Abu'l-Jūd's Answer to a Question of al-Bīrūnī Concerning the Regular Heptagon

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AMONG THE Arabic manuscripts that J. Golius brought from the Middle East to Holland in the 1620s[1] is a famous codex of mathematical and astronomical content, now MS Or. 168 in the University Library of Leiden. Most of the treatises in the codex date back to the 10th century A.D. and are not otherwise extant. The codex itself is undated but certainly older than the 16th century, as is attested by the owner's mark of Taqī al-Dīn ibn Maʿrūf (1526–1585)[2] on the front page.

Fols. 45r–54r of the codex contain four answers by the 10th-century geometer Abu'l-Jūd to questions asked by al-Bīrūnī (973–*ca.* 1050).[3] Abu'l-Jūd's date of birth is not known, but there is a reference[4] to a treatise on the regular heptagon written by Abu'l-Jūd in the autumn of A.D. 969, four years before al-Bīrūnī was born. Thus it seems plausible that al-Bīrūnī asked the questions early in his life. This is confirmed by the beginning of the text on fol. 45r, which we quote in translation:

> Answer by the eminent master (*shaykh*) Abu'l-Jūd Muḥammad ibn al-Layth, may God support him, to what he was asked by the eminent fellow (*akh*) Abu'l-Rayḥān Muḥammad ibn Aḥmad al-Bīrūnī. I have come across what I was asked by the eminent fellow, may God make his happiness permanent, and I shall answer it (*i.e.*, the questions) by means of what came to my mind.[5]

The answers to the first and third question have already been discussed by Woepcke.[6] This article concerns the second question and its answer, which have not yet been studied by modern historians. Following a short introduction we will present the Arabic text and an English translation with notes.

Al-Bīrūnī's second question concerns the regular heptagon. It is well-known that a regular heptagon cannot be constructed by means of ruler and compass only. But the side of a regular heptagon (s_7) (FIG. 1) inscribed in a circle is very well approximated by half of the side of the inscribed equilateral triangle ($\frac{1}{2}s_3$), which is easily constructed by means of ruler and compass. The relative error of the approximation is only 0.2%. It is not known when and how the approximation was first discovered, but the fact that s_7 is very

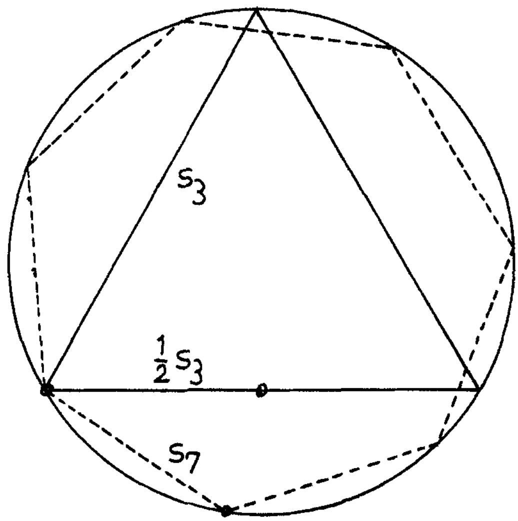

FIGURE 1.

nearly equal to $\frac{1}{2}s_3$ was mentioned by Heron of Alexandria already in the first century A.D.[7]

In his second question Al-Bīrūnī asks Abu'l-Jūd to prove that s_7 is not equal to $\frac{1}{2}s_3$. Abu'l-Jūd gives a proof based on a *reductio ad absurdum*. The proof is essentially as follows:

In FIGURE 2, suppose that $s_7 = \frac{1}{2}s_3 = AB$. Choose G on the circle as in the figure such that triangle AGB is isosceles. Then $\angle A = \angle B = 3\alpha$, $\angle G = \alpha$, with $\alpha = \frac{1}{7} \cdot 180°$. Choose D on BG and E on AG such that AB = AD = DE. Easy calculations of angles show that $\angle EDG = \alpha$, whence GE = DE = AD = AB.[8] Drop perpendiculars ET and AZ onto GB. Then EG : GT = AG : GZ, so EG.GZ = AG.GT. Thus far the argument rests only on the assumption AB = s_7.

Next Abu'l-Jūd performs a number of calculations in order to prove that if $s_7 = \frac{1}{2} s_3$, then EG.GZ cannot possibly be equal to AG.GT. Put AB = 1. Since AB = $\frac{1}{2}s_3$ by assumption, we have $s_3 = 2$. It is now possible to calculate successively the diameter of the circle, AG, GZ and GT. Most of the calculations are in fact missing, because the text has a considerable lacuna between fol. 50r and fol. 50v. But the final results are correctly stated on fol. 50v:

$$AG^2 = 2\tfrac{2}{3} + \sqrt{5\tfrac{7}{9}}$$

AG is the square root of this expression,

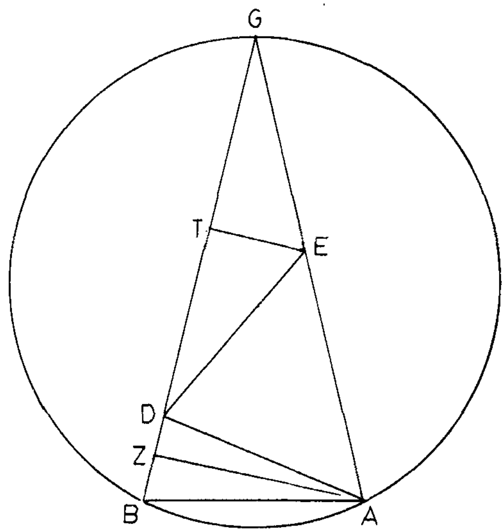

FIGURE 2.

$$GZ = \sqrt{2\tfrac{2}{3} + \sqrt{5\tfrac{7}{9}}} - \sqrt{\tfrac{1}{2} - \sqrt{\tfrac{13}{64}}}$$
$$GT = \sqrt{\tfrac{2}{3} + \sqrt{\tfrac{3}{9} + \tfrac{1}{36}}} - \sqrt{\tfrac{1}{2} - \sqrt{\tfrac{13}{64}}} \ .$$

The results of these calculations recall similar expressions in the second half of the *Algebra* of Abū Kāmil (*ca.* 850–930).[9] Unlike Abū Kāmil, however, Abu'l-Jūd adds two references to the classification of irrationals in Book X of the *Elements* of Euclid: AG^2 is called a "fourth binomial," and its square root AG is called a "major."[10] The references are in principle correct, but Euclid would have used the expression "fourth binomial" only for certain straight segments, not for an "area" as AG^2. Abu'l-Jūd seems to apply the Euclidean classification to irrational numbers rather than irrational straight lines. The same tendency can be observed in other Arabic treatises, for example the commentary on Book X of the *Elements* by Abu'l-Jūd's contemporary Abū ʿAbdallāh al-Ḥasan ibn al-Baghdādī.[11]

After stating the values of AG, GZ and GT Abu'l-Jūd concludes without further explanation that $EG : GT \neq AG : GZ$, that is to say $EG.GZ \neq AG.GT$. In fact $EG.GZ = 2.0297$ and $AG.GT = 2.0352$ correct to four decimals, so a more detailed proof of the inequality of EG.GZ and AG.GT would not have been out of place. Such a proof could have been supplied in different ways:

1. By *reductio ad absurdum*. Calculate AG.GT and equate the result to EG.GZ (=1.GZ). Remove the square roots by successive squarings, subtrac-

tions and additions. This process leads eventually to an absurd equality between two rational numbers.

2. In the style of Book X of the *Elements*, AG.GT = $\frac{5}{6} + \frac{1}{3}\sqrt{13}$ is a "fifth binomial",[12] whereas $EG^2.GZ^2 = \frac{19}{6}(1 + \frac{1}{4}\sqrt{13})$ is a "fourth binomial", so that EG.GZ is a "major" (*Elements* X:57). Because a binomial cannot be equal to a major (*Elements* X:111) EG.GZ and AG.GT cannot be equal.

3. The expressions for AG, GT and GZ only involve square roots, so AG.GT and EG.GZ can be calculated with any desired accuracy. A calculation to two places of sexagesimals shows that EG.GZ and AG.GT are not equal.

However, Abū'l-Jūd would probably not have argued in the way of proof no. 3. The inequality of $\frac{1}{2}s_3$ and s_7 can already be shown using the approximations $\frac{1}{2}s_3 = 51^p57'42''$ and $s_7 = 52^p3'17''$, derived from the table of chords in the *Almagest* of Ptolemy.[13] It seems that Abū'l-Jūd composed his proof in order to circumvent all approximations. I am unable to decide whether Abū 'l-Jūd had a proof such as no. 1 or no. 2 in mind.

The initial hypothesis $s_7 = \frac{1}{2}s_3$ has now led to the absurd conclusion that EG.GZ and AG.GT are at the same time equal and unequal. So the initial hypothesis cannot be true, hence s_7 cannot be equal to $\frac{1}{2}s_3$, Q.E.D.

In the *Qānūn al-Masʿūdī* (written between 1030 and 1040) al-Bīrūnī briefly mentions the work on the regular heptagon done by two outstanding geometers of his time, Abū Sahl al-Kūhī and Abu'l-Jūd. Just before he mentions these two geometers al-Bīrūnī says about the "chord of one-seventh of the circle" (*i.e.*, s_7):

> it is as a chord of unknown quantity, belonging to an arc not (exactly) expressible in them (degrees, minutes, seconds etc.), and similar to the irrational roots.[14]

When writing this passage al-Bīrūnī probably remembered Abu'l-Jūd's answer to the second of his four questions, which he had asked many years earlier.

ARABIC TEXT AND TRANSLATION OF MS LEIDEN OR. 168, FOLS. 49v–50v

Explanation of signs: <   > should be added, according to the editor; [   ] should be deleted, according to the editor; (   ) contains explanatory addition made by the editor.

Numbers in the translation refer to the notes at the end of the article. Numbers in the Arabic text refer to the apparatus. Orthographical changes and trivial emendations involving slashes written above letters referring to the figure are not indicated in the apparatus.

السؤال الثاني . ما البرهان على استحالة قول القائل أن وتر سبع كل دائرة مساو لنصف[١] وتر ثلثها .

الجواب ليكن **ا ب** نصف ضلع المثلث المتساوي الأضلاع الواقع في دائرة **ا ب ج** ونخرج من نقطتي **ا ب** خطي **ا ج** ، **ب ج** متساو يين إلى المحيط. وننزل أن **ا ب** وتر سبع الدور فتكون زاو ية **ا ج ب** سبع زاو يتين قائمتين وكل من زاو يتي **ا ب ج** **ب ا ج** ثلثة أسباع زاو يتين قائمتين . ونخرج **ا د** إلى **ب ج** مساو يا لـ **ا ب** فتكون زاو ية **ا د ب** أيضاً ثلثة أسباع زاو يتين قائمتين ، فتبقى[٢] زاو ية **د ا ب** سبع زاو يتين قائمتين فزاو ية[٣] **ج ا د** سبعي زاو يتين قائمتين . ونخرج خط(f. 50a) [خط] **د ه** مثل **ا د** إلى **ا ج** فتكون زاو ية **ا ه د** أيضاً سبعي زاو يتين قائمتين . وكنا أنزلنا زاو ية **ا ج ب** سبع زاو يتين قائمتين فزاو ية **ه د ج** أيضاً سبع زاو يتين قائمتين فـ **ج ه** مثل **ه د** وخطوط **ج ه** **ه د** **ا د** **ا ب** الأربعة متساو ية .

فنرسل من نقطتي **ا** **ه** عمودي **ا ز**[٤] **ه ط** على **ب ج** فتكون نسبة **ه ج** إلى **ج ط** كنسبة **ا ج** إلى **ج ز** وضرب **ه ج** في **ج ز** مثل ضرب **ا ج** في **ج ط** .

وأيضاً ليكن **ا ب** واحداً فيكون وتر ثلث الدائرة اثنين وعمود المثلث الواقع فيها ⟨جذر⟩ ثلثة وقطر الدائرة جذر خمسة أجزاء (f. 50b) ⟨وثلث .

....

فتبين أن مربع **ا ج** اثنين وثلثين وجذر خمسة أجزاء⟩ وسبعة أتساع وهو ذو الاسمين الرابع وجذره وهو **ا ج** الأصم المسمى الأعظم وأن خط **ج ز** جذر اثنين وثلثين[٥] وجذر خمسة وسبعة أتساع إلا جذر نصف واحد إلا جذر ثلثة عشر ثمن ثمن وأن خط **ج ط** جذر ثلثي واحد وجذر ثلثة أتساع وربع تسع إلا جذر نصف واحد إلا جذر ثلثة عشر ثمن ثمن فليست[٦] نسبة **ج ه** الواحد إلى **ج ط** كما تبين مقداره كنسبة **ا ج** إلى **ج ز** ولا ⟨ضرب⟩ **ج ه** الواحد في **ج ز** المذكور مقداره مثل ضرب **ا ج** في **ج ط** المذكور مقدارهما .

وقد كان في الحكم الأول ضرب **ج ه** في **ج ز** مثل ضرب **ا ج** في **ج ط** وهذا خلف .
فليس وتر سبع الدائرة بمساو لنصف وتر ثلثها وذلك ما أردنا بيانه .

١ مساو لنصف : يسا نصف /
٢ وتبقى / ٣ وزاو ية / ٤ **او** / ٥ يلي / ٦ وليست/

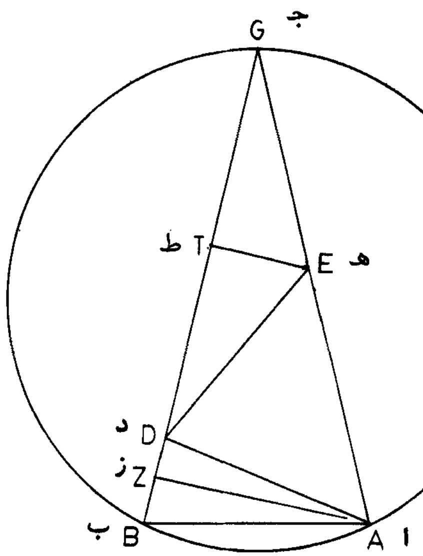

FIGURE 3.

The second question (see FIG. 3). What is the proof of the impossibility of the statement that the chord of one-seventh of any circle is equal to half of the chord of one-third of it?

Answer: Let AB be half of the side of the equilateral triangle inscribed in circle ABG. We draw from points A and B two equal lines AG and BG to the circumference.

We assume that AB is the chord of one-seventh of the circle. Then angle AGB is one-seventh of two right angles, and angles ABG and BAG are both three-sevenths of two right angles.

We draw (line) AD to (meet) BG, equal to AB. Then angle ADB is also three-sevenths of two right angles, so, by subtraction, angle DAB is one-seventh of two right angles. So angle GAD is two-sevenths of two right angles.

We draw line (fol. 50a) DE equal to AD to (meet) AG. Then angle AED is also two-sevenths of two right angles. But we assumed that angle AGB is one-seventh of two right angles.[15] So angle EDG is also one-seventh of two right angles. So GE is equal to ED, and the four lines GE, ED, AD and AB are equal.

We draw from points A and E perpendiculars AZ and ET to BG. Then the ratio of EG to GT is equal to the ratio of AG to GZ, and the product of EG and GZ is equal to the product of AG and GT.

Again, let AB be one. Then the chord of one-third of the circle is two, and the altitude of the inscribed (equilateral) triangle is <the root of> three, and the diameter of the circle is the root of five parts (i.e. units) (fol. 50b) <plus one-third .... so it has become clear that the square of AG is two and two-thirds plus the root of five parts (*i.e.* units)>[16] plus seven-ninths; it is a fourth binomial,[17] and its root AG is the irrational called major,[18] and that line GZ is the root of two and two-thirds plus the root of five and seven-ninths minus the root of one-half minus the root of thirteen sixty-fourths, and that line GT is the root of two-thirds of a unit plus the root of three-ninths plus one thirty-sixth minus the root of one-half minus the root of thirteen sixty-fourths.[19]

So the ratio of GE, one, to GT, the quantity of which has become clear, is not equal to the ratio of AG to GZ, and the <product of> GE, one, and GZ, the quantity of which has been mentioned, is not equal to the product of AG and GT, the quantities of which have been mentioned. But in our first judgment[20] the product of GE and GZ was equal to the product of AG and GT. This is absurd.

So the chord of one-seventh of the circle is not equal to half of the chord of one-third of it. That is what we wanted to prove.

ACKNOWLEDGMENTS

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NOTES

1. For an overview of the history of Dutch interest in Arabic studies, see Brugman & Schröder.
2. For biographical information on Taqī al-Dīn, see Ḥasan, Ch. 1. For examples of Taqī al-Dīn's signatures from Leiden manuscripts, see Ünver, 101–4.
3. On al-Bīrūnī, see the article by E. S. Kennedy in *DSB*, II: 147–58. For information on the manuscript, see Voorhoeve, 431, and *GAS*, V. 354, no. 4, and *CCO*, III: 63, no. 1013. On Abu'l-Jūd, see *GAS*, V, 353–54, updated in Hogendijk, 1984. § 5.1.
4. The reference is in a marginal remark in MS Oxford Bodleian Thurston 3, fol. 129r: see Hogendijk 1984. § 5.3.4.
5. Fol. 45r: lines 2–5.
6. Woepcke, 114–15 (first question), and 125–26 (third question).
7. Heron, *Metrica*. See Bruins, vol. I: 151–52; II: 101; III: 230. Further references can be found in Tropfke, 259.
8. The same argument is found elsewhere in the work of Abu'l-Jūd. Compare *Hogendijk*, 1984 5.2.8.
9. See Levey.

10. For the definitions of a major and a fourth binomial see *Elements*, Book X, prop. 39 and the definitions after prop. 47 (Heath, III: 87–88 and 101–2).
11. See *GAS*, V: 329. The Arabic text is in MS Bankipore 2468, fols. 145v–169r, published in *Rasā'il* (no. 9).
12. For the definition of a fifth binomial see *Elements*, Book X, definitions after prop. 47 (Heath, III: 101–2).
13. *Cf.* for example, Ptolemy, *Almagest*, I.11: Toomer, 57–58.
14. *Bīrūnī*, I: 297, lines 15–16.
15. $\angle AGB = \frac{1}{7} \cdot 180^\circ$ is equivalent to the assumption $AB = s_7$.
16. Let r be the radius of the circle, let h be the altitude of triangle AGB (that is the length of the perpendicular drawn from G onto AB) and let k be the altitude of the equilateral triangle inscribed in the circle. In the missing part of the text, Abu'l-Jūd may have argued as follows: Since $(s_3)^2 = k.2r$ (by *Elements* VI:8), $s_3 = 2$ and $k = \sqrt{3}$, we have $2r = \frac{4}{\sqrt{3}} = \sqrt{5\frac{1}{3}}$. So $h = r + \sqrt{r^2 - \frac{1}{4}AB^2} = \sqrt{\frac{4}{3}} + \sqrt{\frac{13}{12}}$. Hence $GA^2 = h.2r = \frac{8}{3} + \sqrt{5\frac{7}{9}}$ (again by *Elements* VI:8). Because triangle GBA is similar to triangle ADB we have $GA.BD = AB^2 = 1$. Hence $BD = \frac{1}{GA} = \sqrt{2 - \sqrt{\frac{13}{4}}}$, so $BZ = \frac{1}{2}BD = \sqrt{\frac{1}{2} - \sqrt{\frac{13}{64}}}$. GZ and GT can be calculated using $GZ = GB - BZ$ and $GT = \frac{1}{2}GB - BZ$.
17. Abu'l-Jūd calls $AG^2 = 2\frac{2}{3} + \sqrt{5\frac{7}{9}}$ a fourth binomial because $2\frac{2}{3} > \sqrt{5\frac{7}{9}}$ and $(2\frac{2}{3})^2 - 5\frac{7}{9}$ is not the square of a rational number. Compare with the definitions in *Elements* X:36 and after X:47.
18. *Elements* X:57. For the definition of a major see *Elements* X:39.
19. See the Introduction.
20. The "first judgment" is based on the hypothesis $AB = s_7$.

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