition of the *Book of the Levels in the Explanation of the Measurements* may not be worth the effort. However, a summary of the text, such as given in this paper, seems to be useful for the following reasons.

Not much is known about the tradition of practical mathematics in Islamic civilization, but there is abundant unpublished source material available. Thus one could in principle obtain a detailed knowledge of this tradition, which is conceptually rather distant from modern mathematics. The first priority is in my opinion to obtain an overview of the existing texts, that is to say not only a list of titles, but also a description of the contents, and a characterization of each text. This paper is a small contribution towards this goal. In a later stage one can decide which texts and which themes have to be studied in more detail.

Regarding the interest of the *Book of the Levels in the Explanation of the Measurements*, the following can be said. The work contains a few interesting mathematical statements, such as: the three altitudes of a triangle pass through one point (II:8), an approximation formula for the radius of a circle circumscribing a regular polygon (III:8), and numerical examples based on the unusual value $\pi = 3.2$ (III:10). However, the author does not give mathematical proofs. The main interest of the text seems to be its systematic classification of the figures, and the geometrical terminology that was used, especially for solids. Translations of these geometrical terms can be found in the titles of the chapters in Section 2 of this paper, and the Arabic titles are in Section 5. The philosophical first sections (on the definition of a point, etc.) are also interesting, and rather unusual for a treatise of a predominantly practical nature.

I conclude this introduction by a summary of the preface of the *Book of the Levels in the Explanation of the Measurements*.

The text begins as follows: "Qāḍī (judge) Abū Bakr said:"

The author then says that some "brethren" (ikhwān) suggested a discussion of the measurement of figures, together with examples, in which the fundamental figures are treated systematically and in a simple and correct wording. In response to this request he wrote the present work, and he took care to make it concise, so that it is easily mastered. The work contains not only a large number of technical terms, but also "fine tricks and subtle curiosities", which will lead the student to other knowledge, which distinguishes a human being from an animal. The book has various uses, and the intelligent student will quickly master its contents. The science of measurement is a noble science, because it is a proved science and its assertions are correct and verified. It is useful in connection with financial and legal matters, such as inheritances and testaments (several examples are mentioned). The student should master a section completely before proceeding to the next, because the sections are logically dependent.

The author then says "I called it the *Book of the Levels in the Explanation of the Measurements.*" He then gives a table of contents, which I do not render here, because translations of all titles of the chapters can be found in the next section of this paper. The Arabic text of the table of contents is to be found in Section 5 of this paper.

He then says that "each section contains the necessary definitions, proofs, synthesis, analysis and reasons"[6]. He adds that the book is based on writings of the scholars and tricks of the experts in the field, and he mentions the "Book of Elements known as Euclid" (Kitāb al-Ustuqiṣāt al-ma'rūf bi-Ūqlīdis) in this connection. This concludes my summary of the preface of the *Book of the Levels in the Explanation of the Measurements*.

[6] Here the words analysis and synthesis are not to be taken in a technical geometrical sense. See for a similar occurrence of these words Ibn al-Akfānī, *Irshād al-Qāṣid*, ed. Witkam, no. 141–145, p. 17.

2. Survey of the mathematical part of the *Book of the Levels in the Explanation of the Measurements*

The *Book of the Levels in the Explanation of the Measurements* is divided into four "levels" (ṭabaqāt). These levels are in turn divided into sections (fuṣūl). A notation such as IV:3 indicates the third section of the fourth level. I now render the full titles of each level and each section, together with a reference to the line in the manuscript where the level or section begins, and a summary of the contents of the section. The titles are rendered twice in the manuscript, at the beginning of each section and in the table of contents in the preface. The table of contents appears in Section 5, but here I have translated the titles from the headings of the sections. The two occurrences of one title are sometimes slightly different, for example III:10 is in the table of contents: "On the arcs and their areas" and in the heading of the section (187a:13) "On the knowledge of the arcs an the description of their areas". Such insignificant deviations can be reconstructed by comparing the table of contents with the translation. The few significant differences between the two versions of the titles will be noted below.

I (183a:3–4). "The first level, including the circumstances of the point, the line, the surface and the solid. It is in four sections."

I:1 (183a:4). "The first section, on the knowledge of the point." See Section 4 of this paper for a literal translation.

I:2 (183a:18). "The second section on the knowledge of the line." The line is created by the motion of a point. Several definitions and properties of a straight line are explained. Then the author defines seven kinds of straight lines: the perpendicular ('amūd), base (qā'ida), arm (sāq), side (jānib), diagonal or diameter (quṭr), chord (watar) and arrow (sahm). He then mentions the circle, the arc and the curved line (al-khaṭṭ al-munḥanā), on which he does not give any further information.

I:3 (183b:14). "The third section on the knowledge of the surface." This section includes definitions of a plane (basīṭ mustawā or saṭḥ), a concave (muqa''ar) and a convex (muḥaddab) surface.

I:4 (183b:23). "The fourth section on the knowledge of the solid." The solid is the convexity of a surface, and it is the most complete magnitude.

II (183b:38) "The second level, including the circumstances of triangles, their properties, the derivation of the lines from each other, and the discussion of their measurements."

II:1 (184a:1). "The first section of the knowledge of the kinds of figures and angles." The text mentions rectilinear and non-rectilinear figures; triangles, quadrilaterals, circular figures and figures consisting of arcs; equilateral, isosceles and scalene triangles, right-angled, obtuse-angled and acute-angled triangles. The sum of any two sides of a triangle is greater than the remaining side. Only one angle of a triangle can be right or obtuse.

II:2 (184a:16). "The second section, on giving examples of the three angles." The author presents figures of acute, right and obtuse angles.

II:3 (184a:29). "The third section, on the knowledge of the right angle in another way." If in triangle ABG $AB^2 + BG^2 = AG^2$, $\angle B$ is a right angle. Example: AB = 6, BG = 8, AG = 10.

II:4 (184a:37). "The fourth section, on the knowledge of shapes in the two right-angled triangles." The two kinds are the scalene right-angled triangle (such as ABG) and the isosceles right-angled triangle (such as EDZ with ED = 10, DZ = 10, EZ = "root twohundred" (jidhr mi'atān).

II:5 (184b:4). "The fifth section, on the areas of both right-angled triangles." The area is half the product of the two sides containing the right angle. The author remarks that the scalene right-angled triangle is half of a rectangle, and that the isosceles right-angled triangle is half of a square.

II:6 (184b:29). "The sixth section on the knowledge of the circumstances of acute-angled triangles." The author distinguishes between equilateral, isosceles and scalene triangles. The area is half the product of the base and the altitude. He refers to the next sections for the computation of the altitude.

II:7 (184b:33). "The seventh section on the derivation of the altitude in the equilateral and isosceles triangles, and the discussion of their areas." If AD is the altitude in triangle ABG and AB = AG, then $AD^2 = AG^2 - (\frac{BG}{2})^2$. Examples: AB = AG = 10, BG = 12, AD = 8; AB = AG = BG = 10, $AD = \sqrt{75}$. He remarks that this quantity is irrational,

"so the answer is the same as the question, so if it is said (by someone) how much is root 75, it is said to him: root 75, or something that approximates it, but the approximation is not exact" (185a:13–15). The author does not explain how square roots can be approximated.

II:8 (185a:16). "The eighth section, on the knowledge of the derivation of the foot of the altitude and of the altitude in the acute-angled scalene triangle, the knowledge of the shapes of acute-angled triangles, and the discussion of their areas."

The rule is explained with reference to an acute-angled triangle KLD with KL = 15, KD = 13 and LD = 14. For sake of simplicity of expression I introduce the notation X for the foot of the altitude drawn from K. One computes $\frac{KL^2 - KD^2}{2LD}$, which is the distance from X to the midpoint of LD. In the example this distance turns out to be 2, so that XL = 9 and XD = 5. The altitude KX = 12 can now be computed by means of the theorem of Pythagoras (Abū Bakr does not mention the name Pythagoras). The author also says in general terms that $LX^2 = \frac{LD^2 + KL^2 - KD^2}{2LD}$, and that the three altitudes of an acute-angled triangle pass through one point located inside the triangle.

II:9 (185b:5). "The ninth section, on the knowledge of the obtuse-angled triangles, and the discussion of their shapes." The isosceles and scalene obtuse-angled triangles are discussed, and he gives the example of a triangle with sides 10, 10 and 16.

II:10 (185b:22). "The tenth section, on the knowledge of the derivation of the altitude in the isosceles obtuse-angled triangle, and the discussion of its area." This is easy, like II:7.

II:11 (185b:33). "The eleventh section, on the knowledge of the derivation of the foot of the altitude and the altitude in the obtuse-angled scalene triangle, and the discussion of its area." The method, which is analogous to II:8, is illustrated by means of the triangle OFQ with OF = 13, FQ = 11, QO = 20. He draws the altitude OK, which intersects QF extended at K and he shows FK = 5, OK = 12.

III (186a:13). "The third level, containing the types of quadrangles, and the circumstances of them with respect to each other, the discussion of polygons, circles and arcs, and the discussion of their areas. It is in ten sections."

III:1 (186a:14). "The first section, on the knowledge of the square, and the discussion of its area." The author divides the quadrilaterals into (a) parallelograms and (b) other quadrilaterals. The category (a) is further divided into four kinds, to be discussed in III:1, III:2, III:3 and III:4. The square is equilateral and rectangular, and its area is the product of two sides. He discusses the example of a square with sides 10 and diagonal "root 200". For "square" he uses the term *murabba' muṭlaq*, literally "absolute quadrilateral". Apparently he wanted to avoid confusion that could arise because his term for quadrilateral, *murabba'*, is often used for square. He also explains that parallel lines are lines which, if extended indefinitely, do not meet.

III:2 (186a:28). "The second section on the knowledge of the rectangle and the discussion of its area." The rectangle is a quadrilateral with right angles and unequal sides. He discusses a rectangle with sides 6 and 8, and he shows that the area is 48 and the diagonal 10.

III:3 (186b:1). "The third section, on the knowledge of the rhombus and its area." A rhombus is a parallelogram with equal sides and unequal angles. The area is half the product of the two diagonals. In order to find the area the author says it is necessary to know the side and at least one diagonal, the other diagonal can be found (by means of the theorem of Pythagoras). This is illustrated by means of a rhombus with sides 10 and diagonals 12 and 16.

In the end of the section the author describes a discovery of his own in a rather unclear way:

"We inform you here of a subtlety that we have not found written in any place, and that we have not heard. It is as follows: The square is exactly the same figure (as the rhombus), as if it (the square) is stretched out in the imagination on both sides of its two opposite angles, so that it is transformed into the rhombus. If we imagine that it is stretched out as far as possible, its four sides coincide and the two remaining angles that were not stretched (also) coincide. The verification of this is as follows: if we return it in the imagination to the quadrilateral with right angles and the quadrilateral with equal sides, it is possible."

III:4 (186b:20). "The fourth section on the knowledge of the rhomboid and the discussion of its area." The rhomboid is a parallelogram which is neither equilateral nor equiangular, but its opposite sides and angles are equal. The author says that this quadrilateral is

"as if transformed from the rectangle. The description of it is the description of the rhombus, and therefore it is called the rhomboid."

He gives a lengthy description of an example KLMD with a perpendicular KF drawn from K to side MD, such that KL = MD = 10, ML = KD = 5, KF = 4. The area is 40, and FD = 3 can be computed by the theorem of Pythagoras.

III:5 (187a:1). "The fifth section on the knowledge of the trapezium in which there are two right angles, one obtuse angle and one acute angle, and the discussion of its area." The author now continues his classification of quadrilaterals in III:1. The quadrilaterals that are not parallelograms are divided into (b1) trapezia with two parallel sides, discussed in III:5 and III:6, and (b2) trapezoids, with no parallel sides, discussed in III:7. In III:5 he treats a trapezium AGDB in which A and G are right angles and AB and GD are parallel sides, such that AG = 8, AB = 12, GD = 18 and BD = 10 (he fails to note that the length of BD follows from the other data). The area is AG times $\frac{AB+DG}{2}$.

III:6 (187a:13). "The sixth section on the knowledge of the trapezium with two acute and two obtuse angles, and the discussion of its area." He discusses the example of a trapezium EWZH with EW//ZH, EW = 6, WZ = 13, ZH = 20, HE = 15. For sake of simplicity I draw EX//WZ such that X is on ZH. The distance between the parallel sides is equal to the altitude in triangle EHX, which can be computed as in II:8 (the lengths of the sides are the same, the altitude is 12). The author does not say this but he repeats the method of II:8. The area of the trapezium is the product of the altitude and $\frac{EW+HZ}{2}$.

III:7 (187a:36). "The seventh section on the knowledge of the trapezoid and the discussion of its area." The author discusses only the example of quadrilateral FQML such that M is a right angle, QM = 6, ML = 9, LF = 8 and FQ = 11. In order to compute the area the author says one has to divide FQML into two obtuse-angled triangles FQM and MLF, and one has to compute the areas of these triangles. He does not carry out the computation himself, and he does not show that angles Q and L are obtuse.

I now render my own discussion of the example. The method of computation is incorrect, because one does not know the diagonal MF. As a matter of fact, one has to divide the quadrilateral into triangles by means of the diagonal QL. By the theorem of Pythagoras, $QL = 3\sqrt{13}$, and the areas of QML and QFL now turn out to be 27 and $3\sqrt{183}$ respectively. Trigonometrical computations show that angles Q and L are 99°30′ and 104°13′, so that the quadrilateral has indeed two obtuse angles.

III:8 (187b:7). "The eighth section on the knowledge of polygons and the discussion of their areas." The author divides the polygons into regular (equilateral and equiangular) and irregular ones. He says that it is possible to circumscribe and to inscribe a circle in every regular polygon. In order to explain his rules in a simple way, I now consider a regular polygon of n sides, and I introduce the notations s for the side, p for the perimeter ($p = ns$), r for the radius of the incircle and R for the radius of the circumcircle. His rules can be summarized thus:

1. The area of the polygon is $\frac{pr}{2}$.
2. r can be computed using $(2r)^2 = (2R)^2 - s^2$.
3. *R* can be computed from $(2R)^2 = (3 + \frac{n(n-1)}{2})s^2 \cdot \frac{2}{9}$.

The author then applies these rules to a regular hexagon of side 10.

Formulas 1 and 2 are correct. Formula 3 only holds for n = 3, 4, 6, for the other polygons it is an approximation. A mathematical equivalent of this approximation is also found in an anonymous Arabic text described by Wiedemann.[7]

III:9 (188a:7). "The ninth section on the knowledge of the circle, the derivation of the diameter from the circumference and of the circumference from the diameter[8], and the description of its area." The author says that the area of a circle is $1 - \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{7}$ times the square of its diameter, that the circumference is $3\frac{1}{7}$ times the diameter, and that the diameter is the circumference divided by $3\frac{1}{7}$. He discusses a circle with diameter 7.

III:10 (188a:26) "The tenth section on the knowledge of the arcs and the discussion of their areas." The author distinguishes

[7] Wiedemann, *Beiträge* LVIII, p. 266, reprinted in *Aufsätze* II, p. 465.

[8] I have corrected the word "circle" in the manuscript.

between the semicircle, an arc less than a semicircle and an arc greater than a semicircle. To compute the area of a circular segment, he uses the following rules, which I have rephrased in modern notation. Assume that the length of the arc a, the base b and the arrow s are given (the arrow is the line segment which joins the midpoints of the arc and the base). Let A be the area of the segment. Then

for the semicircle $A = \frac{a}{2} \cdot \frac{b}{2}$.

For the other arcs one first finds the diameter d of the circle using $d = s + (\frac{b}{2})^2 \cdot \frac{1}{s}$

Then $A = \frac{a}{2} \cdot \frac{b}{2} \pm \left|\frac{d}{2} - s\right| \cdot \frac{b}{2}$.

The plus (minus) sign should be used for an arc greater (less) than a semicircle.

These formulas are correct. The author then gives the following three interesting examples:

1. (semicircle) $a = 16$, $b = 10$, $A = 40$;
2. $a = 11$, $s = 2$, $b = 8$, $A = 15\frac{1}{2}$
3. $a = 21$, $s = 8$, $b = 8$, $A = 64\frac{1}{2}$

These examples are interesting, because they all concern segments of a circle with diameter 10 and circumference 32, so that the underlying value of π is 3.2 and not $3\frac{1}{7}$ as before. The author shows no awareness of this inconsistency.

At the end of the section the author says

"The areas of circles and arcs are (known) only by approximation. The exact (quantity) is only attained by God, the Sublime." Similar remarks in the margin of a manuscript of the Algebra of Al-Khwārizmī have been discussed by Rosen[9].

IV (188b:25). "The fourth level containing the circumstances of the spherical, cylindrical, conical and prismatical solids, their properties, their subdivisions, and the description of their measurements (surface areas and volumes). It is in eight sections."

IV:1 (188b:26). "The first section on the knowledge of the solid sphere and the description of its measurement."

[9] Rosen, *The Algebra of Mohammed ben Musa*, p. 200.

The author distinguishes between the full (muṣmat) sphere and the hollow (mukhawwaf) sphere, that is the solid contained between two concentric spheres. He points out that there can be two kinds of measurement[10], namely the surface area and the volume. The surface area of a sphere with diameter d is

$$4d^2(1 - \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{7}),\text{ and its volume is } d^3 (1 - \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{7}).$$

These formulas are based on $\pi = 3\frac{1}{7}$. He discusses the example $d = 7$.

He finally mentions the cupola, that is the solid between two concentric hemispheres and one plane through the centre. To compute the volume one has to subtract the volume of the "emptiness" (hawā', i. e. the inner hemishere) from the volume of the outer hemisphere.

IV:2 (189a: 14). "The second section on the knowledge of the solid cylindrical figures, and the description of the cylinder of which the bases are two equal circles, and the discussion of their volumes." (The words are in slightly different order in the table of contents in the preface.)

The author mentions three kinds of "solid cylindrical figures" (ashkāl mujassama usṭuwāniyya):

a. the round solid cylinder, with two equal circular bases, and produced by the rotation of a rectangle around one of its sides

b. "the conical solid figure, which is the figure contained by planes elevated from some plane, which become cone-shaped and converge into one point". This is a pyramid.

c. the prismatical (manshūr, "sawed-off") figure, contained by five planes, namely two parallel triangles and three rectangles.

He says that the volume of a cylinder defined in a. is the product of the circumference of its base and its height. He discusses a cylinder with height 10, and with a circular base of diameter 2. He correctly says that the circumference is $6\frac{2}{7}$, but he then arrives at the incorrect figure $61\frac{3}{7}$ for the volume.

IV:3 (189a: 30) "The third[11] section on the knowledge of the solid cylinder of which the two bases are ⟨two circles⟩, one of

[10] The word misāḥa therefore means area of a plane figure, surface area of a solid, or volume of a solid.

[11] The manuscript wrongly calls IV:3 the fourth section. Section IV:4 is also called the fourth section.

which is greater than the other, and the discussion of its volume." I have restored the word ⟨two circles⟩, which is found in the table of contents in the preface.

The "solid cylinder" in question is a truncated cone, and its volume is computed according to the formula

$$\frac{h}{3} (d_1^2 + d_2^2 + d_1d_2) (1 - \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{7}),$$

where h is the height and d_1 and d_2 are the diameters of the circles at the base and at the top. He discusses the example $d_1 = 2$, $d_2 = 4$, $h = 10$. In the figure in the manuscript the four sides joining the base and the top are incorrectly labeled 10.

IV:4 (189b: 4). "On the knowledge of the square cylinder of which the two bases and the sides are equal – it is the solid which is called the cubical solid – and the discussion of its volume." (In the table of contents, the word order is "the sides and the two bases".) He computes the volume of a cube with side 4.

IV:5 (189b: 11). "The fifth section on the knowledge of the square cylinder with different sides and bases, and the discussion of its volume." This "square cylinder" is in fact a truncated pyramid with square base and top. He computes the volume according to the formula $h(d_1^2 + d_2^2 + d_1d_2) \cdot \frac{1}{3}$, where d_1 and d_2 are the sides of the base and the top and h is the height. He discusses the example $d_1 = 4$, $d_2 = 2$, $h = 10$.

IV:6 (189b:20). "The sixth section, on the knowledge of the conical solid of which the base is a circle, and the discussion of its measurement." He first says that the "conical solid" is also called pyramid (shakl nārī, fire-figure), and that the base can be a polygon or a circle; in the latter case the cone is called round (mustadīr). He considers a cone with circular base of diameter $d = 6$, circumference $c = 18\frac{6}{7}$, and height $h = 10$. He then gives the curved surface area of the cone as $\frac{c \cdot h}{2}$ and the volume as

$$\frac{d^2}{3} (1 - \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{7}) \cdot h.$$

In both cases his answer is $94\frac{2}{7}$. The second rule is correct, but the surface area is in fact $\frac{c}{2}\sqrt{h^2 + (\frac{d}{2})^2}$.

IV:7 (189b: 33). "The seventh section on the knowledge of the conical solid of which the base is a square and the discussion of its volume." The volume of a pyramid having a square base with side s and height h is $\frac{s^2h}{3}$. He gives $s = 4$, $h = 10$ as an example.

IV:8 (189b:38, continued in the margin of 189b). "The eighth section on the knowledge of the prismatical (manshūr, "sawed-off") solid. He says that a prism is always half of a square cylinder. Thus he only considers right triangular prisms of which the base is a right-angled triangle. He then introduces a prism with height 20 and base half of a square with side 10. He states that the volume of this solid is $\frac{1}{2} \cdot 10 \cdot 10 \cdot 20 = 1000$. This is the end of the *Book of the Levels in the Explanation of the Measurements*.

3. Comparison of the *Book of the Levels in the Explanation of the Measurements* with the *Liber Mensurationum*

In this section I will refer to problems in the *Liber Mensurationum* by means of the numbering that was introduced by Busard in his edition.

The *Liber Mensurationum* consists of various groups of problems on a single geometrical figure. The text begins with 19 problems on a square with side 10. These problems are not limited to the computation of the area if the side is given, or the computation of the side if the area is given. In problem no. 3, for example, the sum of the side and the area of the square is supposed to be 110, and one is required to compute the side and the area. (Of course one knows in advance what the answer will be, but the text is concerned with general mathematical methods.) The discussion of this and most other problems includes a treatment by means of quadratic equations, as in the Algebra of Al-Khwārizmī. The 19 problems on a square are followed by 33 problems on a rectangle with sides 6 and 8 and diagonal 10, and so on.

The *Book of Levels in the Explanation of Measurements* (henceforth abbreviated to *Book of Levels*) contains no trace of algebra and problems involving quadratic equations. It is therefore clearly not the same text as the *Liber Mensurationum*.

One could argue that there must be some indirect relationship, because the following numerical examples are used in both texts (I write LM for the *Liber Mensurationum*, and BL for the *Book of Levels*):

A. a square with side 10 in LM nos. 1–19 and BL III:1.

B. a rectangle with sides 6 and 8 and diagonal 10 in LM nos. 20–52 and BL III:2.

C. a rhombus with sides 10 and diagonals 16 and 12 in LM nos. 53–64 and BL III:3.

D. an equilateral triangle with sides 10 in LM nos. 111–116 and BL II:7.

E. an isosceles triangle with sides 10, 10, 12 in LM nos. 117–124 and BL II:7.

F. an acute-angled triangle with sides 13, 14, 15 and altitude 12 in LM nos. 125–133 and BL II:8.

G. a right-angled triangle with sides 6, 8, 10 in LM nos. 134–140 and BL II:3.

H. a truncated pyramid with height 10, base a square with side 4 and top a square with side 2 in LM no. 152 and BL IV:5.

However, five of these examples (B, D, F, G and H) also occur in another Arabic work on mensuration, namely the second part of the *Algebra* of Al-Khwārizmī.[12] Thus the above-mentioned resemblances prove nothing more than that the authors of the *Liber Mensurationum* and the *Book of Levels* were working in the same tradition. The *Liber Mensurationum* also contains many numerical examples different from those in the *Book of Levels*. Sometimes the difference is slight, as in the following case[13]. The *Liber Mensurationum* and the *Book of Levels* both give a circular segment with base 8 and height 2, but the length of the circular arc is 10 in the *Liber Mensurationum* and 11 in the *Book of Levels*, and the areas are 13 and 15 1/2 respectively. On other occasions the examples are completely unrelated. The arrangement of the two texts is also completely different. For example, the *Liber Mensurationum* treats the quadrilaterals before the triangles, unlike the *Book of Levels*. The classification of figures is not the same either; the *Liber Mensurationum* distinguishes between two types of trapezia with two acute and two obtuse angles (in problems nos. 81–94, where the two obtuse angles are adjacent, and 105–110, where the two obtuse angles are opposite), whereas they are treated as one category in the *Book of Levels* III:6.

[12] Rosen, Arabic text p. 56, 58, 59, 61, 63, translation p. 76, 78, 80, 82, 84.

[13] in III:10 and Busard no. 148 (p. 120).

The *Liber Mensurationum* and the *Book of Levels* are texts of a different level, and they betray a different interest on the part of their authors. The *Liber Mensurationum* was written by a mathematician who was interested in algebra and in theoretical problems. On the other hand, the *Book of Levels* displays no sign of such interest, and it contains mistakes, not only in the computation but also of a more fundamental nature (in III:7 and IV:6). The author's own discovery (announced in III:3) that a rhombus can be obtained by stretching a square is rather trivial, and would not have been of interest to the mathematicians of his time. From the preface we can conclude that most or all other material in the *Book of Levels* is derivative. The interest of its author seems to have been in philosophy and in terminological matters, as can be seen from the preface and the first section, which is translated in Section 4 of this paper. These considerations suggest to me that the Abū Bakr who wrote the *Liber Mensurationum* was probably not the same person as the judge Abū Bakr al-Qāḍī, who authored the *Book of Levels*.

We have no further clue to the identity of these two Abū Bakrs. The Abū Bakr who wrote the *Liber Mensurationum* must have lived before the middle of the twelfth century, when Gerard of Cremona translated his work into Latin. The leaf immediately preceding the text of the *Book of Levels* was written in "the first decade" of the month Safar of the year 587 of the Hijra (Jan. 30 to Feb. 8, A. D. 1191) in the city of Mosul in Iraq, so that the judge Abū Bakr must have lived in the 12th century A. D. or before. The judge Abū Bakr refers in his preface to Euclid's Elements, but he must have used other sources, which I have not been able to identify. Perhaps a reader will be able to identify a source of his first section, on the philosophical discussion of a point (see the next section of this paper).

4. Translation of the first section of the *Book of the Levels in the Explanation of the Measurements*

In the following translation I have used pointed brackets to indicate translations of words that I have restored to the text. Round brackets contain my own explanatory additions. The Arabic text is found in Section 5 of this paper.

The first section on the knowledge of the point. The point is a thing which has no part ⟨and no⟩ part of it can be imagined. Because a point is in reality the source of a line – that is to say that when it is moved in length, there is produced by its motion (in the mind of) a person who imagines a (magnitude) possessing length – it becomes the two extremities of the line.

Since this is so, the point is the origin of the three dimensions, namely length, breadth and depth, and it is the origin of the line and the support of it, like the unit, because it is the origin of number and the support of it, but it is not a number, and therefore the point is not a line. A person could say here: "So that line was produced by the motion of a point but it is not a point, how is this possible?" To him is said: "what is produced from a thing does not become part of what produces it." That is clear and obvious. More than this is not suitable on this occasion, so we return to the detailed discussion of the point.

We say: the point corresponds with the unit in indivisibility, but it differs from the unit in composition and decomposition. That is to say: we put together the nine by repeating the units nine times, and then we call it a number. But one cannot imagine the line by patching points together and repeating them. Lines cannot be decomposed into points in the way of the decomposition of a number into units. However, the lines end at points, because they are extremities of lines, without being in the middle.

One can also find in solids that they (points) are the extremities of extremities in them. We will explain these extremities in their proper place. Since the above-mentioned aspects have become clear, it is correct (to say) that it is a thing which has no part, and that it has no dimensions, and that it potentially carries many lines. For the sensory perception there is no space in it, so that if we imagine the superposition of a number of points at one point, they do not become any magnitude or height. They are not said (to be) magnitude or part, because anything with portion or part possesses magnitude and dimension . . . . . . (two words illegible here), and these principles belong to partition and division.

5. Arabic text of the beginning of the *Book of the Levels in the Explanation of Measurement*

Manuscript: Istanbul, Fatih 3439, 182b:1–183a:18. The manuscript is densely written, there are 39 lines to a page. Illegible words have been indicated by dots. Words in the manuscript that have to be deleted appear in square brackets [ ]. Angular brackets ⟨ ⟩ contain words that have been restored to the text by me. The following text is given here for the purpose of possible identification of other manuscripts and fragments, and because of its mathematical terminology. The text is not intended to be part of a final critical edition. The orthography has sometimes been modernized without notice.

بسم الله الرحمان الرحيم رب يسر الحمد لله حمد الشاكرين والصلوة على المصطفى محمد وآله،

قال القاضي أبو بكر رحمه الله . . . الإخوان ضاعف الله إرشادكم وكثر أعدادكم إنكم اقترحتم عليّ ذكر مساحة الأشكال مقترنا بالمثال والقسم نظام أولياتها وتركيبها بتسهيل العبارة وتهذيبها فيها . . . في الإجابة بهذا الكتاب . . . لذوى الذكاء والألباب وتحرينا فيه حسن الإيجاز والاختصار بحيث يكون خفيف الادّخار والافتخار وهو يتضمن بلوغ الأسامي في هذا الفن للغرض الإنساني بل يشتمل على نكت لطيفة وغرائب طريفة يدعو الناظر فيه إلى علوم أخر التي يتميز بها من البهائم البشر ومع هذا لا . . . النفوس عن مطالعته ولا تنبوا الطباع عن مسارعته وهو في حدة ذو منافع كثيرة ويعرف المتعلم الذكي ما فيه بمدة يسيرة.

وإن علم المساحة من العلوم الجليلة ولا ينكر ما قلناه ذو . . . الطويلة وكيف لا يكون وهو علم مبرهن والقول فيه صحيح ممتحن.

وتتعلق به الأمور الدينية والأحكام الشرعية كأقسام الديار والأفرجة المختلفة لأشكال المشتركة بين الشركاء والورثة وغير ذلك من المواريث والوصايا واستخراج حقوق الأنصاف والأثلاث وما بعدها بحسب المقتضيات الشرعية والبدايات العقلية.

وينبغي للمتعلم أن لا يتجاوز فصلا ما إلا بعد إحكامه وفهمه لئلا ينبهت في الفصل الآخر في بحر معرفته لأن معارف العبارات فيه متسلسلة والموضوعات اللغوية فيه متصلة متصلة كأنها الزوايا والخطوط فمن وقف على معرفة متقدمه وتحققه وأحكمه فلم يتعذر عليه معرفة متأخره.

وسميناه كتاب الطبقات في شرح المساحات وهي أربع طبقات.

الطبقة الأولى مشتملة على أحوال النقطة والخط والبسيط والجسم وهي أربع فصول، الأول في معرفة النقطة، الثاني في معرفة الخط، الثالث في معرفة البسيط، الرابع في معرفة الجسم.

الطبقة الثانية مشتملة على أحوال المثلثات وخواصها واستخراج الخطوط بعضها من بعض وذكر مساحاتها ⟨وهي أحد عشر فصلا⟩. الأول في معرفة أنواع السطوح ⟨والزوايا⟩، الثاني في تمثيل الزوايا الثلاث، الثالث في معرفة الزاوية القائمة من جهة أخرى، الرابع في معرفة صور في المثلثين القائمي الزاويتين، الخامس في معرفة مساحة كلى المثلثين القائمي الزاويتين، السادس في معرفة أحوال المثلثات الحادات الزوايا، السابع في معرفة استخراج العمود من المثلثين المتساوى الأضلاع والمتساوى الساقين وذكر مساحتهما، الثامن في معرفة استخراج نقطة مسقط الحجر والعمود[1] من المثلث الحاد الزوايا المختلف الأضلاع ومعرفة صور المثلثات الحادات الزوايا وذكر مساحتها، التاسع في معرفة المثلثات المنفرجة الزوايا وذكر صورها، العاشر في معرفة استخراج العمود من المثلث المنفرج الزاوية المتساوى الساقين وذكر مساحته، الحادى عشر في معرفة استخراج نقطة مسقط الحجر والعمود من المثلث المنفرج الزاوية المختلف الأضلاع وذكر مساحته.

الطبقة الثالثة يتضمن أنواع المربعات وأحوال بعضها عند بعض وذكر ذوى الأضلاع الكثيرة والدوائر والقسي وذكر مساحاتها وهي عشرة فصول، الأول في معرفة

[1] مسقط العمود والحجر

المربع المطلق وذكر مساحته، الثاني في معرفة المربع المستطيل وذكر مساحته، الثالث في معرفة المربع المعيّن ومساحته، الرابع في المربع الشبيه بالمعيّن ومساحته، الخامس في المربع المنحرف الذى فيه زاويتان قائمتان وزاوية منفرجة وزاوية حادة وذكر مساحته، السادس في المربع المنحرف الذى فيه زاويتان منفرجتان وزاويتان حادتان ومساحته، السابع في المربع الشبيه بالمنحرف ومساحته، الثامن في الأشكال الكثيرة الأضلاع وذكر مساحتها، التاسع في الدائرة واستخراج القطر من المحيط والمحيط من القطر[2] ومساحتها، العاشر في القسي ومساحتها.

الطبقة الرابعة يحوى أحوال المجسمات الكرية والأسطوانية والمخروطة والمنشورة وخواصها وأقسامها ومساحتها وهي ثمانية فصول،

الأول في الكرة المجسمة ومساحتها، الثاني في معرفة الأسطوانة المجسمة التي قاعدتاها دائرتان متساويتان وذكر الأشكال المجسمة الأسطوانية وذكر مساحاتها، الثالث في الأسطوانة المجسمة التي قاعدتاها دائرتان إحداهما أعظم من الأخرى ومساحتها، الرابع في الأسطوانة المربعة المتساوية الجوانب والقاعدتين وهي التي تسمى شكلا مكعبا ومساحته، الخامس في الأسطوانة المربعة المختلفة الجوانب والقاعدتين ومساحتها، السادس في الجسم المخروط الذى قاعدته دائرة ومساحته، السابع في الجسم المخروط الذى قاعدته مربعة ومساحته، الثامن في الجسم المنشور.

هذه عدة فصول الكتاب التي حوتها الطبقات الأربع وهي ثلاث وثلثون [وثلثون] فصلا وكل فصل يتضمن ما يقتضيه من الحدود والبراهين والتركيب والتحليل والعلل وقد جمعنا في هذا (f. 183a) الكتاب من اللطائف . . . ما كان زبدة من تصانيف العلماء ونكت الفضلاء في هذا الفن. فمن عرفه حق المعرفة وقرب علم حل الأشكال من كتاب الأستقصات المعروف بأوقليدس وقد بينا الغرض

[2] الدائرة

والمنفعة . . . فهذا ابتداء خوضنا في الطبقة الأولى وقلت مستعينا بالله ومصليا على نبيه وآله الطبقة الأولى مشتملة على أحوال النقطة والخط والبسيط والجسم.

الفصل الأول في معرفة النقطة. النقطة هي شيء ما لا جزء له ⟨ولا⟩ يتوهم له الجزء، ولأن النقطة في الوجود ينبوع الخط وذلك أنها لما تحركت طولا حدث من تحركها من متوهم ذو طول فسمي خطاً فصارت نهايتي الخط.

فإذا كان هذا هكذا فالنقطة إذاً بديئة من الأبعاد الثلاثة وهي الطول والعرض والعمق وهي مبدأ الخط وركن له كالوحدة فإنه مبدأ العدد وركن له وليست من العدد فالنقطة إذاً ليست من الخط.

ولقائل أن يقول في هذا الموضع فذلك الخط إنما حدث من حركة النقطة ليس منها كيف يمكن، قيل له الحادث من الشيء لا يصير جزئا لحدثه، وهذا واضح بين. ولا يليق بهذا الموضع أكثر من هذا رجعنا إلى استقصاء الكلام في النقطة. فنقول النقطة قد توافق الوحدة في اللا تجزى وتخالفها في التركب والانحلال. وذلك أن التسعة نجمعه من تكرار الوحدات مراراً تسعاً فنسميه عدداً، وليس لمتوهم أن يتوهم الخط من التئام النقطة وتكررها، ولا يمكن [الخط] حل الخطوط إلى النقطة كانحلال العدد إلى الوحدة بل ينتهي الخطوط إلى النقط لأنها نهايات الخطوط من غير توسط.

وقد يوجد أيضا في الأجسام أنها نهاية النهايات فيها، وسنبين هذا النهايات في موضعها.

فإذا اتضح الجهات المذكورة صح أنها هي شى ما لا جزء له، وأنها ليست بذات بعد، وأنها حاملة في القوة لخطوط كثيرة. وليس للحس فيها مجال حتى أنا لو توهمنا انطباق عدة من النقط على نقطة واحدة لما صار لها قدر ارتفاع ما. ولا يقال لها قدر أو جزء لأن كل شيء له بعض أو جزء فهو ذو مقدار وبعد وهي . . . . فهذه الأحكام هي داخلة في التجزئة والانقسام.

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