THE TENTH-CENTURY GEOMETER ABŪ ʿABDALLĀH AL-SHANNĪ ON THE AREA RULE FOR A CYCLIC QUADRILATERAL IN TERMS OF ITS SIDES

JAN P. HOGENDIJK*

1. Introduction

A cyclic quadrilateral is a quadrilateral whose four angular points lie on a circle, as $ABGD$ in Figure 1. The area of a cyclic quadrilateral is in modern notation $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where a, b, c, d are the four sides and $s = \frac{1}{2}(a+b+c+d)$ is half the perimeter.

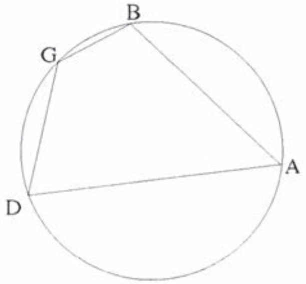


Figure 1

The area rule was unknown in ancient Greek geometry, and it first appeared in India in the seventh century AD. The rule was proved by Abū ʿAbdallāh al-Shannī in the late tenth century AD in a treatise which is now lost. Al-Shannī's proof, which is one of the highlights of the Arabic-Islamic geometric tradition, has reached us through the *Extraction of Chords* by al-Bīrūnī (973-1048 AD). This work is available in two different versions. The purpose of this paper is to publish edited Arabic texts with English translations of al-Shannī's proof in the two versions. We have included the two versions of a related proof by al-Shannī which was also cited by al-Bīrūnī, showing that the area of a triangle with sides a, b, c is $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$.

* Mathematics Department, Utrecht University, P.O. Box 80.010, 3508 TA Utrecht, Netherlands, www.jphogendijk.nl

2. The historic context

Cyclic quadrilaterals occur in ancient Greek mathematics in Euclid's *Elements* III:22 [8, vol. 2, p. 51] and in Ptolemy's *Almagest* I:10 [19, p. 30], where they are used to prove trigonometric rules. The area rule for the cyclic quadrilateral was stated in the Brāhmasphuṭasiddhānta of Brahmagupta (628 AD) in Sanskrit verses without proof.[1] A proof of the area rule is found in a sixteenth-century work in Malayalam by Jyeṣṭhadeva (ca. 1530 AD) [25, vol. 1, pp. 109-133, 239-255]. The rule was transmitted from India to the Islamic world before the tenth century. The proof by Abū ʿAbdallāh al-Shannī has been preserved in the work by Abū Rayḥān al-Bīrūnī (973 - 1048 AD) entitled "Extraction of Chords in a Circle by the Properties of the Broken Line Inscribed in it" *(istikhrāj al-awtār fi'l-dā'ira bi-khawāṣṣ al-khaṭṭ al-munḥanī al-wāqiʿ fīhā)* [26, p. 381 no. 3]. We will abbreviate this title to *Extraction of Chords*. The work exists in two substantially different versions, which we will call the Patna version and the Leiden version. A German translation of the Leiden version appeared in 1911 in [30], and the Arabic version of the Patna version was printed in 1948 in the unreliable edition [2]. Al-Shannī's proof has recently been analyzed in modern notation by Eisso Atzema in an excellent historical survey [1] of proofs of the area rule for the cyclic quadrilateral.

Cyclic quadrilaterals were studied in Western Europe from the fifteenth century onwards. The area rule was first stated in Europe without proof by the Dutch mathematician Willibrord Snel in 1615 [5, p. 189-190]. Atzema discovered the earliest Western proof in a work printed in 1706 by the Dutch mathematician Abraham de Graaf [1, p. 28]. Most proofs in the Western tradition involve long algebraic computations in terms of the sides a, b, c, d, and the preliminary determination of the length of one of the diagonals. The famous mathematician Leonhard Euler (1707-1783) wrote about the difficulty of finding a strictly geometric proof of the area rule.[2] Thus al-Shannī's straightforward geometric proof of the area rule for a cyclic quadrilateral may also interest a modern student of elementary geometry.

Immediately before the proof for the cyclic quadrilateral, al-Bīrūnī presents a related proof of the area rule for the triangle, often called

[1] See [15] for references and the possibility that Brahmagupta proved the rule.

[2] See for the quotation [1, p. 31].

Heron's rule, after Heron of Alexandria (ca. 70 AD).[3] Al-Bīrūnī attributes the rule to Archimedes (ca. 200 BC) and the proof is attributed to al-Shannī in the Leiden version. The two versions of this proof are also published in this paper, and I will argue below that the attribution of the proof to al-Shannī is plausible. Heron's area rule for the triangle is not found in the extant Greek and Arabic works by Archimedes, and therefore al-Shannī and al-Bīrūnī must have had access to an Arabic translation of a work by Archimedes which is now lost.

Nothing is known about the life of Abū ʿAbdallāh Muḥammad ibn Aḥmad al-Shannī, and only three of his works are extant. The longest of these is the *Book of the Disclosure of the Fallacy of Abu'l-Jūd in the Matter of the Two Lemmas for His Alleged Construction of the Heptagon*.[4] The work deals with constructions of the regular heptagon discovered around 970 AD, and thus al-Shannī must have been active around that time. Thus he was at least one generation older than al-Bīrūnī,[5] who was born in 973 AD. Al-Shannī also authored two brief texts on the rule for the area of a triangle in terms of its sides. His *Book on the Measurement of Every Triangle with Different Sides by Means of its Sides (kitāb misāḥat kull muthallath mukhtalif al-aḍlāʿ min jihat aḍlāʿihi)* has been summarized in [11]. It includes a proof of the triangle rule which is different from al-Shannī's proof in the present paper although there are some similarities in method.[6] The other text is the *Book on the Measurement of Every Triangle by Means of its Sides (kitāb misāḥat kull muthallath min jihat aḍlāʿihi)*. In this unpublished text, al-Shannī proves the area rule by first determining the length of the altitude in a triangle in terms of the sides.[7] These two extant works of al-Shannī do not contain the proofs which are attributed to him in the *Extraction of Chords*.

The present paper continues with a brief analysis of al-Shannī's geometric proofs of the two area rules. Then the two different versions of the *Extraction of Chords* and their publication history will be discussed,

[3]For an English translation of Heron's proof see [12, pp. 499-500].

[4]See [21, p. 832-863], [22, p. 671-688] and [10, p. 287-288].

[5]On the life and work of Abū Rayḥān al-Bīrūnī see [13].

[6]In this text he proves $((a+c)^2 - b^2) : 4[ABG] : 4[ABG] : (b^2 - (a-c)^2)$ for a triangle ABG with sides a, b and c and area $[ABG]$, using mean proportionals in much the same way as in his proofs in the present paper.

[7]I have consulted the manuscript Cairo, Dār al-Kutub, Muṣṭafā Fāḍil Riyāḍa 41m, ff. 148b-150a and ff. 152b-153b, see [16, p. 46].

as well as the Arabic manuscripts and the editorial procedures that have been used. The paper concludes with literal English translations of the two proofs in the two versions and edited Arabic texts.

3. Summary of the geometric proofs

We begin with some general information on al-Bīrūnī's *Extraction of Chords*. This work is about properties of a "broken line" *(khaṭṭ munḥanī)* ABG, that is, a combination of two chords AB, BG of a circular arc which meet at one point B on the arc (Figure 2). Al-Bīrūnī's "first property" and "fourth property" of a broken line will concern us here.

Figure 2

Here is the first property (Figure 3). Suppose that $AB > BG$ and let D be the midpoint of arc AG. Drop a perpendicular DE onto AB. The first property says that this perpendicular bisects the broken line, that is to say $AE = EB + BG$. [8]

In both versions of the *Extraction of Chords*, the Patna version and the Leiden version, al-Bīrūnī gives more than twenty proofs of the first property, citing at least eight mathematicians, including Archimedes [2, pp. 4-24], [30, pp. 12-26]. We now present the proof by the Iranian geometer Abu'l-Ḥasan Ādharkhurā [26, p. 342], whom al-Bīrūnī knew personally and who provided him with information on Persian chronology. This is the first proof in the Patna version [2, p. 6-7] and the fifteenth proof in the Leiden version [30, p. 20]. (Figure 3) Drop the perpendicular DP to the rectilinear extension of GB, and draw DA, DB, DG.

[8]We use the notation AB also for the length of the segment AB; thus AB=BA.

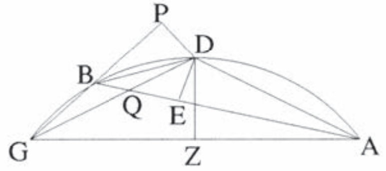


Figure 3

We first show that there are two pairs of congruent triangles: AED, GPD and DBE, DBP. In triangles AED, GPD, angles E and P are right angles, and the hypotenuses AD and GD are equal because D is the midpoint of arc AG. Angles EAD and PGD are equal because they stand on the same arc BD of the circle, by Euclid's *Elements* III:21 [8, vol. 2 p. 49]. Thus the triangles AED, GPD are congruent, hence $DE = DP$. Since $DB = DB$ and angles E and P are right angles, the triangles DBE, DBP are congruent as well. By means of the congruent triangles we conclude $AE = PG = PB + BG = = EB + BG$.

In order to introduce the "fourth property" of the broken line,[9] we call Q the intersection of DG and AB in Figure 3.

The fourth property says that the difference between the areas of triangles ADG and ABG is equal to the area of the rectangle contained by EB and ED, that is the product of the lengths of EB and ED, which we will write as $EB \cdot ED$ from now on.

The fourth property can be proved in a simple way by extending the proof of Abu'l-Ḥasan Ādharkhurā:[10] The area of triangle ADG is the sum of the areas of the three triangles ADE, EDQ and AQG. The areas of ADE and GDP are equal because the triangles are congruent. Thus the area of triangle ADG is the sum of the area of triangle ABG plus the area of kite $EBPD$. The kite consists of the two congruent triangles DBE and DBP with right angles at E and P. The area of each triangle is half the product of the base EB times the altitude ED, and therefore the total area of the kite is $EB \cdot ED$. Thus the fourth property is proved.

[9] The fourth property is discussed in the Patna version with several proofs [2, p. 45-49]; in the Leiden version it is mentioned only implicitly in the "third proof by Abū 'Abdallāh al-Shannī" of the first property [30, p. 20].

[10] The following proof is not found in the *Extraction of Chords*.

We now turn to al-Shannī's proof that the area of a triangle is in modern notation $\sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the three sides and $s = \frac{1}{2}(a+b+c)$ is half the perimeter. In Figure 3, drop perpendicular DZ onto AG. Then Z is the midpoint of AG, therefore the area of triangle ADG is equal to the rectangle $ZA \cdot ZD$. By the fourth property of the broken line, the area of triangle ABG is the difference between the areas of the two rectangles $ZA \cdot ZD$ and $EB \cdot ED$. Because the angles DBA and DAG stand on equal arcs AD and GD, $\angle DBE = \angle DAZ$, thus $ZD : ZA = ED : EB$, and the two rectangles are similar. In the vein of Greek and Islamic geometry, we now express the difference between the areas of the two similar rectangles as the area of a third rectangle similar to the other two, as VWX in Figure 4. Then rectangle VWX has the same area as triangle ABG.

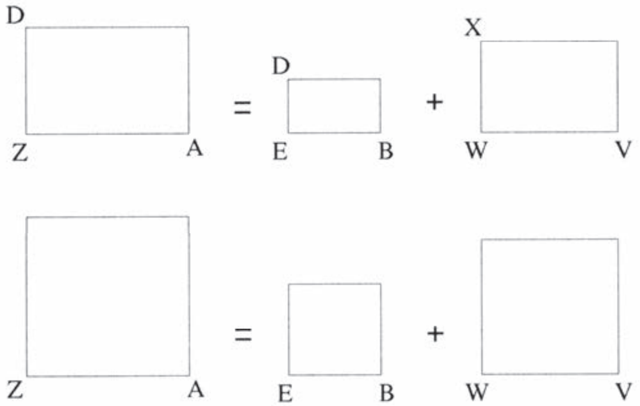


Figure 4

The sides of the third rectangle can be easily found, as suggested by Figure 4: by multiplying the vertical sides of the rectangles by a suitable factor (ZA/ZD), the three rectangles become squares and the equality between the areas is maintained, thus $ZA^2 = EB^2 + WV^2$. By a similar argument for the horizontal sides $ZD^2 = ED^2 + WX^2$.

In Figure 3 put $AB = c, BG = a, GA = b$ and $s = \frac{1}{2}(a+b+c)$ half the perimeter; by assumption $AB > BG$, thus $c > a$.

Then $ZA = \frac{1}{2}b$ and by the first theorem on broken lines, $AE = \frac{1}{2}(c+a)$, hence $EB = c - \frac{1}{2}(c+a) = \frac{1}{2}(c-a)$.

Thus $WV^2 = ZA^2 - EB^2 = (\frac{1}{2}b)^2 - (\frac{1}{2}(c-a))^2 = (s-a)(s-c)$. By the theorem of Pythagoras, $AD^2 = ED^2 + EA^2 = ZD^2 + ZA^2$. Therefore $WX^2 = ZD^2 - ED^2 = EA^2 - ZA^2 =$
$= (\frac{1}{2}(c+a))^2 - (\frac{1}{2}b)^2 = s(s-b)$. Thus the area of triangle ABG is equal to the area of a rectangle with sides $\sqrt{(s-a)(s-c)}$ and $\sqrt{s(s-b)}$, that is $\sqrt{s(s-a)(s-b)(s-c)}$.

Al-Shannī's proof is a bit more complicated in the technical details. No modern notation is used, and auxiliary points H on AE and K on AZ are defined in Figure 7 below, in order to create line segments $EH = s-b, GK = s-a, AK = s-c$. In order to subdivide the area of the square of AD, the text also defines an auxiliary point T on AD such that $AT = AZ$. Secondly, four-dimensional magnitudes are not used in Euclid's *Elements*, and this is why Al-Shannī (according to al-Bīrūnī) avoids multiplication of four line segments until the very end. Instead, he expresses the area of triangle ABG (in my notation: $[ABG]$) as the mean proportional of the rectangles $s(s-b)$ and $(s-a)(s-c)$ as $s(s-b) : [ABG] = [ABG] : (s-a)(s-c)$, in a way acceptable in Greek geometry. To obtain the triangle area as a mean proportional, the text concludes from $ZD : ZA = ED : EB$ that
$ZD : ZA = ZD^2 : ZD \cdot ZA = ZD \cdot ZA : ZA^2 = ED : EB =$
$= ED^2 : ED \cdot EB = ED \cdot EB : EB^2$ and hence by repeated use of a theorem[11] on proportions:
$ZD : ZA = (ZD^2 - ED^2) : (ZD \cdot ZA - ED \cdot EB) =$
$= (ZD \cdot ZA - ED \cdot EB) : (ZA^2 - EB^2)$. where the area of triangle ABG is $(ZD \cdot ZA - ED \cdot EB)$ by the fourth property.

The proof in the Leiden version is similar to that in the Patna version but the explanations in the Leiden version are sometimes unclear, as will be outlined in the footnotes to the English translation of the Leiden version. At the end, the Leiden version explains the easy case of the area of the isosceles triangle, as triangle ADG in Figure 3.

The Leiden version attributes the proof to Abū ^{c}Abdallāh al-Shannī, but the Patna version does not contain such an attribution. Nevertheless, I think that the attribution to al-Shannī is plausible, for the following reasons.First, the tenth-century geometer Abu'l-Wafā' al-Būzjānī wrote a small treatise on the triangle rule by Archimedes with a proof

[11]The theorem is to the effect that if p, q, r and s are four magnitudes of the same kind in such a way that $p > r$ and $p : q = r : s$, then $p : q = (p-r) : (q-s)$.

which al-Būzjānī introduced by "Archimedes said." Inspection of the manuscript shows [12] that the Archimedean proof uses the inscribed circle of the triangle and the radius and centre of one of the excircles, and thus it is different from the proof in the *Extraction of Chords*. The Archimedean proof cannot be transferred to the cyclic quadrilateral, but the proof of the triangle rule in the *Extraction of Chords* can be generalized to the proof for a cyclic quadrilateral, which proof is attributed to al-Shannī in both versions. Secondly, al-Shannī was interested in the triangle rule as we can see from his extant works, which have been mentioned above. Al-Shannī knew the "fourth property" of the broken line which plays a crucial role in the proof of the triangle rule in the *Extraction of Chords*.[13] Finally, al-Shannī's proof uses angles and similar triangles as in Euclid's *Elements* and thus it is different from geometric proofs of the rule which might have been developed in the Indian tradition.[14]

According to al-Bīrūnī in the Patna version, Al-Shannī's proof of the area rule for the cyclic quadrilateral was inspired by the previous proof for the triangle. I now summarize the proof in the Patna version.

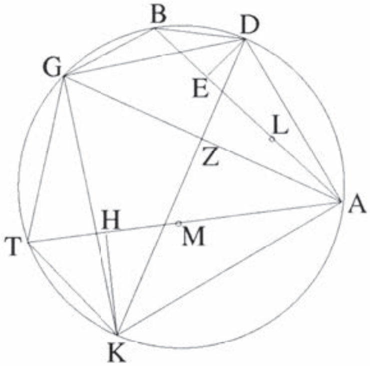


Figure 5

[12]The treatise is in the manuscript Oxford, Bodleian Library, Thurston 3, f. 55b-56a, [26, p. 135 no. 8]. The proof by Archimedes is similar to the proof in proposition 7 of the *Book on the Measurement of the Plane and Spherical Figures* by the Banū Mūsā (ca. 860), see [6, pp. 278-289] and [29, pp. 264-265].

[13]One of the proofs of the fourth property in the *Extraction of Chords* is by al-Shannī; see [2, pp. 47-48]. This proof resembles proof q in [30, p. 20] which is also by al-Shannī and which involves the fourth property without mentioning it by name.

[14]See [15, p. 30] for characteristics of Indian geometric proofs.

Figure 5 shows a cyclic quadrilateral $ABGT$. Again we write the area of a figure using square brackets, thus: $[ABGT]$ is the area of quadrilateral $ABGT$. Al-Shannī assumes $AB > BG$ and $AT > TG$. Points D and K are midpoints of arcs ABG and ATG. He draws DK, which is a diameter of the circle. It is easy to show that DK intersects AG perpendicularly, call Z the point of intersection. Al-Shannī drops perpendiculars DE and KH onto sides AB and AT.

Now ABG is a broken line and D is the midpoint of arc ABG, thus $[ADG]-[ABG] = EB \cdot ED$. Similarly ATG is a broken line and K is the midpoint of arc ATG and therefore $[AKG] - [ATG] = HT \cdot HK$. Adding the two equalities we obtain
$[ADGK] - [ABGT] = EB \cdot ED + HT \cdot HK$.

According to the previous proof, triangle BED is similar to triangle AZD, and also triangle KHT is similar to triangle KZA. Al-Shannī realized that the two triangles AZD and KZA are similar to the big triangle KAD. Thus the three triangles BED, KAD and KHT are similar. Therefore the following three rectangles are similar: (1) the rectangle contained by AK and AD; (2) the rectangle contained by EB and ED; (3) the rectangle contained by HK and HT. See Figure 6.[15] The rectangle $AK \cdot AD$ is the area of the kite $ADGK$.

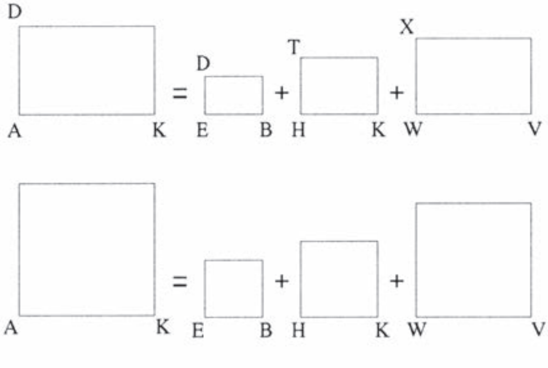


Figure 6

We express the difference $AK \cdot AD - EB \cdot ED - HK \cdot HT$ as the area of a fourth rectangle $WV \cdot WX$ similar to these three rectangles, as in Figure 6. By the same arguments as above,

[15]In Figure 5 $\angle DBE = \angle AKD$, $\angle HTK = \angle ADK$ and $EB : ED =$
$= AK : AD = HK : HT$.

$AK^2 = EB^2 + HK^2 + WV^2$, and $AD^2 = ED^2 + HT^2 + WX^2$ and the required area $[ABGT]$ is the area of the rectangle $WV \cdot WX$.

Returning to Figure 5, by the theorem of Pythagoras $AK^2 = AH^2 + HK^2$ and $AD^2 = ED^2 + AE^2$, and thus we can simplify to $AH^2 = EB^2 + WV^2$ and $AE^2 = HT^2 + WX^2$. Putting $c = AB$, $a = BG, b = GT, d = TA$ and $s = \frac{1}{2}(a + b + c + d)$, we have by assumption $c > a$ and $d > b$, and

$AE = \frac{1}{2}(c + a)$, $EB = \frac{1}{2}(c - a)$, and similarly

$AH = \frac{1}{2}(d + b)$, $HT = \frac{1}{2}(d - b)$, whence

$WV^2 = AH^2 - EB^2 = (\frac{1}{2}(d+b))^2 - (\frac{1}{2}(c-a))^2 = (s-a)(s-c)$,

$WX^2 = AE^2 - HT^2 = (\frac{1}{2}(c+a))^2 - (\frac{1}{2}(d-b))^2 = (s-b)(s-d)$,

and finally $[ABGD] = WV \cdot WX = \sqrt{(s-a)(s-b)(s-c)(s-d)}$.

As above, al-Shannī avoids four-dimensional magnitudes until the very end. He uses a theorem on proportions as above to obtain $(s - b)(s - d) : [ABGT] = [ABGT] : (s - a)(s - c)$. He does not use modern notation and he introduces auxiliary points L on AE and M on AT such that $HM = EB$ and $EL = TH$ in order to get line segments with lengths $AM = s - c$ and $AL = s - d$.

According to al-Bīrūnī. al-Shannī assumed $c > a$ and $d > b$. Other cases such as $c > a$ and $d < b$ can be dealt with in similar ways.

The proof in the Leiden version is similar to that in the Patna version but the exposition in the Leiden version is less clear, although Suter was able to reconstruct the correct argument in his translation and notes [30, pp. 40-42, 70]. The codex Or. 513, which contains the proof, was bought in Istanbul by the German-born orientalist Levinus Warner (1618-1665), bequeated by him to Leiden University, and shipped to the Netherlands before 1674. The *Extraction of Chords* is mentioned in the 1674 catalogue [28, p. 374] as "Birounii Sylloge de Subtensis, cum demonstrationibus Archimedis et aliorum" (al-Biruni's Discussion of Chords, with proofs by Archimedes and others). Thus the manuscript was in the Netherlands before Abraham de Graaf published the first Western proof of the area rule for a cyclic quadrilateral in 1706. We have not found evidence that Abraham de Graaf or anyone else in Western Europe was aware of Al-Shannī's proof before Suter started to work on the Leiden manuscript in the early twentieth century.

4. The two versions of the *Extraction of Chords* by al-Bīrūnī

Al-Shannī's proofs of the area rules for the triangle and cyclic quadrilateral constitute two chapters among the more than sixty chapters in al-Bīrūnī's *Extraction of Chords*. Al-Bīrūnī begins the work with more than twenty chapters on properties of the "broken line" (see above, Figure 2) and several proofs of each of these properties by different mathematicians, whose names are mentioned. Then he presents around fifteen chapters on various applications of these properties, including al-Shannī's proofs of the area rules for the triangle and the cyclic quadrilateral. The work ends with chapters on the application of the properties of the broken lines to the computation of chords in the circle, which are crucial to trigonometry.

The *Extraction of Chords* is available in four Arabic manuscripts which will be indicated by the symbols P, L, M and C.

P = Patna, India, Khuda Bakhs Public Oriental Library no. 2468 (more recent numbering 2519), ff. 282a-298b + 326 a, see [18, p. 92]. The text is complete; the intermediate leaves 299a-325b contain parts of other texts, unrelated to the *Extraction of Chords*. Manuscript P is dated on f. 326a, where the anonymous scribe writes "I finished writing it in Mosul at the end of Dhu'l-Qaʿda of the year 631 of the Hijra" (August 1234 AD).The end of the text also contains in red the marginal note "Abu'l-Rayḥān, may God have mercy on him, finished writing it in Rajab of the year 418" (August 7 - Sept. 8, 1027 AD). The text in the manuscript is written in black (or dark brown) ink, and the titles of the chapters are in red ink. The geometrical figures are in black ink, the letters labelling the points are in red ink.

L = Leiden, University Library, Or. 513, ff.108b-129a, see [33, pp. 225-226]. The manuscript is undated but owner's marks show that it was written before 982 A.H / 1575 CE. The manuscript is written in black ink with titles of the chapters in red. The lines in the geometrical figures are drawn in red ink, letters labelling the points are in black ink.

M = Istanbul, Süleymaniye Library, Murat Molla 1396, ff. 52b-72b, see [17, p. 480]. The manuscript is undated. Manuscript M is closely related to manuscript L because in the two propositions that have been edited in this paper, L and M have in common two redundant passages resulting from scribal error, as well as many common mathematical errors. Manuscript M was available in a black and white copy.

C= Cairo, Dār al-Kutub, Muṣṭafā Fāḍil Riyāḍa 41m, ff. 74b-82a, see [16, pp. 48, 112], written in the eighteenth century by the Egyptian mathematician Muṣṭafā Ṣidqī. Manuscript C contains only a small part of the *Extraction of Chords*, which does not include al-Shannī's proofs of the area rules for the triangle and the cyclic quadrilateral. Thus it has not been used in the edition.

Inspection shows that the four manuscripts contain two essentially different versions of the *Extraction of Chords*. One version is only extant in P and will be called the Patna version. The second version is extant in L, M and C; it will be called the Leiden version because L is superior in quality to M, and C contains only part of the text.

It is not the purpose of this article to investigate the exact relationship between the Patna and the Leiden versions. Some of the differences will be listed here. For the Patna version we refer to the unreliable Hyderabad edition [2], and for the Leiden version to the German translation of the Leiden manuscript by Suter [30].

1. There are many structural and editorial differences between the two versions. For the first property of the broken line (Figure 3 above), the Patna and Leiden versions present the proofs in different order. In the Patna version, the proofs are grouped and analyzed according to structure of the argument; structurally different proofs by the same mathematician may occur in different chapters. In the Leiden version, most of the proofs are grouped according to the name of the mathematician, and therefore proofs with similar structure and the same geometrical figure may be repeated in different places. The Patna version discusses four different properties of the broken line, all provided with proofs. The Leiden version discusses only two properties, which correspond to the first two properties of the broken line in the Patna version. In the Leiden version, the third and fourth properties are not mentioned explicitly although they implicitly occur in some of the proofs of the first property. In a few geometrical figures, points are labelled in different ways in the two versions, as in Figures 8 and 10 below. The preface to the Patna version is more elaborate, and the references to Muḥammad ibn Zakariyyāʾ al-Rāzī (865-925 CE) in the Patna version [2, p. 3 lines 5, 10] are missing in the Leiden version [30, p. 11-12].

2. Two large parts of the Patna version are missing in the Leiden version. The first of these parts includes the proofs of the first property of the broken line by the tenth-century geometers Sulaymān ibn ʿIṣmat al-

Samarqandī, Abu'l-Ḥasan ibn Bāmshādh, Abu'l-Ḥasan al-Samarqandī and Abū Ja'ar al-Khāzin [2, p. 36-45 line 3]. The second missing part in the Leiden version consists of the two chapters at the end of the Patna version on the computation of the chord of one degree and the trisection of the angle, including geometrical figures [2, p. 107 line 1 - p. 108 line 14, followed by p. 224 line 2 - p. 226 line 4]. The Leiden version has only one sentence on the subject [30, pp. 63-64]. The Leiden version may ultimately be based on a manuscript of the *Extraction of Chords* from which some leaves were missing.

3. In the Leiden version, two proofs are attributed to Abū ʿAbdallāh al-Shannī where no such attribution is made in the Patna version. The first case concerns the proof of the triangle rule by Archimedes which is published in this paper. Above it has been argued that the attribution to al-Shannī is plausible. The second case concerns a geometrical construction of two line segments beginning at two given points A and B and meeting at a point G in such a way that $\angle AGB$ is given and the product $AG \cdot BG$ is given, compare [30, p. 35] where al-Shannī is mentioned, with [2, p. 53] where no attribution is made.

In the two proofs which are published in this paper, the following differences between the Patna and Leiden versions can be noticed:

4. The Patna version is concise and clear, while the Leiden version is more elaborate and sometimes confused. For details see the footnotes to the translation of the Leiden version.

5. At the end of the proof of the area rule for the triangle, the Leiden version discusses the case where the triangle is isosceles. This case is not mentioned in the Patna version.

The two versions of the *Extraction of Chords* were compared by Saidan [24, p. 771] and by Bulgakov and Rosenfeld [4, p. 9]. Saidan calls the Leiden version a "much abbreviated copy" of the original, which is the Patna version in his opinion. Bulgakov and Rosenfeld add that the Leiden version may have been revised by an anonymous editor, but since the Patna manuscript is imperfect, the Leiden version is necessary for the reconstruction of al-Bīrūnī's original.[16] The precise

[16]The Iranian historian of mathematics A.Q. Qurbani argued in [20] that the Leiden version is a different work by Biruni entitled "Collection of current ways for the determination of chords in the circle" (*Jamʿ al-ṭuruq al-sāʾira fī maʿrifat awtār al-dāʾira*). On the last page f. 326a of the Patna manuscript, al-Bīrūnī says that the *Jamʿ al-ṭuruq*

relationship between the two versions can only be established in the context of a critical edition of the entire *Extraction of Chords* and a comparison with relevant other works of al-Bīrūnī.

The *Extraction of Chords* has a complicated publication history. The first publication of the work in a Western language was Suter's German translation [30] on the basis of the Leiden manuscript only. The Arabic text in the Patna manuscript was first published in 1948 in the Hyderabad edition [2], which is problematic for two reasons. The editors did not check the mathematical meaning of the text, and therefore they did not catch the scribal errors in the imperfect Patna manuscript. Secondly, the editors believed that the whole manuscript text from 282a through 326a belonged to the *Extraction of Chords*. In 1960 A.S. Saidan discovered that the text in the Hyderabad edition on [2, p. 108 line 14] continues on [2, p. 224 line 2] and that the more than hundred pages intermediate text in [2] (corresponding to the leaves 299a-325b in the disorganized Patna manuscript) are fragments at least one other work by al-Bīrūnī and one work by the tenth-century geometer Ibrāhīm ibn Sinān.[17] In 1965 A. S. al-Demirdash published in [7] an edition of the Arabic text of the *Extraction of Chords* on the basis of the Patna version. The Arabic text on the area of the rules for the triangle and the cyclic quadrilateral is in [7, p. 104-119], and al-Demirdash included the proof of the triangle rule in the Leiden version on the basis of manuscript M. The edition by al-Demirdash is not a critical edition but his text is mathematically much superior to the Hyderabad edition [2]. Unfortunately al-Demirdash mixed his own commentaries in Arabic with the Arabic text of al-Bīrūnī. Al-Demirdash was apparently unaware of the article [23], and thus he did not realize that part of his edited text [7, p. 172-286] did not belong to the *Extraction of Chords* but to other works by al-Bīrūnī and by Ibrāhīm ibn Sinān.

An excellent Russian translation of the *Extraction of Chords* with commentary was published in 1987 in [4, pp. 27-77, 259-278]. The

al-sā'ira fī ma'rifat awtār al-dā'ira contains additional material, not found in the Patna manuscript, from his own commentary on the reasons of the Zīj (astronomical handbook) of Ḥabash, related to the approximate determination of the chord of one degree [2, p. 226 lines 1-4]. No such material is found in the Leiden version, and therefore neither the Patna version nor the Leiden version can be the *Jam' al-ṭuruq al-sā'ira fī ma'rifat awtār al-dā'ira*. Qurbani did not have access to the Patna manuscript but based himself on the unreliable Hyderabad edition [2].

[17]See [23] and for more details [9, pp. 148-159].

translators Bulgakov and Rosenfeld did not know Saidan's article [23] but they reached the same conclusions and excised the text in the leaves 299a-325b of the Patna manuscript. The Russian translation of the text on the area rules for the triangle and the cyclic quadrilateral is in [4, pp. 53-56, 269-270]; the translation is based on the Patna version, to which a few sentences from the Leiden version were added in italics.

Most of the secondary literature in Western languages on the *Extraction of Chords* is based on Suter's translation [30] of the Leiden version. Although Suter reconstructed the correct argumentation in the footnotes to his translation and in his notes [30, pp 39-42, 70], the geometric ideas are less than clear. In his influential *Geschichte der Elementarmathematik*, the German historian Johannes Tropfke called the proof complicated and difficult, and too long to be summarized [32, p. 156]. By means of the texts and translations in this paper, the reader may convince himself of the essential simplicity of the proof.

5. Editorial procedures

The Arabic edition of the Patna version of the proofs of the two area rules is based on manuscript P ff. 291a-292a. The manuscript P is not perfect and I have done my best to reconstruct al-Bīrūnī's original text by means of a few corrections which are paleographically possible. These corrections are indicated in the apparatus at the end of the text, with the following exceptions. In references to the labels of points in the geometrical figures, the text in manuscript P often reads ḥā' (H) where it should be jīm (G) and vice versa, and initial bā' (B) is often indistinguishable from initial lām (L). In such cases I have selected the mathematically correct label, without burdening the apparatus with these trivial corrections. I have indicated all instances where my reconstructed text differs from the Hyderabad edition [2, pp. 61-67], which I have abbreviated to H in the apparatus to the Arabic text, and to Hyd in the translation. My corrections which are also found in the edition of al-Demirdash [7, pp. 104-119] have been credited to him, using the abbreviation D in the apparatus and Dem in the translation. Angular brackets in the Arabic text and the translation < ...> include a few passages which I have restored to the text in P in order to reconstruct al-Bīrunī's original. These restorations are inspired by the Leiden version, which is ultimately based on the same original. Such reconstructed text inside angular brackets is not found in the Hyderabad edition [2]. Page numbers are indicated in square brackets, using the abbreviations P for the Patna manuscript and H for the Hyderabad edition [2]. I have adapted the Arabic text to modern orthography and have added a few full stops.

The Arabic edition of the Leiden version of the two proofs of the area rules is based on the two manuscripts L ff. 119b-121a and M ff. 63b-65a. Page numbers in these manuscripts appear in square brackets in the text. I have attempted to construct an Arabic text which is mathematically correct, and have put the variant readings in the manuscripts in the apparatus, but with the same exceptions H/G and B/L as for the Patna version. It should be noted that the Leiden version may be the product of an editor whose mathematical abilities may have been less than perfect. Therefore the mathematically corrected text may well reflect the original which this anonymous editor used rather than his edited Leiden version. I have adapted the Arabic text to modern orthography and have added a few full stops.

In the translations of the Patna and the Leiden versions and in Figures 7 to 10, the labels in the geometrical figures (which the Arabic-Islamic authors considered as numbers in the abjad-system rather than letters) have been transcribed according to the following key: alif=1=A, bāʾ=2=B, jīm=3=G, dāl=4=D, hāʾ=5=E, zāʾ =7=Z, ḥāʾ=8=H, ṭāʾ=9=T, kāf=20=K, lām=30=L, mīm=40=M. The labels wāw=6 and yāʾ=10 were not used in our text; these two labels were rarely used anyway.

In the translations, parentheses ( ) include explanatory additions made by me. Footnotes to the translation of the Patna version contain mathematical explanations and references to the *Elements* of Euclid which was (in Arabic translation) the standard geometrical textbook in the time of al-Bīrūnī. Such mathematical explanations have not been repeated in the Leiden version. In the translation of the Leiden version I have added footnotes in all cases where the reconstructed text differs from the two manuscripts L and M at the same time, with due references to Suter's translation [30, p. 39-42]. In such cases, the incorrect text in the two manuscripts may have been produced by the anonymous editor of the Leiden version. In the translations of the two versions, words written in red ink in the manuscript text are printed in *italics* in the translation.

6. Translation of the Patna version.

Proof of the procedure of Archimedes in measuring the triangle by means of the excess.

Archimedes said: half the sum of the three sides of the triangle is multiplied by its excess over one of them, and the outcome by its excess over the second, and the outcome by its excess over the third, and the square root of the outcome is taken, and that is the area of the triangle.

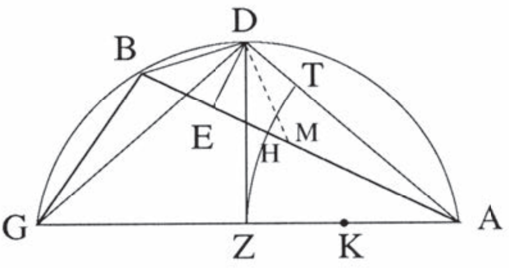


Figure 7

Proof of it: the triangle is ABG. We circumscribe around it a circle. We drop from the midpoint D of arc ABG perpendicular DE onto AB, $<$ and perpendicular DZ onto AG. $>$ We describe with centre A and radius AZ arc ZHT.

Then since the square of DA is equal to the (sum of the) squares of DZ, ZA,[18] (thus) the square of DT and $<$ twice $>$[19] the product DT by TA and the square of TA are (together) equal to the (sum of the) squares of DZ, ZA,[20] which is equal to TA. Thus if we subtract the squares of AT, AZ, which are equal, then the square of DT and twice the product DT by TA are (together) equal to the square of DZ.

In a similar way, the square of AD is equal to the (sum of the) squares of DE, EA. Therefore the (sum of the) squares of DT, TA and twice the product DT by TA are (together) equal to the square of DE and the (sum of the) squares of EH, HA and $<$ twice $>$[21] the

[18] $DA^2 = DZ^2 + ZA^2$ and $DA^2 = DE^2 + EA^2$ by the theorem of Pythagoras, Euclid *Elements* I:47 [8, vol. 1 p. 349].

[19] Added by Dem [7].

[20] $DA^2 = DT^2 + 2DT \cdot TA + TA^2$ and $EA^2 = EH^2 + HA^2 + 2EH \cdot HA$ by Euclid, *Elements* II:4 [8, vol. 1 p. 379.]

[21] Added by Dem [7].

product EH by HA. But HA is equal to AT. Thus if we subtract the two equal squares of them, the square of DT and twice the product DT by TA are (together) equal to the (sum of the) squares of DE, EH and twice the product EH by HA, and that is also equal to the square of DZ.

Triangle DZA is similar to triangle DEB because angle DGZ, which is equal to angle DAZ, is (also) equal to angle DBE which is (i.e., stands) together with it (DGZ) on the same arc.[22] Therefore the ratio of DE to EB is equal to the ratio of DZ to ZA.[23] But the ratio of DZ to ZA is equal to[24] the ratio of the square of DZ to the product DZ by ZA and equal to the ratio of the product DZ by ZA to the square of ZA. Just like this also, the ratio of the square of DE to the product DE by EB is equal to the ratio of the product DE by EB to the square of EB.

If one subtracts from mutually proportional magnitudes (other) mutually proportional magnitudes in their (same) ratios, the ratios of the remainders are the same as their (ratios).[25]

Thus we subtract the square of DE from the square of DZ, then the remainder is the square of EH and twice the product EH by HA, that is the product EH by the sum of EA[26] and one time HA.[27] We subtract the product DE by EB from the product DZ by ZA, then the remainder is the area of triangle ABG, because it has been explained

[22] $\angle DBA = \angle DGA$ because they stand on arc AD, by Euclid's *Elements* III:21 [8, vol. 2 p. 49]. The triangles DBE and DAZ are similar because $\angle DBE = \angle DGZ = \angle DAZ$ and $\angle E$ and $\angle Z$ are right angles.

[23] In the similar triangles DBE and DAZ we have $DE : BE = DZ : ZA$ by Euclid, *Elements* VI:4 [8, vol. 2 p. 200].

[24] We have $DZ : ZA = DZ^2 : DZ \cdot ZA = DZ \cdot ZA : ZA^2 = DE : EB = DE^2 : DE \cdot EB = DE \cdot EB : EB^2$ by Euclid, *Elements* VI:1 [8, vol. 2 p. 191].

[25] The text means the following in modern notation. Suppose $a_1 \dots a_4$ and $b_1 \dots b_4$ are "mutually proportional magnitudes" of the same kind, that is to say that $a_1 : a_2 = a_3 : a_4$ and $b_1 : b_2 = b_3 : b_4$. If they are in the same ratio, that is to say if $a_1 : a_2 = b_1 : b_2$ and if one assumes $a_1 > b_1, a_2 > b_2$ etc., then $(a_1 - b_1) : (a_2 - b_2) = (a_3 - b_3) : (a_4 - b_4) = a_1 : a_2 = a_3 : a_4 = b_1 : b_2 = b_3 : b_4$ by Euclid, *Elements* V:19 [8, vol. 2 p. 174]. In the text the theorem is used for $a_1 = DZ^2, a_2 = a_3 = DZ \cdot ZA, a_4 = ZA^2, b_1 = DE^2, b_2 = b_3 = DE \cdot EB, b_4 = EB^2$ to obtain $(DZ^2 - DE^2) : (DZ \cdot ZA - DE \cdot EB) = (DZ \cdot ZA - DE \cdot EB) : (ZA^2 - EB^2)$.

[26] P and Hyd [2] have EH.

[27] Above it has been proved that $DZ^2 = DE^2 + EH^2 + 2EH \cdot HA$, and we have $EH^2 + 2EH \cdot HA = EH(EH + HA + HA) = EH(EA + HA)$.

that triangle AGD is equal to the sum of triangles ABG, BDM.[28] Let ZK be equal to EB. We subtract the square of ZK, that is EB, from the square of ZA, then the remainder is equal to the product GK by KA.[29] These remainders are proportional, that is to say that the ratio of the product EH by the sum of EA, AZ to the area of triangle ABG $<$ is equal to the ratio of the area of triangle ABG $>$ to the product GK by KA.

AE is half the (sum of the) sides AB, BG and AZ is half of side AG, thus the sum of EA, AZ is half the sum of the sides of the triangle.[30] Therefore EH is the excess of (the sum of) EA, AZ, half the sum of the sides, over the sum of HA, AZ, that is to say AG, and this is one of the excesses. Since ZK and EB are equal, the sum of AE, ZK is equal to side AB. Therefore AK is the excess of the sum of EA, AZ over EA, ZK, that is to say AB, and this is the second excess. And since (the sum of) EB, BG is equal to AE, therefore (the sum of) EB, BG, AZ is equal to half the sum of the sides, thus the excess of it over BG is (the sum of) EB, AZ. But KZ is equal to EB and ZG is equal to AZ, thus KG is the excess of half the sum of the sides over BG, and this is the third excess.[31] And when we multiply area EH by (the sum of) EA, AZ, which is one of the extreme terms in the ratio, by the area GK by KA, the other extreme term, the outcome is the square of the mean (term), which (term) is the area of the triangle.[32]

[28]The fourth property of the broken line is to the effect that the area of triangle AGD is equal to the area of the triangles ABG plus the rectangle $DE \cdot EB$. Here the text represents $DE \cdot EB$ as the area of a triangle BDM, and we can therefore assume with [4, p. 269 n. 235] that point M is located on the line segment AE in such a way that $EM = EB$. Point M and line DM are not found in the figure in the manuscript. and therefore line DM is displayed as a broken line in Figure 7.

[29]We have $GZ = ZA$ and therefore $ZA^2 - ZK^2 = (GZ + ZK) \cdot (ZA - ZK) = = GK \cdot KA$ by Euclid, *Elements* II:5 [8, vol. 1 p. 382].

[30]If we write the sides in modern notation as $a = BG, b = AG, c = AB$ and the half perimeter $s = \frac{1}{2}(a + b + c)$ as in the introduction, we have $EA = \frac{1}{2}(c + a)$ by the first property of the broken line. Since $GZ = ZA$, also $ZA = \frac{1}{2}b$ and therefore $EA + ZA = s$.

[31]Using the same notation it is easy to check the computations. Because $AH = = AZ = \frac{1}{2}b$, therefore $EH = AE - AH = \frac{1}{2}(a + c) - \frac{1}{2}b = s - b$. Further $ZK = EB = \frac{1}{2}(c - a)$ by the definition of point K, therefore $AK = AZ - ZK = = \frac{1}{2}b - \frac{1}{2}(c - a) = s - c$. Finally $KG = KZ + ZG = \frac{1}{2}(c - a) + \frac{1}{2}b = s - a$.

[32]The text proves $EH \cdot (EA + AZ) : [ABG] = [ABG] : GK \cdot AK$, or in

It is the same if we multiply EH, the first excess, by (the sum of) EA, AZ, half the sum of the sides, and (then) multiply AK, the second excess by GK, the third excess, and then multiply the two products, or if we multiply (the sum of) EA, AZ, half the sum of the sides, by KA, and then the outcome by EH, and then the outcome by GK, for the two (final) outcomes are the same, and that is the square of the area of the triangle. Thus, if we take the square root of it, we obtain what was desired.

Proof of the procedure of India for the measurement of the trapezium[33] *in the circle, by Abū ʿAbdallāh al-Shannī.*

Abū ʿAbdallāh al-Shannī built on this[34] in the proof of the method of India for the area of the quadrilateral in the circle. It is as follows: they multiply the (four) excesses of half the sum of its sides over each side by one another and they take the root of the outcome, then it is the area of the trapezium, and let it be $ABGT$.

modern notation $s(s-b):[ABG]=[ABG]:(s-a)(s-c)$. We write the area of triangle ABG as $[ABG]$. The multiplication of two areas or four line segments shows $[ABG]^2 = s(s-a)(s-b)(s-c)$, that is to say $[ABG] = \sqrt{s(s-a)(s-b)(s-c)}$, but note that a multiplication of four line segments is not possible in Greek geometry and this is probably the reason why the multiplication is delayed until the very end.

33 Although the Arabic word is *munḥarif*, meaning trapezium, the first line of the text shows that an arbitrary "quadrilateral in the circle" is meant.

34 In Figure 8, the letters A, B, G, D, E, Z are labels of the same points as in Figure 7, and for these points the same assumptions hold. We can show that DZ is perpendicular to AG: since DK is a diameter of the circle and arc AD = arc DG, therefore arc AK = arc GK, thus $\angle ZAG = \angle ZDG$ and therefore triangles ZAD and ZGD are congruent because also $AD = GD$ and $ZD = ZD$ by *Elements* I:4 [8, vol. 1 p. 247]. We conclude $AZ = ZG$ and $\angle AZD = \angle GZD$ which must therefore be right angles.

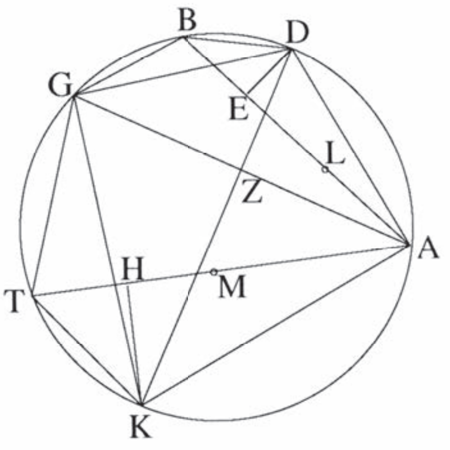


Figure 8

We draw AG and we draw from the midpoint D of arc ABG the diameter DZK, and we drop two perpendiculars DE, KH onto AB, AT. Then, because of the similarity of triangles DEB, DZG and DB being shorter than DG, therefore ZG,[35] that is to say AZ, is greater than EB.[36] But AZ, half of side AG, is less than AH, half of the sum of the sides AT,[37] TG.[38] Therefore AH is much greater than EB. Similar to this it is shown that AE is greater than HT. Thus we cut off EL equal to HT and HM equal to EB.[39]

It is known that the excess of area $ADGK$ over area $ABGT$ is equal to the product DE by EB together with the product KH[40] by HT.[41]

[35]Hyd [2] reads AG, restored to GZ by Dem [7].

[36]Triangles DEB and DZG are similar and $DB : EB = DG : ZG$ as in the previous proposition. Because arc BD is less than arc DG, $DB < DG$, and therefore $EB < ZG$ by Euclid, *Elements* V:14 [8, vol. 2 p 162].

[37]Hyd [2] reads AG, restored in [4, p. 55]. Here al-Shannī uses the fact that ATG is also a "broken line" in the circular arc AKG with midpoint K.

[38]The argument is based on *Elements* I:20, Heath vol. 1, p 286.

[39]Since $EL = HT < AE$ and $HM = EB < AH$, point L is located between E and A and point M is located between H and A.

[40]Manuscript P and Hyd [2] have KG, restored to KH by Dem [7].

[41]Since ABG is a broken line in arc ADG with midpoint D, the area of triangle ADG is the sum of the area of triangle ABG plus the rectangle $EB \cdot ED$. Since ATG is a broken line in arc AKG with midpoint K, the area of triangle AKG is the sum of the area of triangle ATG plus the rectangle $HT \cdot HK$. By addition we obtain the area of kite $ADGK$ as the sum of the area of the quadrilateral $ABGT$ plus the sum of the two rectangles.

Triangle DAK is similar to each of the two triangles DEB, KHT.[42] Therefore the ratio of DA to AK is equal to the ratio of DE to EB and (also) equal to the ratio of TH to HK. But the ratio of DA to AK is equal to the ratio of the square of DA to the product DA by AK <and equal to the ratio of the product DA by AK to the square of AK.> The product of DA and AK is equal to twice triangle DAK and that is area $ADGK$. Thus it (area $ADGK$) is a mean proportional between the product AD by DG and the product AK by KG.[43]

The sum of proportional magnitudes together with other proportional magnitudes in the same ratio or the differences between them - each (magnitude) combined with the corresponding (magnitude), are proportional in the same way.[44] Therefore the ratio of (1) the sum of the squares of DE, TH to (2) the sum of the product DE by EB and the product of TH and KH is the same as the ratio of the sum of these two areas to (3) the sum of the squares of EB, KH.

And if (1) the sum of the squares DE, TH,[45] that is to say EL, is subtracted from the square of AD, and (2) the sum of the product DE by EB and the product of TH and KH (is subtracted) from the area $ADGK$ and (3) the sum of the squares of EB, KH (is subtracted) from the square of AK, the remainders are proportional.[46] As for the first remainder, it is the product AL by the sum of LB, BG[47] because if the square of DE is subtracted from the square of AD, the square of AE remains, therefore if from it one subtracts the square of HT, which is equal to EL, the remainder is the square of AL and twice the product AL by LE, and that is equal to the product AL by the sum of AE, EL, that is to say the product AL by the sum of GB, BE, which

[42]This follows from $\angle DBA = \angle DKA$, $\angle KDA = \angle KTA$ and $\angle BED = = \angle DAK = \angle THK$ which are right angles, using Euclid, *Elements* III: 27, 31, VI:4 [8, vol. 2 pp. 58, 61, 200].

[43]The text means $AD \cdot DG : [ADGK] = [ADGK] : AK \cdot KG$, which follows from $AD^2 = AD \cdot AK = AD \cdot AK = AK^2$.

[44]This is a generalisation of the theorem in the previous proposition: if $a_1 : a_2 = a_3 : a_4 = b_1 : b_2 = b_3 : b_4$ then $(a_1 \pm b_1) : (a_2 \pm b_2) = (a_3 \pm b_3) : (a_4 \pm b_4) = = a_1 : a_2 = a_3 : a_4 = b_1 : b_2 = b_3 : b_4$. The addition case can be proved by repeated applications of Euclid, *Elements*, V:16, 18, [8, vol. 2 pp. 164, 169].

[45]Hyd [2] reads TG, restored to TH by Dem [7].

[46]The text means $(AD^2 - DE^2 - TH^2) : ([ADGK] - DE \cdot EB - TH \cdot KH) = = ([ADGK] - DE \cdot EB - TH \cdot KH) : (AK^2 - EB^2 - KH^2)$.

[47]Hyd [2] reads LG, restored to GB by Dem [7].

(sum) is equal to AE, (added) to LE. As for the second remainder, it is trapezium $ABGT$. As for the third (remainder), it is in a way similar to what has been explained for the first (remainder), the product AM by the sum of MT, TG.[48]

It is known that AE is half < the sum of AB, BG, and that AH is half > the sum of AT, TG, Therefore the sum of EA, AH is half the sum of the sides of the trapezium $ABGT$. Because of the equality of EB and HM, the excess of the sum of EA, AH over side AB, that is to say, (over the sum of) AE, HM, is AM, (which is therefore) the first excess, and (the excess of the sum of EA, AH) over AT is in the same way AL,[49] the second excess. Again, the sum of lines EB, BG, GT, TH is also half the sum of the sides of the trapezium, therefore the excess of it over side BG is the sum of GT, TH together with HM which is equal to EB, and it (the sum of GT, TH, HM) is the third excess, and the excess of it over side GT is in a way similar to this, (the sum of) LB, BG,[50] the fourth (excess).[51]

As has been explained before,[52] the trapezium $ABGT$ is a mean proportional between the product AL, the second (excess), by the sum of LB, BG, the fourth (excess), and the product of AM, the first excess, by the sum of MT, TG, the third excess.[53]

It is the same if we multiply one of these two products by the other, or multiply the first excess by the second, the outcome by the third, and the outcome (of this) by the fourth, for by both of these (methods) one obtains the square of the mean, that is the trapezium. Therefore if we take the root of it, it is the desired (area).

[48]By the theorem of Pythagoras $AD^2 - DE^2 = AE^2$ and $AK^2 - KH^2 = AH^2$. Then the previously defined points L and M are used: $AE^2 - TH^2 = AE^2 - EL^2 =$
$= (AE - EL)(AE + EL) = (AE - EL)(GB + BE + EL) = AL \cdot (GB + BL)$
and $AH^2 - EB^2 = AH^2 - HM^2 = (AH - HM)(AH + HM) =$
$= (AH - HM)(GT + TH + HM) = AM \cdot (GT + TM)$.

[49]P and Dem [7] have AL, Hyd [2] reads incorrectly *ilā*, "towards".

[50]Hyd [2] has LG, restored in [4, p. 56].

[51]In modern notation we can verify these results as follows. Put $c = AB$, $a = BG, b = GT, d = TA$ and $s = \frac{1}{2}(a + b + c + d)$. Then we have
$AE = \frac{1}{2}(c + a), HM = EB = \frac{1}{2}(c - a), AH = \frac{1}{2}(d + b), EL = HT = \frac{1}{2}(d - b)$,
whence $s = AE + AH = EB + BT + TG + GH$ and therefore
$AM = AH - HM = s - c, AL = AE - EL = s - d$,
$MT + TG = AH + HM = s - a, LB + BG = AE + EL = s - b$.

[52]Hyd [2] has errors in the following sentence, see the apparatus to the Arabic text.

[53]The text means $AL \cdot (LB + BG) : [ABGT] = [ABGT] : AM \cdot (MT + TG)$.

7. Translation of the Leiden version.

Proof of the rule of Archimedes for measuring triangles by means of the excess of their sides, by Abū ʿAbdallāh al-Shannī.

Archimedes said: if we want the measurement (i.e., area) of any triangle by means of its sides, we take the excess of half the sum of its sides over each of its sides. We multiply one of these three excesses by another of them and then the outcome by the third (excess) and then the outcome by half the sum of the sides of the triangle, and we take the root of the outcome, then the result is the measurement (i.e., area) of that triangle.

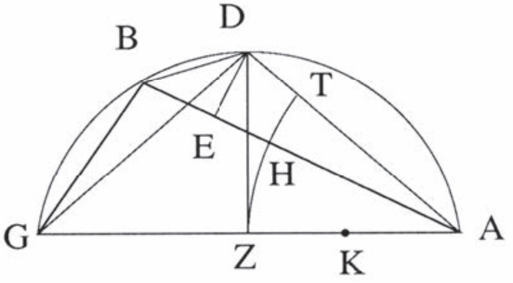


Figure 9

Abū ʿAbdallāh said: Proof of this: we make the triangle (such that) on it (are labels) ABG and we circumscribe on it circle $ADBG$, and let point D be at the midpoint of arc $ADBG$. We drop from it onto line AB perpendicular DE, and onto line AG[54] perpendicular DZ, and we draw DA, DG, DB. We make point A the centre and we draw with radius AZ arc ZHT of a circle, which intersects line AE at point H and line AD at point T.

Then the square of DA is equal to the (sum of the) squares of DE, EA. The two lines AT, AZ[55] are equal. The square of DT and twice the product DT by TA, and that is equal to the square of DZ since we have made AT equal to AZ.[56] The square of DZ is equal to the square

[54] The manuscripts have AD, corrected to AG by Suter [30] and Dem [7].

[55] The manuscripts have AT, AD, Suter [30] and Dem [7] have correctly AT, AZ.

[56] The sentence is grammatically defective. Mathematically, $DT^2 + 2DT \cdot TA = DZ^2$ can be obtained by subtracting $AT^2 = AZ^2$ from $DT^2 + 2DT \cdot TA + AT^2 = DA^2 = DZ^2 + AZ^2$.

of DE together with the square of EH and twice the product EH by HA.[57]

Since triangle DEB is similar to each of the triangles DZG, DZA because angles AGD,[58] ABD, GAD are equal because each of them is the chord of[59] half of arc $ADBG$, thus the ratio of DE to EB is equal to the ratio of DZ to ZA. But the ratio of DZ to ZA is equal to the ratio of the square of DZ to the area DZ by ZA and equal to the ratio of area DZ by ZA to the square of ZA. Therefore[60] also the ratio of the square of DE to area DE by EB is equal to the ratio of area DE by EB to the square of EB.

If from proportional (magnitudes) one subtracts proportional (magnitudes) in the ratio of them, the remainders are proportional.

Thus, if we subtract the square of DE from the square of DZ, the remainder is, by what we have explained, the square of EH and twice the product EH by HA, that is the product EH by the sum of EA, AZ.[61] And if we subtract the product DE by EB from the product DZ by ZA, which is the area of triangle DAG, the remainder is the area of triangle ABG, and that is by what we explain among the properties of the broken line.[62] And if we subtract the square of EB, that is the square of KZ if we make ZK equal to EB, from the square of ZA, the remainder is the product GK[63] by KA. Thus, by subtraction, the triangle ABG is a mean proportional between the area EH by the sum of EA, AZ and the area GK by KA.[64]

As for EH, it is the excess of the sum of EA, AZ, which is half the sum of the sides of triangle ABG, over side AG, since we have

[57] Here the manuscripts repeat: "thus the square of DZ is equal to the square of DE and the square of EH and twice the product EH by HA".

[58] The manuscripts have ADZ, corrected to AGD by Suter [30].

[59] The Arabic verb *tuwattiru*, from the root *watar* (chord), is terminologically incorrect because and angle can never be the chord of an arc.

[60] The manuscripts have "therefore", *li-dhālika*, but the mathematical meaning requires "just like this", *ka-dhālika*, as reconstructed by Suter [30] ("ebenso") and Dem [7, p. 109 line 13].

[61] We have $DZ^2 - DE^2 = EH^2 + 2EH \cdot HA = EH(EH + 2HA) =$
$= EH(EA + HA) = EH(EA + AZ)$.

[62] The property in question is the fourth property of the broken line in the Patna version. In the Leiden version this property is mentioned in the third proof of the first property of the broken line by al-Shannī [30, p. 20].

[63] The manuscripts have DK, restored to GK by Suter [30] and Dem [7].

[64] The manuscripts have KB, restored to AK by Suter [30] and to KA by Dem [7].

made AH equal to ZG, and since the perpendicular DE bisects the sum of the two lines AB, BG at E, and AG is also bisected at Z.[65] AK is also the excess of the sum of EA, AZ over AB because we have made[66] KZ equal to EB, and EH, AK, KG are the excesses of half the sum of the sides of triangle ABG over each of its sides.

It is the same to multiply the area AK by KG by the area EH by the sum of EA, AH, and to multiply AK by KG, that is, one of the excesses by the other, and then the outcome by EH, the third excess, and then the outcome by the sum of EA, AZ, half the sum of the sides. The outcome in all of them is the area of triangle ABG multiplied by itself. That is what we wanted to explain.

And if the triangle is isosceles, as triangle ADG,[67] then the ratio of the square of DZ to the area DZ by ZA is equal to the ratio of exactly the same area to the square of ZA. As for the square of DZ, we have explained that it is equal to the square of DT[68] and twice the product DT by TA, that is to say, equal to the product DT by the sum of DA, AZ. As for the area DZ by ZA, it is the area of triangle ADG. As for the square of AZ, it is equal to the area AZ by ZG. Therefore we multiply AT by GZ,[69] and that is one of the excesses by the other, then the outcome by DT, the third excess, then the outcome by the sum of DA, AZ, and that is the sum of half the sides, and the outcome is the square of area DZ by ZA, and that is what we wanted to explain.[70]

Proof of a rule attributed to India for the measurement of the trapezium circumscribed in a circle, by Abū ʿAbdallāh al-Shannī.

If we want the measurement of the trapezium[71] which is circum-

[65]The manuscripts have ZA, restored to Z by Dem [7].

[66]Here the manuscripts repeat: "AH equal to ZG Since the perpendicular DE bisects the sum of the two lines AB, BG at E, and AG is also bisected at Z, and AK is also the excess of the sum of EA, AZ over AB, because we have made."

[67]The manuscripts have ABG, restored to ADG by Suter [30].

[68]The manuscripts have DK, restored to DT by Suter [30] and Dem [7].

[69]The manuscripts have KZ, restored to ZG by Dem [7].

[70]The special case of the isosceles triangle is not discussed in the Patna version. In the Patna version, the point T is used for dividing the square of AD into the two squares of AT and TD and twice the rectangle $AT \cdot TD$.

[71]The Leiden and Patna versions use the same word *munḥarif*, trapezium, even though an arbitrary cyclic quadrilateral is meant. This is one of the arguments why the two versions are ultimately based on a single original.

scribed by a circle in which it is contained, we take the excess of half of the sum of its sides over each of its sides. Then one of these excesses is multiplied by another excess, and then the outcome by the third excess, and then the outcome by the fourth excess, and then the root of the amount is taken, and the result is the measurement (i.e., area) of that trapezium.

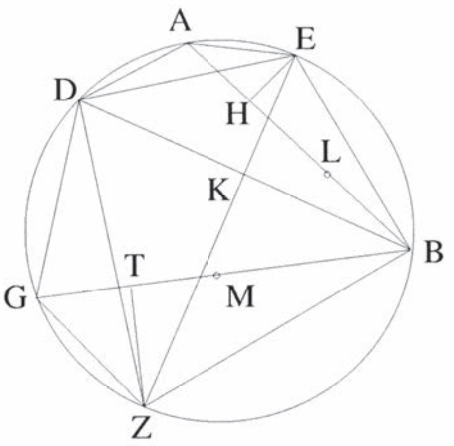


Figure 10

For the proof let us draw a quadrilateral $ABGD$ in a circle $ABGD$. We bisect each of the arcs BAD, BGD at points E, Z and we draw lines BE, ED, DZ, ZB, EA, EZ. We drop perpendicular EH onto AB and perpendicular ZT onto BG. We mark at the intersection of lines BD, EZ the point K. Then triangles EHA, EKD are similar, and ED is longer than EA, therefore KD is longer than HA. But BT is half of the sum of BG and DG as we have explained before, therefore BT is longer than DK, and thus it is much longer than AH. Similarly, we can also explain in the same way of this proof that BH is longer than TG. We cut off from BH line HL equal to TG, and from BT line TM equal to AH.

Since[72] the excess of area $EBZD$ over area $ABGD$ is the product EH by HA together with the product ZT by TG, and triangle EHA is similar to triangle EBZ, the ratio of EH to HA is equal to the ratio

[72]Note that the word "since" is out of place here because the statement on the excess of areas is not used to prove the similarity of triangles EHA and EBZ.

of EB to BZ[73]. Similarly also, triangle TZG[74] is similar to triangle EBZ, thus the ratio of ZT[75] to TG[76] is equal to the ratio of ZB to BE. The ratio of EB to BZ is equal to the ratio of the square of EB to the area EB by BZ and equal to the radio of area EB by BZ to the square of BZ.[77] But the square of BE is equal to the product BE by ED, and similarly the square of BZ is equal to the product BZ by ZD, and the product EB by BZ is the area $EBZD$. Therefore area $EBZD$ is a mean proportional between the product BE by ED and the product BZ by ZD.

If proportional (magnitudes) are in the same ratio, and if the corresponding (magnitudes) are added, the sums are proportional. Therefore the ratio of the sum of the squares of EH, GT to the sum of the areas EH by HA and GT by TZ is equal to the ratio of the sum of these two areas to the sum of the squares of AH, ZT.

Again, if from proportional (magnitudes) proportional (magnitudes) are subtracted, the remainders are proportional.[78]

Thus if we subtract (1) the (sum of the) two squares EH, GT, that is to say, (the sum of) the two squares EH, HL because we have made HL equal to TG, from (2) the square of BE, the remainder is area BL by (the sum of) LA, AD, and this is because if we subtract the square of EH from the square of BE the remainder is the square of BH, and if we subtract from the square of BH the square of HL, the remainder is the square of BL and twice the product BL by LH, and that is equal of the product of BL by the sum of BH, BL. As for BH, it is equal to the sum of HA, AD. Therefore the remainder is equal to the product BL times the sum of LA, AD. And if we subtract the sum of area EH by HA and (area) GT by TZ from area $EBZD$, the remainder is area $ABGD$. And if we subtract the two squares of HA, ZT, that is to say, the two squares of ZT, TM, from the square of BZ, the remainder is area BM by (the sum of) MG, GD in analogy with what we have

[73]The Leiden Ms has MD, restored to BZ by Suter [30].

[74]The manuscripts have TZD, restored to TZG by Suter [30].

[75]The manuscripts have ZK.

[76]The manuscripts have KG.

[77]The manuscripts have BE, restored to BZ by Suter [30].

[78]This statement is not generally true. Example: 9:12=12:16 and 4:6=6:9 are proportional magnitudes but (9-4):(12-6) is not equal to (12-6):(16-9). One has to add the condition that the two series of proportional magnitudes are in the same proportion.

explained before. Therefore, by subtraction, area $ABGD$ is a mean proportional between the two areas BL by (the sum of) LA, AD and (area) BM by (the sum of) MG and GD.

Since BH is half the sum of BA, AD, and similarly also BT is half of the sum of BG, GD as we have explained before, the sum of HB,[79] BT is half the sum of the sides of area $ABGD$. The excess of the sum of HB, BT over AB is BM, since we have made TM equal to AH. Again, the excess of it over BG is BL, since we have made LH equal to GT. Again, the sum of HA, AD, DG, GT is half the sum of the sides of area $ABGD$. Therefore the excess of it over AD is the sum of DG, GM as we have explained, and the excess of it $<$ over $>$ GD is the sum of DA, AL as we have also explained. Therefore the lines BL, LAD, DGM and MB are the excesses of half the $<$ sum of the $>$ sides of area $ABGD$ over each of its sides.

We have explained that area $ABGD$ is a mean proportional between area BL by LAD and area BM by MGD.

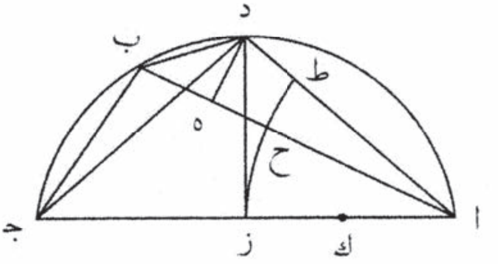
It is the same if we multiply BL by LAD and then the outcome by area BM by MGD or if we multiply BL by LAD and then the outcome by DGM and then the outcome by BM, because the outcome in both of these (procedures) is the area of trapezium $ABGD$ multiplied by itself. Therefore the root is taken, and it is what was desired. That is what we wanted to explain.

8. Arabic text of the Patna version

[$P291a, H61$] برهان عمل أرشميدس في مساحة المثلثات بالتفاضل .

قال أرشميدس يضرب نصف مجموع أضلاع المثلث الثلاثة في فضله
على أحدها وما اجتمع في فضله على الثاني وما بلغ في فضله على الثالث
ويؤخذ جذر المجتمع فيكون تكسير المثلث .

[79] The manuscripts have HD, restored to BH by Suter [30].



برهانه إن المثلث ابج وندير عليه دائرة ونخرج من منتصف قوس ابج
وهو د عمود ده على اب < وعمود دز على اج > وندير على مركز ا
وببعد از قوس زحط .

فلأن اد يقوى على دز زا يكون مربع دط وضرب دط في طا
< مرتين > ومربع طا مساويًا لمربعي دز زا المساوي ل طا فإذا ألقينا
مربعي اط از المتساويين بقي مربع دط وضرب دط في طا مرتين مساويًا
لمربع دز . وكذلك اد يقوى على ده ها فمربعا دط طا وضرب دط
في طا مرتين مساويًا لمربع ده ومربعي هح حا وضرب هح في [H62]
حا < مرتين > لكن حا مساو ل اط فإذا أسقطنا مربعيهما المتساويين
بقي مربع دط وضرب دط في طا مرتين مساويًا لمربعي ده هح وضرب
هح في حا مرتين وذلك أيضًا مساو لمربع دز .

ومثلث دزا شبيه بمثلث دهب لأن زاوية دجز المساوية لزاوية داز
مساوية لزاوية دبه الكائنة معها على قوس واحدة فنسبة ده إلى هب
كنسبة دز إلى زا ونسبة دز إلى زا كنسبة مربع دز إلى ضرب دز في زا
وكنسبة ضرب دز في زا إلى مربع زا . وكذلك أيضًا نسبة مربع ده إلى
ضرب ده في هب كنسبة ضرب ده في هب إلى مربع هب .

وإذا ألقي من مقادير متناسبة مقادير متناسبة على نسبها كانت نسب
البواقي على حالها .

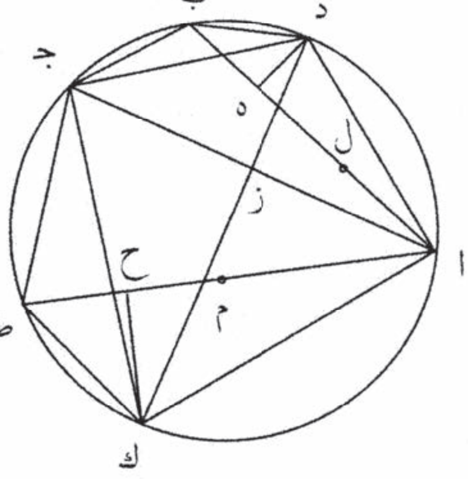
فنلقي مربع ده من مربع دز ويكون ما يبقى مساويًا لمربع هح مع

ضرب هح في حا مرتين أعني ضرب هح في مجموع ها [1] حا مرة ونلقى ضرب ده في هب من ضرب دز في زا فيبقى تكسير مثلث ابج لما تبين من مساواة مثلث ادج مجموع مثلثي ابج بدم . وليكن زك مساويًا لـ هب ونلقي مربع زك أعني هب من مربع زا فيكون الباقي [H63] مساويًا لضرب جك في كا وهذه البقايا متناسبة أعني أن نسبة ضرب هح في مجموع ها از إلى تكسير مثلث ابج < كنسبة تكسير مثلث ابج > إلى ضرب جك في كا .

و اه نصف ضلعي اب بج و از نصف ضلع اج فمجموع ها از هو نصف جماعة أضلاع المثلث فـ هح إذن فضل ها از نصف جماعة الأضلاع على مجموع حا از أعني اج وهو أحد الفضول . ولمساواة زك هب يكون مجموع اه زك مساويًا لضلع اب فـ اك إذن فضل مجموع ها از على ها زك أعني اب وهو الفضل الثاني . ولأن هب بج مساو لـ اه فإن هب بج از مساو لنصف جماعة الأضلاع ففضله على بج هو هب از لكن كز مساو لـ هب و زج مساو لـ از فـ كج هو فضل نصف جماعة الأضلاع على بج وهو الفضل الثالث . ومتى ضربنا سطح هح في ها از إحدى الحاشيتين في سطح جك في كا الحاشية الأخرى اجتمع مربع الواسط [2] أعني تكسير المثلث .

وسواء ضربنا هح الفضل الأول في ها از نصف جماعة الأضلاع وضربنا اك الفضل الثاني في جك الفضل [P291b] الثالث ثم ضربنا أحد المجتمعين في الآخر أو ضربنا ها از نصف جماعة الأضلاع في كا وما اجتمع في هح [H64] وما اجتمع في جك فإن كلي المبلغين يكون سواء وذلك مربع تكسير المثلث فإذا أخذنا جذره كان المطلوب .

برهان عمل الهند في مساحة المنحرف في الدائرة لأبي عبد الله الشني



وعلى هذا بنى أبو عبد الله الشني في البرهان على طريق للهند في تكسير ذي الأربعة الأضلاع في الدائرة وهو أنّهم يضربون فضول نصف جماعة أضلاعه على كل ضلع منه بعضها في بعض ويأخذون جذر المبلغ فيكون تكسير المنحرف وليكن ابجط .

ونصل اج ونخرج من منتصف قوس ابج وهو د قطر دزك وعمودي ده كح على اب اط. فلتشابه مثلثي دهب دزج وقصور دب عن دج يكون زج[3] أعني از أعظم من هب و از نصف ضلع اج [*H*65] أصغر من اح نصف مجموع ضلعي اط [4] طج فـ اح أعظم كثيرًا من هب وبمثل هذا يتبين أن اه أعظم من حط فنفصل هل مساويًا ل حط و حم مساويًا ل هب .

ومعلوم أن فضل سطح ادجك على سطح ابجط مساويًا لضرب ده في هب مع ضرب كح [5] في حط ومثلث داك يشابه كل واحد من مثلثي دهب كحط فنسبة دا إلى اك كنسبة ده إلى هب وكنسبة طح إلى حك ونسبة دا إلى اك كنسبة مربع دا إلى ضرب دا في اك < وكنسبة ضرب دا في اك الى مربع اك > وضرب دا في اك يساوي ضعف مثلث داك

وذلك سطح ادجك فهو إذن وسط في النسبة بين ضرب اد في دج وبين ضرب اك في كج .

ومجموع المقادير المتناسبة مع [6] مقادير أخر متناسبة على نسبها أو فضول ما بينها كل واحد مع نظيره كذلك متناسبة . فنسبة مجموع مربعي ده طح إلى مجموع ضرب ده في هب وضرب طح في كح كنسبة مجموع هذين السطحين إلى مجموع مربعي هب كح .

فإن أسقط مجموع مربعي ده طح [7] أعني هل من مربع اد ومجموع ضرب ده في هب وضرب طح في كح [H66] من سطح ادجك ومجموع مربعي هب كح من مربع اك كانت البواقي متناسبة . فأمّا البقية الأولى فتكون ضرب ال في مجموع لب بج [8] لأنه اذا أسقط من مربع اد مربع ده بقي مربع اه فإن ألقي [9] منه مربع حط المساوي ل هل بقي مربع ال وضعف ضرب ال في له وذلك مساو لضرب ال في مجموع اه هل أعني ضرب ال في مجموع جب به المساوي ل اه إلى له فأمّا البقية الثانية فتكون منحرف ابجط وأمّا الثالثة فتكون لمثل ما تقدم في الأولى ضرب ام في مجموع مط طج فمنحرف ابجط وسط في النسبة بين سطحي ال في لب بج و ام في مط طج .

ومعلوم أن اه نصف مجموع < اب بج و اح نصف مجموع > اط طج فمجموع ها اح نصف جماعة أضلاع منحرف ابجط ولمساواة هب حم يكون فضل مجموع ها اح على ضلع اب أعني اه حم هو ام الفضل الأول وعلى ضلع اط لمثل ذلك ال [10] الفضل الثاني . وأيضًا فإن مجموع خطوط هب بج جط طح هو أيضًا نصف جماعة أضلاع المنحرف ففضله على ضلع [H67] بج هو مجموع جط طح مع حم المساوي ل هب وهو الفضل الثالث [P292a] وفضله على ضلع جط لمثل ذلك هو لب بج [11] وهو الرابع .

ولكن منحرف [12] ابجط [13] كما تقدم وسط في النسبة بين ضرب ال [14] الثاني في مجموع لب بج الرابع وبين [15] ضرب ام الفضل الأول في مجموع مط طج الثالث وسواء ضربنا أحد هذين المضروبين في الآخر أو ضربنا الفضل الأول في الثاني وما اجتمع في الثالث وما اجتمع في الرابع فإن بكليهما يحصل مربع الوسط [16] أعني المنحرف فإذا أخذنا جذره كان المطلوب .

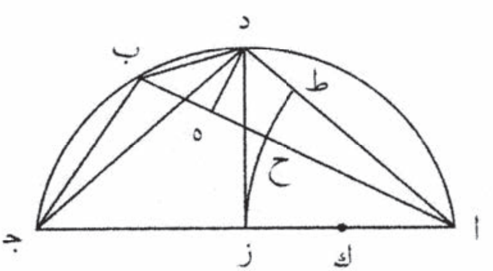
Apparatus to the Patna version

1 ها : هح PH . 2 الواسط P : الوسط H .
3 زجـ P : اجـ H ، جز D . 4 اط P : اجـ H . 5 كح D : كج PH .
6 مع P : ـ H . 7 طح D : طج PH . 8 بج P : لج H ، جب D .
9 فإن ألقى P : فألقى H . 10 ال PD : إلى H . 11 بج P : لج H .
12 منحرف P : ضرب H . 13 ابجط : ابجد PH ط *above* د P .
14 ال P : ل H . 15 وبين P : بين H . 16 الوسط P : الموسط H .

9. Arabic text of the Leiden version

[*L*119*b*, *M*63*b*] برهان عمل أرشميدس في مساحة المثلثات من جهة تفاضل أضلاعها لأبي عبد الله الشني

[*M*64*a*] قال أرشميدس إذا أردنا مساحة كل مثلث من جهة أضلاعه فإنا نأخذ فضل نصف جماعة أضلاعه على كل ضلع من أضلاعه ونضرب أحد تلك الفضول الثلاثة في آخر منها ثم ما بلغ في الثالث ثم ما بلغ في نصف جماعة أضلاع المثلث ونأخذ جذر المبلغ فما كان فهو مساحة ذلك المثلث .



قال أبو عبد الله برهان ذلك إنّا نجعل المثلث عليه ابج ونخط عليه دائرة ادبج وليكن نقطة د على منتصف قوس ادبج ونخرج منها على خط اب عمود ده وعلى خط اج [1] عمود دز ونصل دا دج دب . ونجعل نقطة ا مركزًا وندير ببعد از قوس [L120a] زحط من دائرة تقاطع خط اه على نقطة ح وخط اد على نقطة ط .

فمربع دا مساو لمربّعي ده ها . وخطا اط از [2] متساويان يكون مربع دط وضرب دط في طا مرتين وذلك مساو لمربع دز لأنّا جعلنا اط مساويًا لـ از يكون مربع دز مساويًا لمربع ده مع مربع هح وضرب هح في حا مرتين [3] .

ولان مثلث دهب شبيه بكل واحد من مثلثي دزج دزا لأن زوايا اجد[4] ابد جاد متساوية من أجل أن كل واحدة منها توتر نصف قوس ادبج فنسبة ده إلى هب كنسبة دز إلى زا . ونسبة دز إلى زا [5] كنسبة مربع دز إلى سطح دز في زا وكنسبة سطح دز في زا إلى مربع زا . ولذلك أيضًا نسبة مربع ده إلى سطح ده في هب كنسبة سطح ده في هب إلى مربع هب .

والمناسبة إذا أسقط عنها متناسبة على نسبتها كانت البواقي متناسبة .

فإذا أسقطنا مربع ده من مربع دز كان الباقي لما بيّنا مربع هح وضرب هح في حا مرتين أعني ضرب هح في مجموع ها از . وإذا أسقطنا ضرب ده في هب من ضرب دز في زا وهو تكسير مثلث داج

كان الباقي هو سطح مثلث ابج وذلك لما نبيّن من خواص الخط المنحنى . وإذا أسقطنا مربع هب أعني مربع كز إذا جعلنا زك مساويًا ل هب من مربع زا كان الباقي هو ضرب جك [6] في كا فمثلث ابج إذن يبقى موسطًا في النسبة بين سطح هح في مجموع ها از وبين سطح جك في كا [7] .

فاما هح فهو فضل مجموع ها از وهو نصف جماعة أضلاع مثلث ابج على ضلع اجـ لأنّا جعلنا اح مساويًا ل زجـ ولأن عمود ده يقسم مجموع خطي اب بج بنصفين على ه و اجـ منقسم أيضًا بنصفين [8] على ز [9] .

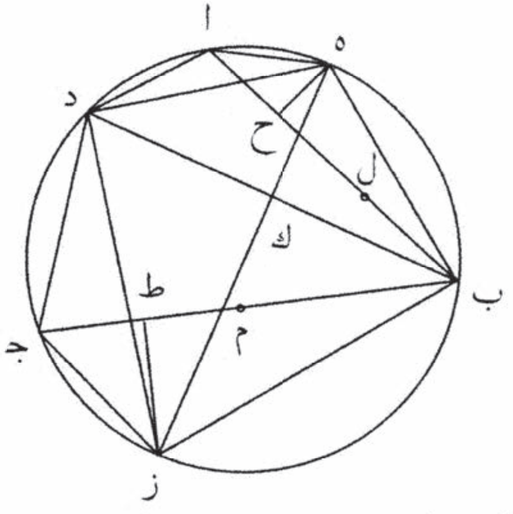
و اك هو فضل مجموع ها از أيضًا على اب لأنا جعلنا [10] كز مساويًا ل هب . و هح اك كج فضول نصف جماعة أضلاع مثلث ابج على كل [M64b] واحد من أضلاعه .

وسواء ضربنا سطح اك في كج في سطح هح في مجموع ها از وضربنا اك في كج وهو أحد الفضول في الآخر ثم ما بلغ في هح وهو الفضل الثالث ثم ما بلغ في مجموع ها از وهو نصف جماعة الأضلاع فإن المبلغ في جميعها هو تكسير مثلث ابج مضروبًا في مثله وذلك ما أردنا أن نبيّن .

وإن كان المثلث متساوي الساقين مثل مثلث ادجـ [11] كانت نسبة مربع دز إلى سطح دز في زا كنسبة هذا السطح بعينه إلى مربع زا . فأما مربع دز فقد بيّنا [L120b] أنه مساو لمربع دط [12] وضرب دط في طا مرتين أعني مساويًا لضرب دط في مجموع دا از . وأما سطح دز في زا فهو تكسير مثلث ادجـ . وأما مربع از فهو مساو لسطح از في زجـ . فلذلك نضرب اط في جز [13] وهو أحد الفضول في الآخر ثم ما اجتمع في دط وهو الفضل الثالث ثم ما بلغ في مجموع دا از وهو مجموع نصف الأضلاع فيكون المبلغ مربع سطح دز في زا وذلك ما أردنا بيانه .

برهان عمل ينسب إلى الهند في مساحة المنحرف الذي تحيط به دائرة لأبي عبد الله الشني

إذا أردنا مساحة المنحرف الذي تحيط به دائرة يقع فيها فإنا نأخذ فضل نصف جماعة أضلاعه على كل ضلع من أضلاعه ثم يضرب أحد تلك الفضول في الفضل الآخر ثم ما اجتمع في الفضل الثالث ثم ما اجتمع في الفضل الرابع ثم يؤخذ جذر المبلغ فما كان فهو مساحة ذلك المنحرف .



فلنخط للبرهان ذا أربعة أضلاع ابجد في دائرة ابجد وننصف كل واحدة من قوسي باد بجد على نقطتي ه ز ونصل خطوط به هد دز زب ها هز. ونخرج عمود هح على اب وعمود زط [14] على بج ونعلم على تقاطع خطي بد هز نقطة ك . فمثلثا هحا هكد متشابهان و هد أطول من ها فكد أطول من حا و بط نصف مجموع بج دج على ما قدمنا فبط أطول من دك [15] فهو إذن أطول كثيرًا من اح وكذلك أيضًا نبين بمثل هذا البرهان بعينه أن بح أطول من طج فنفصل من بح خط حل مساويًا لـ طج ومن بط خط طم مساويًا لـ اح .

ولأن فضل سطح هبزد على سطح ابجد هو ضرب هح في حا مع ضرب زط في طج ومثلث هحا شبيه بمثلث هبز فنسبة هح إلى حا

كنسبة هب إلى بز [16] وكذلك أيضًا مثلث طزج [17] شبيه بمثلث هبز فنسبة زط[18] إلى طج [19] كنسبة زب إلى به [20] . ونسبة هب إلى بز كنسبة مربع هب إلى سطح هب في بز [$M65a$] وكنسبة سطح هب في بز إلى مربع بز [21] ومربع به [22] مساو لضرب به في هد وكذلك مربع بز مساو لضرب بز [23] في زد وضرب هب في بز هو سطح هبزد فسطح هبزد إذن موسط في النسبة بين ضرب به في هد وبين ضرب بز في زد .

والمناسبة على نسبة واحدة فإن النظائر إذا جمعت كانت المجتمعة [24] متناسبة فنسبة مجموع مربعي هح جط إلى مجموع سطحي هح في حا و جط[25] في طز كنسبة مجموع هذين السطحين إلى مجموع مربعي اح زط .

وأيضًا فإن المناسبة إذا أسقط عنها متناسبة كانت البواقي متناسبة .

فإذا أسقطنا مربعي هح جط أعني [$L121a$] مربعي هح حل لأنّا جعلنا حل مساويًا لـ طج من مربع به [26] كان الباقي هو سطح بل في لا اد وذلك لأنّا إذا أسقطنا مربع هح من مربع به كان الباقي هو مربع بح وإذا أسقطنا من مربع بح مربع حل كان الباقي هو مربع بل وضرب بل في لح مرتين وذلك مساو لضرب بل في مجموع بح بل فأما بح فهو مساو لمجموع حا اد فلذلك يكون الباقي مساويًا لضرب بل في مجموع لا اد . وإذا أسقطنا مجموع سطح هح في حا و جط في طز من سطح هبزد كان الباقي هو سطح ابجد . وإذا أسقطنا مربعي حا زط أعني مربعي زط [27] طم من مربع بز كان الباقي هو سطح بم في مج جد على نحو ما بيّنا فسطح ابجد إذن يبقى متوسطًا في النسبة بين سطحي بل في لا اد و بم في مج جد .

ولأن بح نصف مجموع با اد وكذلك أيضًا بط نصف مجموع بج جد على ما قدمنا يكون مجموع حب [28] بط نصف جماعة أضلاع سطح ابجد ويكون فضل مجموع حب بط على اب هو بم لأنّا جعلنا طم مساويًا لـ اح . ويكون أيضًا فضله على بج هو بل لأنّا جعلنا لح مساويًا

ا جط . وأيضًا فإن مجموع حا اد دج جط نصف جماعة أضلاع سطح ابجد ففضله على اد هو مجموع دج جم لما بيّنا وفضله < على > جد هو مجموع دا ال لما بيّنا أيضًا فخطوط بل لاد دجم مب هي فضول نصف < مجموع > أضلاع سطح ابجد على كل واحد من أضلاعه .

وقد بيّنّا أن سطح ابجد موسط في النسبة بين سطح بل في لاد وبين سطح بم في مجد .

وسواء ضربنا بل في لاد ثم ما اجتمع في سطح بم في مجد أو ضربنا بل في لاد ثم ما اجتمع في دجم ثم ما اجتمع في بم فإن المبلغ من كليهما هو تكسير منحرف ابجد مضروبًا في مثله فلذلك يؤخذ جذره فيكون المطلوب وذلك ما أردنا أن نبيّن .

Apparatus to the Leiden version

1 اجـ : اد *LM* . 2 از : اد *LM* . 3 + فمربع دز إذن مساو لمربع ده ومربع هح وضرب هح في حا مرتين *LM* . 4 اجد : ادز *LM* .
5 ونسبة دز إلى ز *L* : ـ *M* . 6 جك *D* : دك *LM* .
7 كا *D* : كب *LM* . 8 بنصفين *L* : — *M* . 9 ز *D* : زا *LM* .
10 + اح مساويًا ا زج ولأن عمود ده يقسم مجموع خطي اب بج بنصفين على ه و اج منقسم أيضًا بنصفين على ز و اك هو فضل مجموع ها از أيضًا على اب لأنا جعلنا *LM* . 11 ادج : ابج *LM* .
12 دط *D* : دك *LM* . 13 جز : كز *LM* . 14 زط *L* : بط *M* .
15 دك *L* : بك *M* . 16 بز : مد *L* ، هد *M* . 17 طزج : طزد *LM* .
18 زط : زك *LM* . 19 طج : كج *LM* . 20 به *M* : بز *L* .
21 بز : به *LM* . 22 به *M* : بز *L* . 23 مساو لضرب بز : : ـ *M* .
24 المجتمعة *L* : المجقعة *M* . 25 جط *L* : هط *M* .
26 به *M* : بز *L* . 27 زط *L* : بط *M* . 28 حب : حد *LM* .

10. References

1. E.J. Atzema, From Brahmagupta to Euler: on the formula for the area of a cyclic quadrilateral. *Journal of the British Society of the History of Mathematics* 30 (2015), pp. 20-34.
2. al-Bīrūnī, *Rasā'ilu'l-Bīrūnī*, containing four tracts. Hyderabad, Osmania Oriental Publication Bureau, 1367 H./ 1948 AD [in Arabic].
   Reprinted in [27, vol. 32]. The *Extraction of Chords* is in tract no. 1.
3. Al-Bīrūnī, *Kitāb al-Qānūn al-Mas'ūdī*. Hyderabad-Deccan, Osmania Oriental Publications Bureau, 1956 [in Arabic], 3 vols.
4. P.A. Bulgakov, B.A. Rozenfeld, tr. and comm. *Abu Raikhan Beruni, 973-1048, Izbrannye Proizvedenia, vol. 7: Matematicheskie i astronomicheskie traktaty.* Tashkent 1987 [in Russian].
5. *Fundamenta Arithmetica et Geometrica* ... auctore Ludolpho a Ceulen Hildesheimensi, e vernaculo in Latinum translata a Wil. Sn. Lugduni Batavorum, apud Iustum a Colster Bibliopolam, Anno MDCXV (1615).
6. M. Clagett, Marshall, *Archimedes in the Middle Ages. Volume I: the Arabo-Latin Tradition.* Madison 1964.
7. A.S. Al-Demirdash, *Istekhrag el awtar fil daïra, author Aboul Rihan El Bironi*. Cairo 1965 [in Arabic].
8. T.L. Heath, *Euclid: The Thirteen Books of the Elements*. New York, Dover Reprints, 1956, 3 vols.
9. J.P. Hogendijk, Rearranging the Arabic Mathematical and Astronomical Manuscript Bankipore 2468. *Journal for the History of Arabic Science* 6 (1982), pp. 133-159.
10. J.P. Hogendijk, Greek and Arabic constructions of the regular heptagon. *Archive for History of Exact Sciences* 30 (1984), pp. 197-330.
11. Y. Id and E.S. Kennedy, A medieval proof of Heron's formula. *The Mathematics Teacher* 62 (1969), pp. 585-587, reprinted in [14, p. 492-494].
12. V.J. Katz and C. Montelle, *Sourcebook in the Mathematics of Ancient Greece and the Eastern Mediterranean*, Princeton 2024.
13. E.S. Kennedy, article: al-Bīrūnī. *Dictionary of Scientific Biography*, ed. C.G. Gillispie, vol. 2, New York 1970, pp. 148-158. Reprinted in [14, pp. 562-572].
14. E.S Kennedy, Colleagues and Former Students, *Studies in the Islamic Exact Sciences*, Beirut 1983.
15. S. Kichenassamy, Brahmagupta's derivation of the area of a cyclic quadrilateral. *Historia Mathematica* 37 (2010), pp. 28-61.
16. D.A. King, *A Survey of the Scientific Manuscripts in the Egyptian National Library*. Winona Lake 1986.
17. M. Krause, Stanbuler Handschriften islamischer Mathematiker. *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, B 3 (1936), pp. 437-532. Reprinted in [27, 83, pp. 237-332], 1998.
18. Abdul Hamid Mawlawi et. al., *Catalogue of the Arabic and Persian Manuscripts in the Oriental Public Library at Bankipore*, vol. 22, Sciences. Patna 1937.
19. Ptolemy, *Ptolemy's Almagest*, translated by G. J. Toomer, second edition. Princeton 1998,

20. A.Q. Qorbani, *Taḥrīr-i Istikhrāj al-Awtār, ta'līf ustād Abū Rayḥān Bīrūnī, riyāḍīdān wa-munajjim-i buzurg-i Īrānī*. Tehran 1355 A.H. Solar (1975-6) [in Persian].
21. R. Rashed, *Les mathématiques infinitésimales du IX^e au XI^e siècle, vol. 3: Ibn al-Haytham: Constructions géométriques, théorie des coniques et géométrie pratique*, London 2000.
22. R. Rashed, *Ibn al-Haytham's Theory of Conics, Geometrical Construction and Practical Geometry: A History of Arabic Science and Mathematics Volume 3*, translated by Judith V. Field, Abingdon 2013.
23. A.S. Saidan, The Rasa'il of Biruni and Ibn Sinan: a Rearrangement. *Islamic Culture* 34 (1960), pp. 173-175.
24. A.S. Saidan, Al-Biruni on Trigonometry. *Erdem* 9 (1966), pp. 769-778.
25. K.V. Sarma et al., *Gaṇita-Yukti-Bhāṣā (Rationales in Mathematical Astronomy) of Jyeṣṭadeva: Malayalam Text Critically Edited with English Translation*, with explanatory notes. New Delhi 2008, 2 vols.
26. F. Sezgin, *Geschichte des arabischen Schrifttums*, Band V: Mathematik bis ca. 430 H. Leiden: Brill, 1974.
27. F. Sezgin, ed., *Islamic Mathematics and Astronomy*. Frankfurt: Institut für Geschichte der Arabisch-Islamischen Wissenschaften, 1997 - . . . , 120 volumes to date.
28. *Catalogus bibliothecae publicae lugduno-batavae noviter recognitus* ed. Fridericus Spanheimius. Leiden 1674.
29. H. Suter, Über die Geometrie der Söhne des Mûsâ ben Schâkir. *Bibliotheca Mathematica*. 3. Folge, 3, pp. 259-272. Reprinted in [27, vol. 76, pp. 137-150] and in [31, vol. 2, pp. 144-157].
30. H. Suter, Das Buch der Auffindung der Sehnen im Kreise von Abū'l-Raiḥān Muḥ. al-Bīrūnī. *Bibliotheca Mathematica*. 3. Folge, 1910-1911, pp. 11-78. Reprinted in [27, vol. 35, pp. 39-106] and in [31, vol. 2, pp. 280-347].
31. H. Suter, *Beiträge zur Geschichte der Mathematik und Astronomie im Islam*, ed. F. Sezgin. Frankfurt 1986.
32. J. Tropfke, *Geschichte der Elementar-Mathematik in systematischer Darstellung*, vierter band, Ebene Geometrie, Dritte Auflage. Berlin - Leipzig 1940.
33. J.-J. Witkam, *Inventory of the Oriental Manuscripts in the Library of the University of Leiden*, vol. 1, manuscripts Or. 1-1000. Leiden 2007.

Abstract

This paper contains edited Arabic texts and English translations of a tenth-century geometric proof by Abū ʿAbdallāh al-Shannī, of the rule for the area of the cyclic quadrilateral in terms of its sides. Al-Shannī's simple geometric proof is of intrinsic mathematical interest; even the famous mathematician Leonard Euler (who did not know al-Shannī's proof) stated in 1750 that it is difficult to find a geometric proof of the area rule for a cyclic quadrilateral in terms of its sides. Al-Shannī's proof has come down to us only in the *Extraction of Chords*, a work by al-Bīrūnī (973-1048 AD) which has survived in two different versions. This paper contains the texts of al-Shannī's proof in the two versions. We have added edited Arabic texts and English translations of the preceding proof by al-Shannī of the area rule (Heron's rule) for the triangle, because of the close relationship between the two proofs.