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AN ARABIC TEXT ON THE COMPARISON OF THE FIVE REGULAR POLYHEDRA: "BOOK XV" OF THE *REVISION OF THE ELEMENTS* BY MUḤYĪ AL-DĪN AL-MAGHRIBĪ

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Introduction

This paper contains a critical edition of the Arabic text, with English translation and notes, of "Book XV" of the *Revision of the Elements* (Taḥrīr al-Uṣūl) by Yaḥyā ibn Muḥammad ibn Abī 'l-Shukr Muḥyī al-Dīn al-Maghribī (died 682 H./A.D. 1283). The text is extant in Arabic manuscripts in Utrecht, Oxford and Istanbul. In "Book XV" al-Maghribī geometrically determines the ratios of (1) the edges, (2) the faces, (3) the surface areas, (4) the perpendicular distances from the centre to a face and (5) the volumes of the five regular polyhedra inscribed in one sphere. Hitherto no solution of this general problem was believed to exist in the ancient Greek and medieval Arabic-Islamic literature. The ratios between the surface areas, the perpendicular distances from centre to face and the volumes of the icosahedron and the dodecahedron were determined in a lost work of Apollonius (ca. 200 B.C.) and also in an extant treatise by Hypsicles (150 B.C.) which was added as "Book XIV" to the Greek text of the *Elements*. Al-Maghribī's "Book XV" is related to, but contains much more than "Book XIV" of Hypsicles.

"Book XV" of the *Revision of the Elements* is not an original work by al-Maghribī; it is an adaptation of a lost Arabic source (**A**), which survives in a different recension in an as yet unpublished Hebrew translation. At present it is still unclear whether **A** was an Arabic translation of a Greek text or the work of an early Arabic-Islamic mathematician who was inspired by Greek sources.

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Al-Maghribī and his Revision of the Elements

Yaḥyā ibn Muḥammad ibn Abī 'l-Shukr Muḥyī al-Dīn al-Maghribī al-Andalusī[1] was born in Spain, but spent most of his scientific career in the Near East. In 1257 he composed the Tāj al-Azyāj (crown of the Zījes, i.e. astronomical handbooks with tables) in Damascus. After Damascus was invaded by the Mongols around 1258, he joined the newly founded astronomical observatory of Marāgha, which was directed by Naṣīr al-Dīn al-Ṭūsī. Al-Maghribī made astronomical observations between 1262 and 1274, and he died in Marāgha in 1283[2]. The Oxford manuscript of the *Revision of the Elements* was written in 659 H./1260-1, at Marāgha, and in his introduction al-Maghribī refers to another edition of the *Elements* by "al-shaykh al-ra'īs" (the master, the chief). If this is Naṣīr al-Dīn al-Ṭūsī, al-Maghribī must have composed his *Revision of the Elements* shortly after he arrived in Marāgha.

Euclid's *Elements* consist of 13 "Books". In late antiquity "Book XIV" (written by Hypsicles around 150 B. C.) and an anonymous "Book XV" were added to the Greek text, and all 15 Books were translated into Arabic. Al-Maghribī's *Revision of the Elements* also consists of 15 books, and the first twelve books in the *Revision* have approximately the same content as the first twelve books of the *Elements*[3]. To indicate the correspondences between the remaining books, I use a notation such as El. XIII for Book XIII in the original text of the *Elements* and M XIV (M for al-Maghribī) for Book XIV in the *Revision*. Book M XIII includes the theorems in El. XIII on plane figures and some related material not found in the *Elements*. The text of two of these additional propositions is to be found in the appendix to this paper. Book M XIV consists of the theorems on solids in El. XIII and of most of the trivial Book El. XV. Book M XV can be considered as a generalisation to all five regular polyhedra of the theory of Book XIV of the *Elements* (by Hypsicles), which is about the comparison of the icosahedron and the dodecahedron only.

[1] See Suter art. no. 376 (p. 155), Matvievskaya and Rozenfeld vol. 2, pp. 418-419, Krause, *Sphärik*, pp. 74-75, S. Tekeli in DSB IX, 555-557.

[2] The preceding information is taken from Saliba, *An observational notebook*, pp. 391-392.

[3] There are interesting additions in the *Revision*, such as a "proof" of the parallel postulate, which was published by A. I. Sabra in 1969.

Al-Maghribī describes the subject of M XIV and M XV in his preface to the *Revision* as follows:

"Then we finished the work with two books, one on the construction of the five solids in one another, and we made the treatment of this detailed, and the second on the ratios between their edges, perpendiculars, faces, surfaces (i.e. surface areas) and solids (i.e. volumes). We completed this with five propositions on finding the five lines proportional to their edges, perpendiculars, surfaces and volumes."[4] Al-Maghribī does not say anything about the sources of his Book XV.

The mathematical contents of Book XV of the Revision

The purpose of Book M XV is to explain the geometrical construction of five groups of five straight line segments, in the ratios of (1) the edges, (2) the surface areas, (3) the faces, (4) the perpendicular distances between centre and face, and (5) the volumes of the five regular polyhedra inscribed in a given sphere. These constructions are preceded by a large number of preliminary theorems. Book M XV can be divided into five parts as follows:

i. M XV:1-6 preliminary theorems on edges of regular polyhedra,
ii. M XV:7-11 theorems on faces,
iii. M XV:12-17 theorems on surface areas,
iv. M XV:18-30 theorems on volumes,
v. MXV:31-37 comparison of the five polyhedra.

The majority of the theorems in the first four groups deal with the ratio of a certain quantity (for example: face, surface area, volume) in two regular polyhedra inscribed in the same sphere. In groups iii. and iv. the combinations are treated in the following order: (1) tetrahedron and octahedron, (2) cube and octahedron, (3) octahedron and icosahedron (with preliminaries for cube and icosahedron, if necessary), and (4) icosahedron and dodecahedron. The structure of groups i. and ii. is not so clear. There are also a number of lemmas not directly

[4] My translation from Ms. Oxford, Bodl. 448, f. 2a. The "five propositions" are M XV:31, 33, 35, 36 and 37. Note that al-Maghribī does not mention the comparison of the faces. A proposition on the comparison of the faces (such as M XV:36) is not found in the Hebrew translation of **A** either.

dealing with polyhedra, such as M XV:1, 4, 7, 22, 28. Most of the proofs are very elaborate, and the manipulations of proportions are given in great detail, in the way of Books V and VI of the *Elements*. The ratios are determined correctly, except for the ratio between the volume of the octahedron and the icosahedron in M XV:29. The proof of this proposition contains an error which must have escaped al-Maghribī. By making minimal changes in the text of M XV:29 one obtains a correct theorem, and it will be shown in the notes to the translation that the proof in the text can be extended to form a correct proof in the style of ancient and medieval geometry. Because Book M XV is a revised version of an older text, which passed through a complicated process of transmission, it is conceivable that the error is due to an editor who altered a proof that was correct.

The text of M XV contains a number of inadequacies, which show that al-Maghribī worked rather carelessly as an editor. Thus he inserted numerous references in his text to previous propositions, but because many of these references are incorrect it seems that he did not check them carefully. Al-Maghribī could have omitted theorems M XV:9, 10, 10 cor. and 22 because they are not used in the rest of the text.

The contents of the 37 propositions of Book XV of the *Revision* will now be rendered in modernized notation. Notes on the proofs can be found in footnotes to the translation. In the following list I include information on the logical structure of Book XV as well as references to Book XIV in the Greek text of the *Elements*. A statement like 5. $r_{12}=r_{20}$ (El. XIV, § 3 = prop. 2 Heath) means that the identity $r_{12}=r_{20}$ is proved in M XV:5 and in § 3 in the edition of the Greek text in Heiberg-Stamatis, *Euclides V,1, Elementa XIV-XV, Scholia in Libros I-V*, pp. 1-22, which corresponds to prop. 2 in the translation by Heath, *Euclid's Elements*, vol. 3, pp. 512-519. The proofs in M XV:5 and El. XIV, §3 need not be identical. The following abbreviations will be used:

e_i, f_i, s_i, p_i, v_i : (i = 4,6,8,12,20)	the edge, face, surface, perpendicular (distance between the centre of the circumscribing sphere and any face) and volume of a regular polyhedron with i faces inscribed in a fixed sphere.
R	radius of the sphere circumscribing the polyhedra,
r_i	radius of the circle circumscribing the face f_i,
d, h	side and altitude of a fixed equilateral triangle,
a, x_1, x_2	a is an arbitrary line segment, divided in extreme and mean ratio into segments x_1 and x_2, i.e. $a=x_1+x_2$ and $a:x_1=x_1:x_2$.
c(n)	side of a regular n-gon inscribed in a fixed circle, i.e. chord of an arc of 360/n degrees,
c(6)	radius of this circle (= side of inscribed hexagon),
c(2.5)	diagonal of the pentagon inscribed in the circle,
cor.	corollary.

Table of contents of the propositions in Book XV of the Revision

1. $h^2=3/4\ d^2$. Used in 2, 26, 32.
2. e_8 is equal to the altitude in the "triangle of the tetrahedron" (i.e. $e_4:e_8=d:h$). Based on 1. Used in 2 cor, 20, 20 cor.

2 cor. $f_8= 3/4\ f_4$. Based on 2. Used in 17, 36.

3. $r_6=r_8$. Used in 3 cor. 1, 3 cor. 2.

3 cor. 1 $p_6=p_8$. Based on 3. Used in 23, 26, 33.

3 cor. 2 $e_8^2= 3/2\ e_6^2$. Based on 3. Used in 8.

4. $c(5)^2+c(2.5)^2=5c(6)^2$ (El. XIV, § 2 = lemma in prop. 2). Used in 5, 15.
5. $r_{12}=r_{20}$ (El. XIV, § 3 = prop. 2). Based on 4. Used in 5 cor, 6, 16, 31.

5 cor. $p_{12}=p_{20}$. Based on 5. Used in 30, 33.

6. $e_6:e_{20}= \sqrt{(a^2+x_1^2)} : \sqrt{(a+x_2^2)}$ (El. XIV, § 9 = prop. 7). Based on 5. Used in 13, 16 cor.
7. The line joining the centre of the circle to the midpoint of a side of the inscribed regular pentagon is equal to {c(6)+c(10)}/2 (El. XIV, § 1 = prop. 1). Used in 25, 33.
8. $f_8:f_6= h/2 : 2d/3$. Based on 3 cor. 2. Used in 12, 23, 36.
9. Let y_i (i=12, 20) be the perpendicular drawn from the centre of a face of the dodecahedron or icosahedron to a side of the face. Then $y_{12}.e_{12} =s_{12}/30$ (El. XIV, § 4 = prop. 3). Used in 10 cor.
10. $y_{20}.e_{20} =s_{20}/30$ (El. XIV, § 5, beginning = prop. 4). Used in 10 cor.

10 cor. $s_{12}:s_{20}=y_{12}.e_{12} : y_{20}e_{20}$ (El. XIV, § 5, end = prop. 5). Based on 9, 10. Not used.

11. The area of the regular pentagon inscribed in a circle of diameter D (=2c(6)) is equal to (3/4)D·(5/6)c(2.5) (El. XIV, § 7 = preliminary to second proof in prop. 6). Used in 16.
12. $s_6:s_8$=d:h. Based on 8. Used in 12 cor, 35.

12 cor. $s_6:s_8$ is equal to the ratio between the area of the square of any line d and twice the equilateral triangle with side d. Based on 12. Used in 14.

13. $s_6:s_{20}=c(5)^2$: 3 1/3 T, where T is the equilateral triangle such that the square of its side is equal to $3c(10)^2$. Based on 6. Used in 13 cor.

13 cor. 3/5 $s_{20}:s_6=2T:c(5)^2$. Based on 13. Used in 14.

14. $s_{20}:s_8=5c(10)^2:c(5)^2$. Based on 12 cor, 13 cor. Used in 15.

15. $s_{20}-s_8:s_8=x_2:a$. Based on 4, 14. Used in 15 cor, 29.

15 cor. $s_{20}:s_8=a+x_2:a$. Based on 15. Used in 35, 36.

16. $s_{12}:s_{20}=e_6:e_{20}$ (El. XIV, § 8 = prop. 6, second proof). Based on 5, 11. Used in 16 cor.

16 cor. $s_{12}:s_{20}=\sqrt{(a^2+x_1^2)} : \sqrt{(a+x_2^2)}$ (compare El. XIV, § 12 = summary of results in Heath's translation, p. 519). Based on 6, 16. Used in 30, 35, 36.

17. $s_8=3s_4/2$. Based on 2 cor. Used in 35, 36.

18. $2f_4 \cdot 2R=9v_4$. Used in 18 cor.

18 cor. $2f_4 \cdot 2R/9 = v_4$. Based on 18. Used in 20.

19. $e_8^2 \cdot 2R=3v_8$. Used in 19 cor.

19 cor. $3e_8^2 \cdot 2R/9=v_8$. Based on 19. Used in 20.

20. $v_4:v_8=d:3h$. Based on 2, 18 cor, 19 cor. Used in 20 cor, 37.

20 cor. $v_4:v_8=e_4:3e_8$. Based on 2, 20. Not used.

21. A regular solid is divided into equal and similar pyramids whose bases are the faces of the solid and whose vertices are at the centre of the circumscribing sphere. Used in 23, 24, 25.

22. One considers a pyramid with a triangular base and a second pyramid with a pentagonal base and the same height as the first pyramid. Theorem: the pyramids are in the same ratio as the bases. Not used.

23. $v_6:v_8=d:h$. Based on 3 cor.1, 8, 21. Used in 37.

24. The parallelepiped with base the pentagon of the icosahedron (i.e. a regular pentagon formed by five angular points) and height 2/3 times the diameter of the sphere is equal to v_{20}. Based on 21. Used in 25.

25. $p_{20}:2R = \{c(10)+c(6)\}/2 : 2H$, where H indicates the altitude of an equilateral triangle with side c(5). Based on 7, 21, 24. Used in 27.

26. $R:p_8=h:d/2$. Based on 1, 3 cor. 1. Used in 27.

27. $p_{20}:p_8=c(6)+c(10):c(5)$. Based on 25, 26. Used in 29, 33.

28. The ratio between two rectangular parallelepipeds is the product of the ratio between the bases and the ratio between the heights. Used in 29.

29. $v_{20}:v_8=\{c(10+c(5)\}:c(5)$. (The statement is mathematically incorrect. In fact $v_{20}:v_8=c(5):c(10)$. See note 29.7 below). Based on 15, 27, 28. Used in 29 cor, 37.

29 cor. $v_{20}:v_8=\sqrt{(a^2+x_1^2)}+x_1 : \sqrt{(a^2+x_1^2)}$. (In fact $v_{20}:v_8=\sqrt{(a^2+x_1^2)}:x_1$.) Based on 29. Not used.

30. $v_{12}:v_{20}=\sqrt{(a^2+x_1^2)} : \sqrt{(a^2+x_2^2)}$ (stated in El. XIV, § 12 = summary of results in Heath's translation, p. 519). Based on 5 cor, 16 cor. Used in 37.

31. Construction of five line segments in the ratio $e_4:e_8:e_6:e_{20}:e_{12}$. Based on 5. (The construction differs from El. XIII:18.). Used in 31 cor.

31 cor. $e_4>e_8>e_6>e_{20}>e_{12}$. Based on 31.

32. $p_4:p_6=h/3 : d/2$. Based on 1. Used in 33.

33. Construction of five line segments in the ratio $p_4:p_8:p_6:p_{20}:p_{12}$. Based on 3 cor. 1, 5 cor, 7, 27, 32.

34. Construction of a segment a such that $a^2+x_2^2=c^2$ for given c. Used in 36, 37.

35. One wishes to construct five line segments in the ratio $s_{12}:s_{20}:s_8:s_6:s_4$. (One actually constructs two segments in the ratio $s_{12}:s_{20}$ and four segments in the ratio $s_{20}:s_8:s_6:s_4$, see note 35.1 below.) Based on 12, 15 cor, 16 cor, 17.

36. Construction of five line segments in the ratio $f_{12}:f_{20}:f_8:f_4:f_6$. Based on 2 cor, 8, 15 cor, 16 cor, 17, 34.

37. Construction of five line segments in the ratio $v_{12}:v_{20}:v_8:v_6:v_4$. Based on 20, 23, 29, 30, 34.

The following two propositions of Book XIII of the *Revision* of al-Maghribī have been edited and translated in the appendix to this paper because they also occur in the Hebrew translation (see the next section); they must therefore have been in al-Maghribī's source.

10 If segment a is divided in extreme and mean ratio into x_1 and x_2, and similarly, a' is divided in extreme and mean ratio into x_1' and x_2', then $a:a'=x_1:x_1'=x_2:x_2'$ (El. XIV, § 11 = lemma in Heath's translation, vol. 3, p. 518). Used in M XV:5, 6, 15.

14 If a=c(6), then x_1=c(10). Used in M XV:5, 6, 13, 29.

The historical background to the text

Book XV of al-Maghribī includes almost the whole mathematical content of "Book XIV" of the Elements, written by Hypsicles. The theorems in M XV could not have been directly copied from (the Arabic translation of) El. XIV, because the points in the geometrical figures are labelled in a different way and because there are many differences in the details of the mathematical proofs. However, there are striking similarities in the theorems themselves. Thus the area of a pentagon is shown to be equal to the product of *three-fourths* of the diameter of the circumscribing circle and *five-sixths* of the diagonal of the pentagon in El. XIV § 7 and in M XV: 11. There are many other ways of expressing the area of the pentagon, and therefore we can assume that the resemblance between Hypsicles and al-Maghribī is not accidental. The proposition El. XIV § 6 (first proof of prop. 6 in Heath, vol. 3, p. 516) is missing in M XV, but al-Maghribī gives in M XV:9-10 the preliminaries to this proposition, which are not used anywhere in M XV. Similarly, the proof of El. XIV § 10 (prop. 8 in Heath, vol. 3, p. 518) does not occur in M XV, but al-Maghribī gives in M XV:22 a preliminary to this proof; again, this preliminary is not used. Thus it seems that the theorems on the icosahedron and the dodecahedron in Book XV of the *Revision* are based on a source which included the entire Book XIV of the *Elements*.

Dr. Langermann has recently discovered a manuscript which is very important in this connection. This is a Hebrew translation by Kalonymos b. Kalonymos[5] (born 1286) of a lost Arabic treatise, which contains almost all theorems of Book XV of the *Revision* in a different order, but often in strikingly similar wording. Dr. Langermann kindly provided me with a preliminary English translation, on which the following information is based.

The Hebrew translation can be divided into the following parts:

I. An introduction, containing references to one Meseclunus (perhaps a rendering of Insiqlāwus, an Arabicized form of Hypsicles that was often used).

II. Theorems on the tetrahedron and the octahedron, corresponding to M XV:1, 2,17-20.

III. Theorems on the octahedron and the cube, corresponding to M XV:3, 8, 12, 23.

IV. Theorems on the octahedron and the icosahedron, and necessary preliminaries for these theorems, corresponding to M XIII:10, 14, XV:4-6, 13-15, 24, 7, 25-27, 29.

V. Theorems on the icosahedron and the dodecahedron proved by Hypsicles (if not already treated in Part IV), corresponding to M XV:9-11, *, 16, 16 cor, 30 (proof missing). The asterisk denotes the proof of El. XIV § 6 (the first proof of prop. 6 in Heath's translation), which was missing in al-Maghribī's text, as has been mentioned above.

VI. Concluding theorems. M XV:31? (a page in the manuscript is missing), 32, 33, 35, 37.

For chronological reasons the *Revision* of al-Maghribī (died 1283) cannot be dependent on the Hebrew translation by Kalonymos b. Kalonymos[5] (born 1286). The Hebrew translation cannot have depended on al-Maghribī either, for there is nothing in the *Revision* which corresponds to the introduction and to the references to Hypsicles in the Hebrew. Thus the two texts must have a common Arabic origin, which I will call **A**. No copies of this text **A** have yet been found. It is likely that the order of the theorems in **A** was essentially that of the Hebrew translation, in which logically connected theorems are grouped together. The order of the theorems in the Hebrew also reflects the character of **A** as a reworking and extension of Hypsicles' treatise. There are striking similarities and striking differences in the mathematical reasoning in the Hebrew and the *Revision*. One cannot be certain that all these differences are the result of the editorial intervention of al-Maghribī, because the Hebrew may have been based on an edited version of **A**. The reconstruction of **A** from the Hebrew translation and the *Revision* is therefore a non-trivial problem, the solution of which will have to await the publication of the Hebrew.

In 1984 Langermann and I suggested that **A** is an Arabic translation of a Hellenistic treatise. Now I consider that it is just as likely that **A** was written by an Arabic-Islamic mathematician, who wanted to extend Hypsicles' treatment of the icosahedron and the dodecahedron in Book XIV of the *Elements* to all regular polyhedra, and who may of course have been inspired by more Greek sources than Hypsicles' Book

[5] Information on the manuscript can be found in Langermann and Hogendijk 1984.

XIV. Al-Kindī (9th century A. D.) is known to have written a *treatise on the correction of Books XIV and XV of the Elements of Euclid*[6].

I do not want to suggest that this was **A**, which is not concerned with Book XV, but the title of Al-Kindī's work shows that there was some interest in the regular polyhedra in the 9th century A.D., and thus **A** could very well have been written in that period. An argument for dating **A** early in the Islamic tradition is the inefficient way in which the proportions are handled. The origin of **A** will be clarified by the publication by Dr. Langermann of the Hebrew translation and related textual material.

One could try to find out more about the history of **A** by looking for possible traces in other Arabic treatises. In pp. 431-433 of the 1594 Rome edition of the *Elements*, attributed to Naṣīr al-Dīn al-Ṭūsī, the following theorems are proved in a corollary to El. XIII:15 (construction of the octahedron):

1. $r_6=r_8$ (cf. M XV:3).
2. $s_6:s_8=v_6:v_8=d:h$, where d and h are two line segments such that $h^2=3/4\ d^2$ (cf. M XV:12, 23).

The proofs in the Rome edition differ from those of al-Maghribī. Thus it is possible but not certain that the author of the Rome edition was influenced by **A** or by al-Maghribī.

The manuscripts

Al-Maghribī's revision of the *Elements* is extant in four Arabic manuscripts, which will be called D, T, F and H. The manuscripts were mentioned by Sezgin in GAS V, 114; D, F and H in no. 50, T in no. 52.

D is MS. Oxford, Bodleian Library 448, f. 160a-175b. This manuscript was identified by A. I. Sabra (p. 14). According to the colophon, the manuscript was completed in Marāgha in 659 H./A. D. 1260-1. Thus D was written during the life of al-Maghribī, probably soon after the completion of the *Revision*. There is no evidence that D is an autograph by al-Maghribī. The manuscript is written in a clear naskhī, and there are 19 lines to a page. There are a number of (usually insignificant) scribal errors in the text; some of these errors are also in the other manuscripts and they may have been in the

[6] Al-Kindī's mathematical works are listed by Sezgin in GAS V, 256.

original of al-Maghribī. The last five leaves in the manuscript bear the numbers 171, 172, 172, 173, 173. I have renumbered these leaves 171-175 for sake of clarity.

T is MS. Utrecht, Universiteitsbibliotheek, 1440 (formerly Or. 21), f. 81a-90a, see Voorhoeve, p. 392, Tiele p. 340. T is dated Saturday 9 Dhū'l-Ḥijja 685 H. (January 25, A. D. 1287). The size of the leaves is ca. 165 x 77 mm, and the size of the text is ca. 150 x 66 mm, there are between 37 and 41 lines to a page. The name of the scribe is Ibrāhīm ibn Ya'qūb ibn Maymūn. A few leaves of this manuscript are missing, including the front leaf, and the name of al-Maghribī is not mentioned anywhere in the manuscript. The text is written in a small but legible naskhī, in black ink, with almost no diacritical marks, and there are very few scribal errors. The figures and the proposition numbers in the margin are drawn in red ink. One leaf of Book XV, containing the text from the end of prop. 12 up to the beginning of prop. 15, is missing.

F is MS. Istanbul, Süleymaniye Library, Aya Sofya 2719 (Krause, p. 506), f. 178a-195a. This manuscript is dated Dhū 'l-Ḥijja 924/A.D. 1518-9 (the date 714, stated by Krause, is incorrect, and I have found no evidence that the ms. was written by Aḥmad ibn al-Sarrāj). Although Krause calls this manuscript "sehr gut", there are many scribal errors, sometimes of the most grotesque kind, and numerous words and passages have been repeated or omitted by mistake.

H is MS. Istanbul, Süleymaniye Library, Mihrishah 337, f. 168a-186a. According to Krause (p. 506), most of this manuscript was written in the 8th century H./14th c. A.D. A few leaves were written in 1159 H./A.D.1746-7 by Muṣṭafā Ṣidqī, a well-known 18th century mathematician of Cairo. This manuscript appears to be a copy of a slightly revised version of the *Revision*, in which some changes were made in the wording and in the mathematical reasoning. Some of these are improvements, but others are changes for the worse. In addition to these deliberate changes there are numerous scribal errors in H.

My analysis of scribal errors has shown that manuscript F is related to, but not dependent on T, and that manuscript H is related to, but not dependent on D.

Editorial principles

I have attempted to restore the original text of Book XV of al-Maghribī's *Revision* on the basis of the two manuscripts D and T, which are the oldest and the best. I have modernized the orthography, added hamzas and made trivial corrections involving slashes and diacritical marks without noticing this. All other scribal errors and variants in D and T have been mentioned in the apparatus. I have not taken account of the fact that in T the last word on a leaf is usually repeated at the beginning of the next leaf.

I have indicated the beginning of new pages in F and H in the text, but I have used these manuscripts only to a very limited extent. Manuscript F is of little use for the reconstruction of the original text of the *Revision*, because F never has a plausible variant in the cases where there are difficulties in the text in D and T. F and H contain a large number of errors, nearly all of which are uninteresting. There are only a few cases where the text in D and T is corrupt and H is (mathematically) correct, or where the context indicates that a passage in D and T must be missing, and the missing passage is supplied by H. In such cases I have emended the text on the basis of H, as indicated in the apparatus (this concerns nos. 33-36, 38, 44, 46, 124, 132, 134). In these cases H does not necessarily represent the original text of al-Maghribī, because H is a copy of a slightly revised version of the *Revision*, and the errors or omissions may have been corrected by the person who revised the text. Apart from these exceptional cases I have not mentioned the variants and errors in F and H, because this would have tripled the size of the apparatus, and very little would have been gained for the reconstruction of the original of al-Maghribī. The text is on the whole very simple, and the edition does not pose any major philological problems.

I have added full stops and commas in the Arabic text for sake of better legibility. Square brackets in the Arabic text contain the page numbers in the manuscripts. In the Arabic text and the apparatus the four manuscripts have been abbreviated as follows: D = د, T = ط, F = ف, H = ه. Angular brackets < > in the Arabic text indicate additions made by me to restore the original, in cases where the text in D and T is incomplete. Passages restored by means of H are also in angular brackets. In Book XV, al-Maghribī often refers to previous propositions in his revision of the *Elements*, using abjad-numbers with slashes. These alphabetical numbers can very easily be confused

with the letters that are used in the text for points in the geometrical figures, because these letters are also written with slashes. To avoid confusion I have put these Arabic references to earlier propositions in parentheses.

The same procedures have been used for the edition of propositions XIII:10 and 14 of the *Revision of the Elements* in the appendix to this paper. These propositions have been edited because they also occur in the Hebrew translation found by Dr. Langermann, and hence they must have been in the lost Arabic source **A**. The text of these propositions is found in the following pages in the manuscripts: D 141a:8-141b:3, 143a:1-5; T 69b:10-28, 70a:32-37; H 149a:13-149b:10, 150b:12-19; F 156b:18-157a:14, 158b:6-12.

Abbreviations used in the critical apparatus

D (د) Ms. Oxford, Bodleian Library Or. 448, 160a-175b, 141a:8-141b:3, 143a:1-5.

T (ط) Ms. Utrecht, Universiteitsbibliotheek, 1440, f. 81a-90a, 69b:10-28, 70a:32-37.

F (ف) Ms. Istanbul, Süleymaniye Library, Aya Sofya 2719, 178a-195a.

H (ه) Ms. Istanbul, Süleymaniye Library, Mihrishah 337, 168a-186a, 149a:13-149b:10, 150b:12-19.

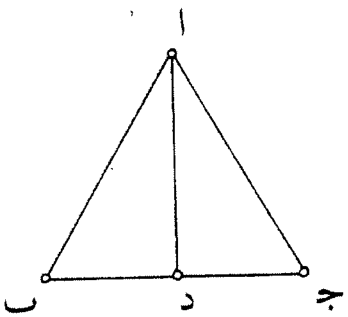
\+ added in one of these manuscripts (as indicated)

\- omitted in one of these manuscripts

*** illegible passage.

المقالة الخامسة عشرة فى المناسبة الواقعة بين المجسمات الخمس[1]

ا. مربع عمود كل مثلث متساوي [هـ ١٦٨ ب] الأضلاع ثلاثة أرباع مربع ضلعه. فليكن مثلث ا ب جـ متساوي الأضلاع وعموده ا د. فأقول إن مربع ا د ثلاثة أرباع مربع ا ب. برهانه إن زاويتي ا ب جـ ا جـ ب متساويتان وزاويتا[2] د قائمتان وخط ا د مشترك لمثلثي[3] ا د ب ا د جـ فخطا ب د د جـ متساويان فمربع ب د ربع مربع ب جـ أي ا ب (د من ب). ومربعا ا د د ب مثل مربع ا ب (مز من ا). لكن مربع د ب ربع مربع ا ب فيبقى مربع ا د ثلثة أرباع مربع ا ب. وذلك ما أردنا بيانه[4].



شكل ١

ب. ضلع ذي الثماني قواعد مساو لعمود مثلث الناري المخطوطين في كرة واحدة. برهانه إن مربع قطر الكرة مثل ونصف مربع ضلع الناري (ا من يد) ومربع قطر الكرة أيضاً ضعف مربع ضلع ذي الثماني قواعد (هـ من يد)

[1] الخمسة (د) || [2] وزاويتا (ط) وزاويتى (د) وزاويتي خ (حاشية في ط) || [3] بين مسلسي (ط) لمثلثي خ (حاشية في ط) || [4] أردناه(ط).

فمثل ونصف مربع ضلع الناري [ط ٨١ ب] مثل ضعف مربع ضلع ذي الثماني قواعد. [ف ١٧٨ ب] فثلاثة[5] أرباع مربع ضلع الناري مثل مربع ضلع ذي الثماني قواعد. وثلاثة أرباع مربع ضلع الناري كمربع عمود مثلثه فمربع ضلع ذي الثماني قواعد مساوٍ لمربع عمود مثلث الناري. فضلع ذي الثماني قواعد كعمود [د ١٦٠ ب] مثلث الناري وذلك ما أردنا بيانه[6].

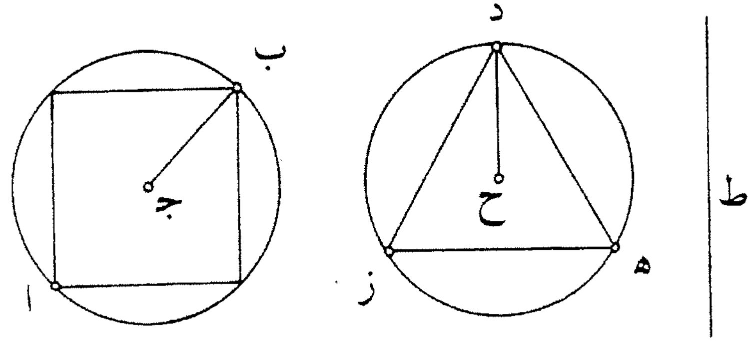
ويستبين من هذا أن مثلث ذي الثماني قواعد ثلاثة[7] أرباع مثلث الناري لأن نسبة مثلث ذي الثماني قواعد[7] إلى مثلث الناري كنسبة مربع ضلعه إلى مربع ضلع الآخر ومربع ضلع ذي الثماني قواعد ثلاثة أرباع مربع ضلع الناري.

جـ. مربع ضلع المكعب ومثلث ضلع ذي الثماني قواعد المعمولين فى كرة واحدة تحيط بهما دائرة واحدة. فليكن مربع ضلع المكعب ا ب ونصف قطر الدائرة المحيطة به جـ ب[8] ومثلث[9] ذي الثماني قواعد دهـ ز ونصف قطر الدائرة المحيطة به ح د[10] وليكن قطر [هـ ١٦٩ ا] الكرة خط ط. فمربع ط ثلاثة أمثال مربع ا ب (جـ من يد) ومربع ا ب ضعف مربع جـ ب (مز من ا) فمربع ط ستة أمثال مربع جـ ب. ومربع ط ضعف مربع د هـ (هـ من يد) ومربع د هـ ثلاثة أمثال مربع ح د (يا من يجـ) فمربع ط ستة أمثال مربع ح د. فمربع جـ ب كمربع ح د فـ جـ ب مثل ح د[11]. فالدائرة المحيطة بمربع ضلع المكعب كالدائرة المحيطة بمثلث ذي الثماني [ف ١٧٩ ا] قواعد وذلك ما أردنا بيانه.

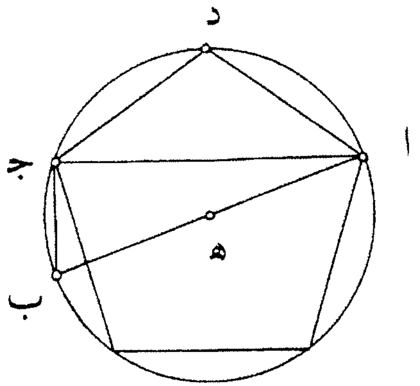
وقد استبان من هذا أن العمود الواقع من مركز الكرة على مربع ضلع المكعب مثل العمود الواقع من مركزها على مثلث[12] ذي الثماني قواعد لأن الأعمدة الواقعة من مركز الكرة على الدوائر المتساوية المرسومة على بسيط الكرة متساوية.

[5] فمثلثه (د) || [6] أردناه (ط) || [7] ثلاثة ... قواعد: (حاشية في د فقط) || [8] جـ ر (د، ط) || [9] ومسل (ط) || [10] ح ط (ط) || [11] ح حـ (ط) || [12] مسل (ط).

واستبان أيضاً أن مربع ضلع ذي الثماني قواعد مثل ونصف لمربع ضلع المكعب لأن مربع ضلع ذي الثماني قواعد ثلاثة أمثال مربع نصف قطر الدائرة المحيطة به أي المحيطة بمربع ضلع [د ١٦١ ا] المكعب ومربع ضلع المكعب ضعف مربع نصف قطر هذه الدائرة فمربع ضلع ذي الثماني قواعد مثل ونصف لمربع ضلع المكعب .



شكل ٣



شكل ٤

د. كل دائرة فإن مجموع مربعي ضلع مخمسها ووتر زاوية المخمس [هـ ١٦٩ ب] خمسة أمثال مربع نصف قطر الدائرة المحيطة بالمخمس. فليكن دائرة قطرها ا ب وضلع مخمسها جـ د ووتر زاوية المخمس ا جـ ومركز الدائرة هـ

فأقول[١٣] إن مربعي ا جـ جـ د خمسة أمثال مربع هـ ب . برهانه إنا نصل جـ ب فـ جـ ب وتر المعشر فمربعا ا جـ جـ ب مثل مربع ا ب (مز من ا) أعني أربعة أمثال مربع هـ ب. ونأخذ مربع هـ ب مشتركاً فمربعات ا جـ جـ ب هـ ب الثلاثة [ط ٨٢ ا] خمسة أمثال مربع هـ ب. ولكن مربعا جـ ب هـ ب مثل مربع جـ د (يجـ من يجـ) فمربعا ا جـ جـ د خمسة أمثال مربع هـ ب وذلك ما [ف ١٧٩ ب] أردنا بيانه.

هـ. مثلث ذي العشرين قاعدة ومخمس ذي الاثنتي عشرة قاعدة اللذان تحيط[١٤] بهما كرة واحدة يقعان في دائرة واحدة. مثاله إنا نجعل مخمس ذي الاثنتي عشرة قاعدة مخمس ا ب جـ د هـ ومركز الدائرة المحيطة به ز ونصف قطرها ز د ومثلث ذي العشرين قاعدة مثلث ح ط ك ومركز الدائرة المحيطة به ل ونصف قطرها ل ك . فأقول إن ل ك مثل ز د.

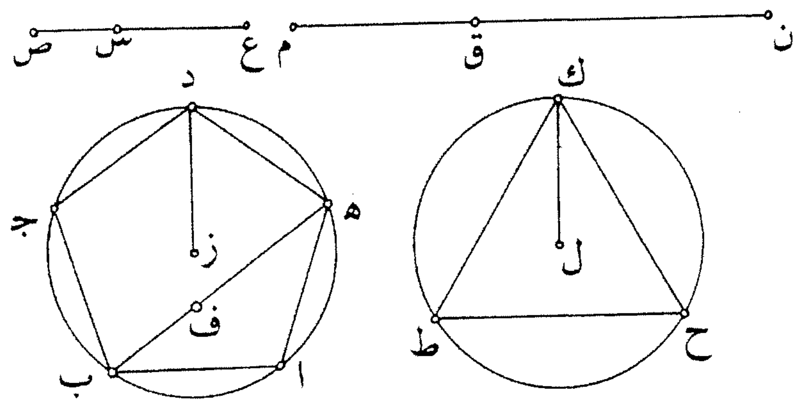
برهانه إنا نصل ب هـ فهو ضلع المكعب. ونجعل قطر الكرة م ن ونصف قطر الدائرة التي قد تبين أن ضلع مخمسها مساوٍ لضلع مثلث ذي العشرين ص ع . ونقسم م ن على نسبة ذات وسط وطرفين على ق وليكن القسم الأعظم ق ن. ونقسم ص ع بنسبة ذات وسط وطرفين [د ١٦١ ب] أيضاً على س وليكن القسم الأعظم س ع فـ س ع ضلع المعشر (يد من يد) . ونقسم ب هـ أيضاً على نسبة ذات وسط وطرفين على ف والقسم الأعظم ف هـ فهو مثل ا هـ (يد من يجـ). فنسبة م ن إلى ص ع كنسبة ق ن إلى س ع (ي من يد). [ف ١٨٠ ا] فنسبة مربع م ن إلى مربع ص ع كنسبة مربع ق ن إلى مربع س ع (كب من و). ومربع م ن قد [هـ ١٧٠ ا] تبين أنه خمسة أمثال مربع ع ص (د من يه) فمربع ق ن خمسة أمثال مربع س ع. فمربعا م ن ن ق خمسة أمثال مربعي ص ع ع س أعني ‹مربع› ضلع مخمس الدائرة التي نصف قطرها

[١٣] أقول (ط) || [١٤] يحيطان (د، ط).

ص ع أي مربع ح ط (يه من يجـ) أعني ثلاثة أمثال مربع ل ك (يا من يجـ). فمربعا م ن ن ق خمسة عشر مثلاً لمربع ل ك.

وأيضاً فإن[١٥] نسبة مربع م ن إلى مربع ب هـ أي ضلع المكعب كنسبة مربع ق ن إلى مربع ف هـ. ومربع م ن ثلاثة أمثال مربع ب هـ (جـ من يد) فمربع ق ن ثلاثة أمثال مربع ف هـ. فمربعا م ن ن ق ثلاثة أمثال مربعي ب هـ هـ ف. ولكن مربعي[١٦] ب هـ هـ ف خمسة أمثال مربع د ز (د من يه) فمربعا م ن ن ق خمسة عشر مثلاً لمربع د ز. فمربعا ل ك د ز متساويتان فخطا ل ك ز د متساويان فدائرتا ا ب جـ د هـ ح ط ك متساويتان وذلك ما أردنا بيانه.

وقد استبان من هذا أن العمود الواقع من مركز الكرة على قاعدة ذي الاثنتي عشرة قاعدة مساوٍ للعمود الواقع من مركز الكرة على قاعدة ذي العشرين قاعدة لأن الأعمدة الواقعة من مركز الكرة على الدوائر المتساوية المرسومة على بسيطها متساوية وذلك ما أردناه [د ١٦٢ ا].



شكل ٥

و. كل خط [هـ ١٧٠ ب] يقسم بنسبة ذات وسط وطرفين فإن نسبة الخط القوي على الخط كله [ط ٨٢ ب] وعلى قسمه الأعظم إلى الخط القوي على

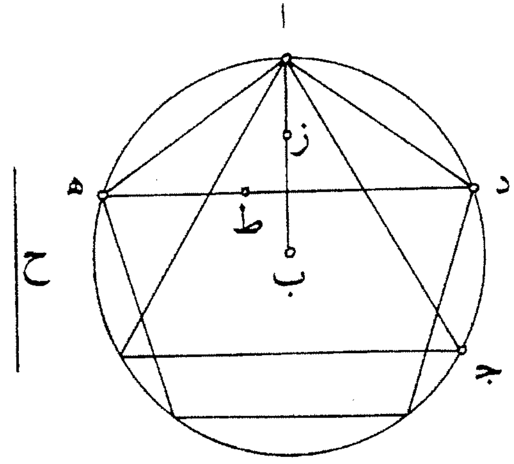
[١٥] فإن: -(ط) || [١٦] مربعات (د) ، مربعا (ط).

الخط كله وعلى قسمه الأصغر كنسبة ضلع المكعب إلى ضلع ذي العشرين قاعدة المخطوطين في كرة واحدة.

مثاله ليكن ا ب نصف قطر الدائرة المحيطة بمخمس ذي الاثنتي عشرة قاعدة ومثلث ذي العشرين قاعدة وليكن ا جـ ضلع مثلثها و ا د [ف ١٨٠ ب] ضلع مخمسها و د هـ وتر خمسيها فهو ضلع المكعب الذي تحيط به الكرة المحيطة بذي الاثنتي عشرة قاعدة وبذي العشرين قاعدة. ونقسم ا ب على نسبة ذات وسط وطرفين على ز وليكن قسمه الأعظم ب ز فـ ب ز ضلع المعشر (يد من يد) فـ ا د يقوى على ا ب ب ز (يجـ من يجـ) وليكن ح القوي على ا ب ا ز. فأقول إن نسبة ا د إلى ح كنسبة ضلع المكعب إلى ضلع ذي العشرين قاعدة.

برهانه إن ح القوي على ا ب ا ز يقوى على ثلاثة أمثال ب ز (ح من يجـ) و اجـ[١٧] يقوى على ثلاثة أمثال ا ب فنسبة مربع ا جـ إلى مربع ا ب كنسبة مربع ح إلى مربع ب ز (يد من يجـ) فنسبة ا جـ إلى ا ب كنسبة ح إلى ب ز (كب من و). فإذا بدلنا كانت نسبة ا جـ إلى ح كنسبة ا ب إلى ب ز. ونقسم د هـ بنسبة ذات وسط وطرفين على ط وليكن قسمه الأعظم د ط فهو مثل ا د (يد من يجـ). فنسبة هـ د إلى د ط أعني إلى ا د كنسبة ا ب إلى ب ز أعني كنسبة [هـ ١٧١ ا] ا جـ إلى ح. فبالتبديل نسبة د هـ ضلع المكعب إلى ا جـ ضلع ذي العشرين قاعدة كنسبة ا د القوي على الخط كله وعلى أعظم [د ١٦٢ ب] قسميه إلى ح القوي على الخط كله وعلى أصغر قسميه. وذلك ما أردنا أن نبين.

[١٧] اح (ط).



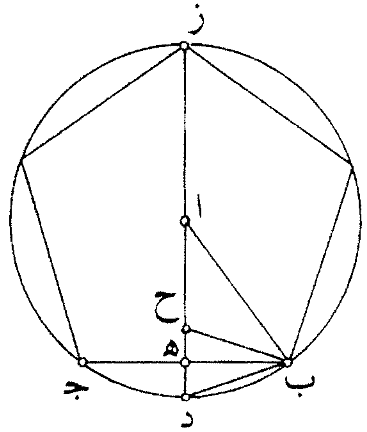
شكل ٦

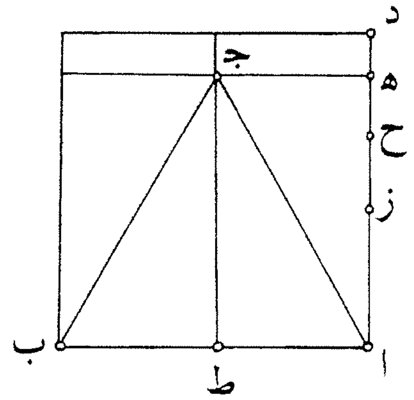
ز. كل دائرة فإن العمود الخارج من مركزها إلى ضلع مخمسها مساوٍ لنصف ضلع معشرها ونصف[18] ضلع مسدسها [ف ١٨١ ا] مجموعين. فلتكن دائرة مركزها ا وضلع مخمسها ب ج وضلع معشرها ب د والعمود الواقع من مركزها على ب ج ضلع المخمس خط ا ه. فأقول أنه مثل نصف ا د ونصف ب د مجموعين.

برهانه إنا نخرج عمود ا ه في الجهتين إلى د و ز ونصل ا ب ونفصل ه ح مثل ه د ونصل ح ب. فلأن قوس ز ب أربعة أمثال قوس ب د فزاوية ز ا ب أربعة أمثال زاوية ب ا د وهي ضعف زاوية ب د ا (ه من ا و لب[19] من ا) فزاوية ب د ا ضعف زاوية ب ا د. ولأن زاويتي ب ه د ب ه ح قائمتان وخطي ه ح ه د متساويان وخط ب ه مشترك فزاوية ب د ه كزاوية ب ح ه (د من ا) فزاوية ب ح ه ضعف زاوية ب ا د وهي كزاويتي ا ب ح ح ا ب (لب من ا) فزاويتا ا ب ح ح ا ب ضعف زاوية ح ا ب فزاويتا ا ب ح ح ا ب متساويتان فخطا [ط ٨٣ ا] ا ح ح ب متساويان (و من ا)

[18] ولنصف (ط) || [19] و لب: أو لب (ط).

فـ اح مثل ب د و ح ه مثل ه د فـ اه مثل ه د دب ونجعل اه مشتركاً [ه ١٧١ ب] فضعف ا ه مثل اد دب فعمود اه مثل نصف اد دب فنصفه أعني اه مثل نصف اد ونصف دب مجموعين أعني نصفي ضلعي المعشر والمسدس وذلك ما أردنا أن نبين.





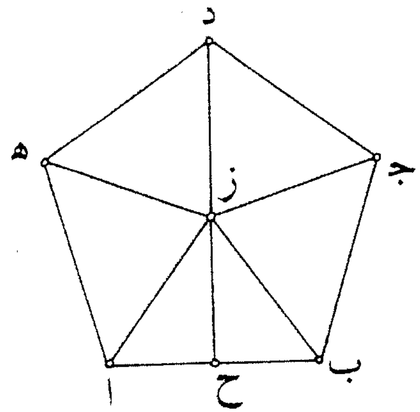
شكل ٨ شكل ٧

ح. نسبة مثلث ذي الثماني قواعد إلى مربع ضلع المكعب كنسبة نصف عمود المثلث [ف ١٨١ ب] إلى ثلثي ضلعه. برهانه إنا نجعل مثلث ذي الثماني قواعد ا ب ج ونعمل على ا ب مربع دب ونخرج من ج خطاً يوازي ا ب ويلقى د ا على ه وننصف ا ه [د ١٦٣ ا] على ز فـ ا ز نصف عمود مثلث ا ج ب ونجعل ا ح ثلثي ا ب. فنسبة ا ز إلى ا د كنسبة نصف سطح ه ب أعني مثلث ا ج ب (ما من ا) إلى مربع دب كنسبة ا ز إلى ا د (ا من و). ونسبة ا د إلى ا ح كنسبة مربع د ب إلى مربع ضلع المكعب لأن مربع ضلع ذي الثماني قواعد مثل ونصف لمربع ضلع المكعب (جـ من يه) و اد[20] مثل ونصف اح. فبالمساواة نسبة از أي[21] نصف عمود مثلث ا ب ج إلى اح ثلثي

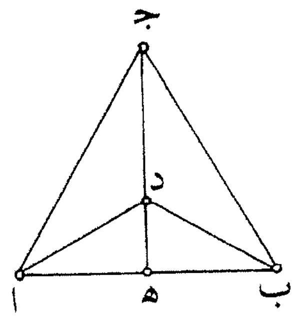
[20] و ا ز (ط) || [21] إلى (ط).

ضلعه كنسبة مثلث اب جـ أعني مثلث ذي الثماني قواعد إلى مربع ضلع المكعب وذلك ما أردنا أن نبين.

ط . مضروب العمود الخارج من مركز مخمس ذي الاثنتي عشرة قاعدة في ضلع المخمس جزء من ثلاثين من سطح ذي الاثنتي عشرة قاعدة. برهانه ليكن مخمس ذي الاثنتي عشرة قاعدة مخمس ا ب جـ د هـ ومركزه ز والعمود الخارج من ز إلى ضلع المخمس [هـ ١٧٢ ا] ز ح . فأقول إن مضروب ز ح[22] في ا ب جزء من ثلاثين من سطح ذي الاثنتي عشرة قاعدة. برهانه إنا نصل بين زوايا المخمس ونقطة ز[23] بخطوط مستقيمة فينقسم المخمس بمثلثات[24] متساويات وكذلك ينقسم كل واحد من قواعد ذي الاثنتي عشرة قاعدة بخمس مثلثات [ف ١٨٢ ا] متساويات مساويات لمثلث ا ز ب وينقسم السطح كله بستين مثلثاً. وضرب ز ح في ا ب ضعف مثلث ا ز ب (ما من ا) فهو جزء [د ١٦٣ ب] من ثلاثين من سطح ذي الاثنتي عشرة قاعدة وذلك ما أردناه.



شكل ٩



شكل ١٠

[22] ا ح (ط) || [23] د (ط) || [24] المخمس بمثلثات: مخمس مثلثات (د).

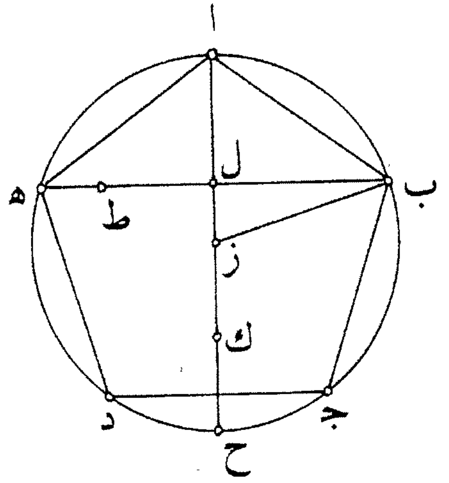
ي. مضروب العمود الخارج من مركز مثلث ذي العشرين قاعدة فى ضلع المثلث جزء من ثلاثين من سطح ذي العشرين قاعدة. مثاله ليكن مثلث ذي العشرين قاعدة مثلث ا ب جـ ومركزه د والعمود الخارج من مركز[25] د إلى ا ب د هـ. ونصل د بزوايا المثلث فينقسم المثلث بثلاث مثلثات متساويات وكذلك ينقسم كل واحد من مثلثات ذي العشرين قاعدة [ط ٨٣ ب] بثلاث مثلثات متساويات فينقسم سطح ذي العشرين قاعدة بستين مثلثاً متساويات وكل واحد منها مساوٍ[26] لمثلث ا د ب. ومضروب د هـ فى ا ب ضعف مثلث ا دب (ما من ا) فهو جزء من ثلاثين جزءاً من سطح ذي العشرين قاعدة وذلك ما أردنا أن نبين.

وقد استبان من هذين الشكلين أن نسبة سطح ذي الاثنتي عشرة قاعدة[27] إلى سطح ذي العشرين قاعدة كنسبة ضرب العمود الخارج من مركز [ف ١٨٢ ب] مخمس ذي الاثنتي عشرة قاعدة [هـ ١٧٢ ب] إلى ضلع المخمس في ضلع المخمس إلى ضرب العمود الخارج من مركز مثلث ذي العشرين قاعدة إلى ضلع المثلث فى ضلع المثلث لأن نسبة الأجزاء كنسبة أضعافها المتساوية (يد من ها) وذلك ما أردنا بيانه[28].

يا . كل مخمس تحيط به دائرة فإن سطحه مساوٍ لضرب ثلاثة أرباع قطر دائرة فى خمسة أسداس وتر زاوية المخمس . فليكن مخمس ا ب جـ د هـ في دائرة قطرها ا ز ح ومركزها ز وليكن ب هـ وتر زاوية المخمس وخمسة أسداسه ب ط. ونقسم ز ح بنصفين على ك فـ ا ك ثلاثة أرباع القطر. فأقول إن ضرب ا ك فى ب ط مثل مخمس ا ب جـ د هـ. برهانه إنا نصل [د ١٦٤ ا] ز ب فمثلث ا ز ب خُمس مخمس ا ب جـ د هـ كما بيّنا غير مرة. وب ل أعني نصف ب هـ ثلاثة أمثال ط هـ و ا ك ثلاثة أمثال نصف ا ز. فنسبة ب ل إلى ط هـ كنسبة ا ك إلى نصف ا ز فضرب ب ل فى نصف ا ز أعني مثلث ا ز ب

[25] مركز: - (د) || [26] مساوي (د) || [27] قاعدة: - (ط) || [28] أردناه (ط).

(ما من آ) كضرب ا ك فى ط هـ (يو[29] من و) فضرب ا ك أي ثلاثة أرباع قطر الدائرة فى سدس وتر زاوية المخمس مثل خُمس مخمس ا ب جـ د هـ. فضرب ا ك فى خمسة أمثال ط هـ أعني ب ط مثل مخمس ا ب جـ د هـ وذلك ما أردنا أن نبين.



شكل ١١

يب. نسبة سطح المكعب إلى سطح ذي الثماني قواعد كنسبة ضلع المثلث المتساوي الأضلاع إلى عموده. برهانه إن نسبة مربع ضلع المكعب إلى مثلث ذي الثماني قواعد كنسبة ثلثي ضلع المثلث المتساوي الأضلاع إلى نصف [ف ١٨٣ ا] عموده (ح من يه). فنسبة ستة أمثال مربع ضلع المكعب أعني سطح المكعب إلى ستة أمثال مثلث ذي [هـ ١٧٣ ا] الثماني قواعد أعني ثلاثة أرباع سطح ذي الثماني قواعد كنسبة ثلثي ضلع المثلث المتساوي الأضلاع إلى نصف عموده (يد من هـ). فنسبة سطح المكعب إلى ثمانية أمثال ‹مثلث› ذي الثماني قواعد أعني سطح ذي الثماني قواعد كنسبة ثلثي ضلع المثلث

[29] ير (ط).

المتساوي الأضلاع إلى ثلثي عموده أعني كنسبة ضلع المثلث [ط : آخر صفحة ٨٣ ب] المتساوي[30] الأضلاع إلى عموده.

واستبان من هذا أن نسبة سطح المكعب إلى سطح ذي[31] الثماني قواعد كنسبة مربع خطٍ إلى ضعف المثلث المتساوي الأضلاع المعمول عليه لأن نسبة كل خط إلى عمود مثلثه [د ١٦٤ ب] كنسبة مربع ذلك الخط إلى ضربه فى عمود مثلثه بالمقدمة وضرب العمود فى القاعدة ضعف ذلك المثلث. ونعني أبداً بمثلث كل خط المثلث المتساوي الأضلاع المعمول عليه وكذا مربعه ومخمسه إلى غير ذلك من الأشكال وذلك ما أردناه.

يج. نسبة سطح المكعب إلى سطح ذي العشرين قاعدة كنسبة مربع ضلع مخمس دائرة ذي العشرين قاعدة إلى ثلاثة أمثال وثُلث مثلث خط يقوى على ثلاثة أمثال مربع ضلع معشرها.

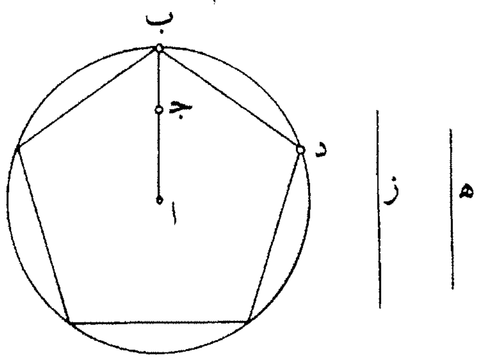
فليكن نصف قطر الدائرة التي[32] ضلع مخمسها ضلع ذي العشرين قاعدة ا ب ونقسمه بنسبة ذات وسط وطرفين على جـ وليكن قسمه الأعظم ا جـ فـ ا جـ ضلع المعشر (يد من يد). وليكن ب د ضلع المخمس [هـ ١٧٣ ب] فهو يقوى على ا ب ا جـ (يجـ من يجـ) وليكن خط هـ القوي على ا ب ب جـ فهو يقوى على ثلاثة أمثال مربع ا جـ أي ضلع معشر الدائرة (ح من يجـ). وليكن ز ضلع المكعب. فأقول إن نسبة سطح المكعب إلى سطح ذي العشرين قاعدة كنسبة مربع ب د إلى ثلاثة أمثال وثُلث مثلث خط هـ.

برهانه إن نسبة ز أي ضلع المكعب إلى ب د ضلع ذي العشرين قاعدة [ف ١٨٣ ب] كنسبة ب د إلى هـ (و من يه) فنسبة مربع ز إلى مربع ب د كنسبة مربع ب د إلى مربع هـ (كب من و). وكل خطين فإن نسبة مربع أحدهما إلى مربع الآخر كنسبة مثلث الأول إلى مثلث الآخر فبالتبديل نسبة مربع أحد الخطين إلى مثلثه كنسبة مربع الآخر إلى مثلثه. فنسبة مربع ب د إلى مثلثه

[30] المتساوي ... ب هـ - (ط). || [31] ذ (د) || [32] الذي (د).

كنسبة مربع هـ إلى مثلثه. وقد كانت نسبة مربع ز إلى مربع [د ١٦٥ ا] ب د كنسبة مربع ب د إلى مربع هـ فبالمساواة نسبة مربع ز إلى مثلث ب د كنسبة مربع ب د إلى مثلث هـ. فنسبة ستة أمثال مربع ز أعني سطح المكعب إلى ستة أمثال مثلث ب د أعني ثلاثة أعشار سطح ذي العشرين قاعدة كنسبة مربع ب د إلى مثلث هـ (يد من هـ). فنسبة سطح المكعب إلى سطح ذي العشرين قاعدة كنسبة مربع ب د إلى ثلاثة أمثال وثلث مثلث هـ وذلك ما أردنا أن نبين.

و قد استبان من هذا أن نسبة ثلاثة أخماس سطح ذي العشرين قاعدة إلى سطح المكعب كنسبة ضعف مثلث الخط القوي على ثلاثة أمثال ضلع المعشر إلى ‹مربع›[٣٣] ضلع المخمس بعد العكس.

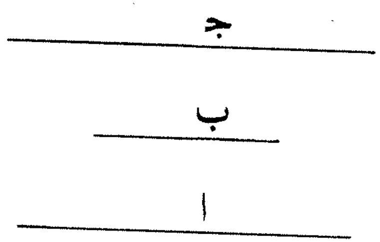


شكل ١٣

يد. نسبة سطح ذي العشرين قاعدة إلى سطح ذي الثماني قواعد [هـ ١٧٤ ا] كنسبة خمسة أمثال مربع ضلع معشر دائرة إلى مربع ضلع مخمسها. فليكن خط ا ضلع مخمس الدائرة وخط ب ضلع معشرها فأقول إن نسبة سطح ذي العشرين قاعدة إلى سطح ذي الثماني قواعد كنسبة خمسة [ف ١٨٤ ا] أمثال مربع ب إلى مربع ا.

[٣٣] مربع (هـ): - (د).

برهانه إنا نجعل خط ج يقوى على ثلاثة أمثال مربع ب (يح من يج) فنسبة ثلاثة أخماس سطح ذي العشرين قاعدة إلى سطح المكعب كنسبة ضعف مثلث ج إلى مربع ا (يج من يه) ونسبة سطح المكعب إلى[٣٤] سطح ذي الثماني قواعد كنسبة مربع ا إلى ضعف مثلثه (يب من يه) فبالمساواة نسبة ثلاثة أخماس سطح ذي العشرين ‹قاعدة›[٣٥] إلى سطح ذي الثماني قواعد كنسبة ضعف مثلث ج إلى ضعف مثلث ا أعني كنسبة مثلث ج إلى مثلث ا (يد من هـ) أعني كنسبة مربع ‹ج›[٣٦] إلى مربع ا. ومربع ج ثلاثة أمثال مربع ب فنسبة ثلاثة أخماس سطح ذي العشرين قاعدة إلى سطح ذي الثماني قواعد كنسبة ثلاثة أمثال مربع ب إلى مربع ا. فنسبة [د ١٦٥ ب] سطح ذي العشرين قاعدة إلى سطح ذي الثماني قواعد كنسبة خمسة أمثال مربع ب ضلع المعشر إلى مربع ا ضلع المخمس (يد من هـ) وذلك ما أردنا أن نبين.



شكل ١٤

يه. نسبة فضلة سطح ذي العشرين قاعدة على سطح ذي الثماني قواعد إلى سطح ذي الثماني قواعد كنسبة القسم الأصغر من الخط المقسوم على نسبة ذات وسط وطرفين إلى الخط كله.

فليكن دائرة مركزها ا وضلع مخمسها ب ج وضلع معشرها ب د ووترا[٣٧]

[٣٤] + ثلاثة أخماس (د) : - (هـ) || [٣٥] قاعدة (هـ، ف): - (د) || [٣٦] ج (هـ): - (د) ||
[٣٧] ووترى (د).

زاويتي مخمسها ب هـ هـ جـ. ونقسم ب هـ على نسبة ذات وسط وطرفين وليكن قسمه الأعظم [هـ ١٧٤ ب] ب ز. فأقول إن نسبة فضلة سطح ذي العشرين قاعدة على سطح ذي الثماني قواعد إلى سطح ذي الثماني قواعد [ف ١٨٤ ب] كنسبة ز هـ إلى هـ ب .

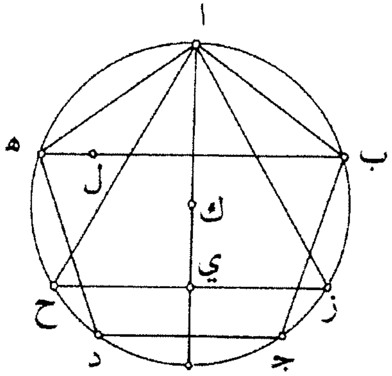
برهانه إنا نخرج خطوط ا ب ا جـ ا د فزاويتا ا متساويتان (كو من جـ) فزاوية ب ا جـ ضعف زاوية ب ا د وهي أيضاً ضعف زاوية ب هـ جـ (يط من جـ) فزاوية ب ا د كزاوية ب هـ جـ فتبقى زاويتا هـ ب جـ هـ جـ ب المتساويتان (هـ من ا) كزاويتي ا ب د ا د ب المتساويتين (هـ من ا) فمثلثا ا ب د هـ ب جـ متشابهان (د من و). فنسبة د ب إلى ب ا كنسبة جـ ب إلى ب هـ .

فنسبة مربع د ب إلى مربع[38] ب ا كنسبة مربع جـ ب إلى مربع ب هـ (كب من و) وكنسبة خمسة أمثال مربع د ب إلى خمسة أمثال مربع ا ب أعني كنسبة مربع ب جـ إلى مربع ب هـ. فبالتبديل نسبة خمسة أمثال مربع د ب إلى مربع ب جـ كنسبة خمسة أمثال مربع ا ب إلى مربع ب هـ[40]. [ط ١٨٤ ا] ولكن خمسة أمثال مربع ا ب أعني مربعي هـ ب ب ز (د من يه) إلى مربع ب هـ ونسبة خمسة أمثال مربع د ب إلى مربع ب جـ كنسبة سطح ذي العشرين قاعدة إلى سطح ذي الثماني قواعد (يد من يه) . فنسبة سطح ذي العشرين قاعدة إلى سطح ذي الثماني قواعد كنسبة مربعي[39] هـ ب ب ز إلى مربع ب هـ. فبالتفصيل نسبة فضلة سطح ذي العشرين قاعدة [د ١٦٦ ا] على سطح ذي الثماني قواعد إلى سطح ذي الثماني قواعد كنسبة مربع ب ز إلى مربع ب هـ[40]. فبالعكس نسبة سطح ذي الثماني قواعد إلى فضلة سطح ذي العشرين قاعدة عليه كنسبة مربع هـ ب إلى مربع ب ز أعني ضرب ب هـ[41] في

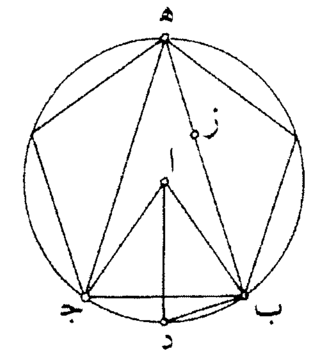
[38] مربع (هـ): - (د) || [39] مربع (ط) || [40] ب د (ط). || [41] ضرب ب هـ: ضرنسبة (د)، ضرب نسبة (ط).

هـ ز أعني [ف ١٨٥ ا] [هـ ١٧٥ ا] نسبة ب هـ إلى هـ ز. فبالعكس نسبة فضلة سطح ذي العشرين قاعدة على سطح ذي الثماني قواعد ‹إلى سطح ذي الثماني قواعد› كنسبة ز هـ إلى هـ ب وذلك ما أردناه.

وقد استبان من هذا أنا إذا ركبنا فنسبة سطح ذي العشرين قاعدة إلى سطح ذي الثماني قواعد كنسبة الخط المقسوم بنسبة ذات وسط وطرفين مع قسمه الأصغر إلى الخط كله.



شكل ١٦



شكل ١٥

يو. نسبة سطح ذي الاثنتي عشرة قاعدة إلى سطح ذي العشرين قاعدة كنسبة ضلع المكعب إلى ضلع ذي العشرين قاعدة. فليكن مخمس ذي الاثنتي عشرة قاعدة مخمس[42] ا ب جـ د هـ ومثلث ذي العشرين قاعدة مثلث ا ز ح ولتحيط عليهما دائرة واحدة قطرها ا ط ومركزها ك (هـ من يه). وليقطع ز ح على ي فـ ك ي[43] ربع القطر. ونصل ب هـ فهو ضلع المكعب (ط من يد). فأقول إن نسبة سطح ذي الاثنتي عشرة قاعدة إلى سطح ذي العشرين قاعدة كنسبة ب هـ إلى ز ح.

[42] مخمس: - (ط) || [43] و ك ي (ط).

برهانه إنا نجعل ب ل خمسة أسداس ب هـ فضرب ا ي في ب ل مثل مخمس ا ب جـ د هـ وضرب ا ي في ز ي مثل مثلث ا ز ح فنسبة مخمس ا ب جـ د هـ إلى مثلث ا ز ح كنسبة ب ل إلى ز ي (آ من و). ومخمس ا ب جـ د هـ جزء من اثني عشر من سطح ذي الاثنتي عشرة قاعدة ومثلث ا ز ح [هـ ١٧٥ ب] جزء من عشرين من سطح [د ١٦٦ ب] ذي العشرين قاعدة فنسبة سطح ذي الاثنتي عشرة قاعدة إلى سطح ذي العشرين قاعدة كنسبة اثني عشر مثلاً ل ب ل إلى عشرين مثلاً ل ز ي أعني كنسبة عشرة أمثال ب هـ إلى عشرة أمثال ز ح أعني كنسبة ب هـ إلى ز ح وذلك ما أردنا أن نبين.

وقد استبان من هذا ومن و من يه[44] أن نسبة سطح ذي الاثنتي عشرة قاعدة إلى سطح ذي العشرين قاعدة كنسبة الخط القوي على الخط المقسوم[45] بنسبة ذات وسط وطرفين وعلى قسمه الأعظم إلى الخط القوي على [ف ١٨٥ ب] الخط كله [ط ٨٤ ب] وعلى قسمه الأصغر وذلك أن هذه النسبة كنسبة ضلع مخمس الدائرة إلى الخط القوي على ثلاثة أمثال مربع ضلع معشرها .

يز. سطح ذي الثماني قواعد مثل ونصف لسطح ذي الأربعة قواعد المخطوطين في كرة واحدة. برهانه إن كل مثلثين من مثلثات ذي الثماني قواعد ربع سطحه وكل مثلث من مثلثات ذي الأربع قواعد ربع سطحه فنسبة كل مثلثين من مثلثات ذي الثماني قواعد إلى مثلث من مثلثات ذي الأربع قواعد كنسبة سطح ذي الثماني قواعد إلى ‹سطح›[46] ذي الأربع قواعد (يد من هـ). لكن كل مثلثين من مثلثات ذي الثماني قواعد مثل ونصف لمثلث من مثلثات ذي الأربع قواعد (ش ب من يه). فسطح ذي الثماني قواعد مثل ونصف لسطح ذي الأربع قواعد المخطوطين في كرة واحدة وذلك ما أردنا أن نبين[47].

[44] ومن و من يه (هـ): ومن و ومن يه (د، ط، ف) || [45] مقوم (د) || [46] سطح (هـ): مثلث من مثلثات (د، ط) || [47] أن نبين: بيانه (ط).

يح. كل مجسم متوازي السطوح قاعدته [هـ ١٧٦ ا] ضعف قاعدة الناري [د ١٦٧ ا] وارتفاعه مثل قطر الكرة المحيطة بالناري فهو تسعة أمثال الناري. برهانه إن المجسم المتوازي السطوح الذي قاعدته مثل قاعدة الناري وارتفاعه كارتفاعه ثلاثة أمثاله[48] (و من يب) فالمجسم المتوازي السطوح الذي قاعدته ضعف قاعدة الناري وارتفاعه كارتفاعه ستة أمثاله فالمجسم الذي قاعدته ضعف قاعدة الناري وارتفاعه مثل ونصف[49] لارتفاعه [ف ١٨٦ ا] هو تسعة أمثاله. لكن مثل ونصف ارتفاع الناري هو مثل قطر الكرة المحيطة به لأن ارتفاع الناري ثلثا قطر الكرة (شا من يد).

فالمجسم المتوازي السطوح الذي قاعدته ضعف قاعدة الناري وارتفاعه مثل قطر الكرة تسعة أمثال الناري. ويلزم من هذا أن المجسم الذي قاعدته ضعف قاعدة الناري وارتفاعه تُسع قطر الكرة مثل المجسم الناري

يط. كل مجسم متوازي السطوح قاعدته مثل مربع[50] ضلع ذي الثماني قواعد وارتفاعه مثل قطر الكرة فهو ثلاثة أمثال ذي الثماني قواعد.

برهانه إن المجسم المتوازي السطوح الذي قاعدته مربع ضلع ذي الثماني قواعد وارتفاعه مثل نصف قطر الكرة المحيطة بذي الثماني قواعد ثلاثة أمثال المخروط الذي قاعدته مربع ضلع ذي الثماني قواعد وارتفاعه مثل نصف قطر الكرة أعني نصف مجسم ذي الثماني قواعد. فالمجسم المتوازي السطوح الذي قاعدته مربع ضلع ذي الثماني قواعد وارتفاعه مثل قطر الكرة ستة أمثال نصف مجسم ذي الثماني قواعد فهو ثلاثة أمثاله. ويلزم من هذا أن يكون المجسم الذي قاعدته مربع ضلع ذي الثماني قواعد [هـ ١٧٦ ب] وارتفاعه ثُلث قطر الكرة مثل مجسم ذي الثماني قواعد.

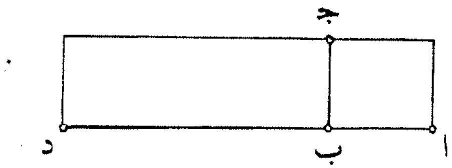
[48] أمثال (د) || [49] مثل نصف (ط) || [50] ربع (ط).

وقد استبان من هذا أن المجسم المتوازي السطوح الذي قاعدته ثلاثة أمثال مربع ضلع ذي الثماني قواعد وارتفاعه مثل تسع قطر الكرة مثل مجسم ذي الثماني قواعد.

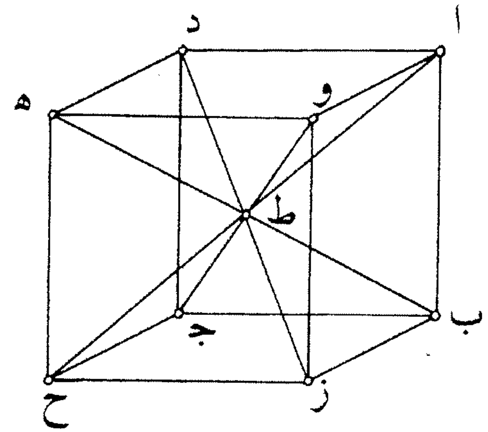
ك. نسبة مجسم ذي الأربع قواعد إلى مجسم ذي الثماني قواعد [د ١٦٧ ب] كنسبة ضلع المثلث المتساوي الأضلاع إلى ثلاثة أمثال عموده. برهانه إنا نجعل اب ضلع [ط ٨٥ ا] مجسم ذي الأربع [ف ١٧٦ ب] قواعد ونخرج من نقطة ب عمود ب جـ ونفصله مثل عمود مثلث الناري فهو مثل ضلع ذي الثماني قواعد (ب من يه). ونخرج اب على استقامة[51] ونفصل منه ب د ثلاثة أمثال ب جـ ونتمم سطحي[52] ا جـ جـ د متوازيي[53] الأضلاع. فسطح ا جـ ضعف مثلث الناري و ب د ثلاثة أمثال ضلع ذي الثماني قواعد. فالمجسم المتوازي السطوح الذي قاعدته ا جـ وارتفاعه تسع قطر الكرة مثل مجسم الناري (يط من يه) والمجسم المتوازي السطوح الذي قاعدته جـ د وارتفاعه تسع قطرالكرة مثل مجسم ذي [هـ ١٧٧ ا] الثماني قواعد (يط من يه). ونسبة المجسم المتوازي السطوح الذي قاعدته ا جـ وارتفاعه تسع قطر الكرة إلى المجسم المتوازي السطوح الذي قاعدته جـ د وارتفاعه تسع قطر الكرة كنسبة ا جـ إلى جـ د أعني ا ب إلى ب د. لكن ب د ثلاثة أمثال ضلع ذي الثماني قواعد. فنسبة مجسم الناري إلى مجسم ذي الثماني قواعد كنسبة ضلع مثلث الناري إلى ثلاثة أمثال عموده.

وقد استبان من هذا أن نسبة مجسم الناري إلى مجسم ذي الثماني قواعد كنسبة ضلع الناري إلى ثلاثة أمثال ضلع ذي الثماني قواعد.

[51] الاستقامة (ط) || [52] سطح (د) || [53] متوازي (د).



شكل ٢٠



شكل ٢١

كا. كل مجسم متساوي الأضلاع والزوايا تحيط به كرة فإنه إذا وصل بين زواياه المجسمة ومركز الكرة بخطوط مستقيمة انقسم بأشكال نارية متساوية متشابهة عدتها كعدة قواعد الشكل المجسم وتكون الأعمدة الواقعة من مركز الكرة على قواعد هذه السطوح متساوية. مثاله ليكن مكعب عليه ا ب [ف ١٨٧ ا] جـ د هـ و ز ح ومركزه ط ونصل خطوط ط ا ط ب ط جـ ط د ط هـ ط و ط ز ط ح المستقيمة فهي متساوية لأنها أنصاف أقطار [د ١٦٨ ا] الكرة[54] المحيطة بالمكعب. وهي أضلاع الناريات التي قواعدها قواعد المكعب ورؤوسها مركز الكرة فقد انقسم المكعب بناريات ستة عدتها كعدة قواعده الست. وهذه الناريات متساويات القواعد والخطوط الخارجة من ط أعني رؤوسها إلى زوايا قواعدها متساوية لأن[55] [هـ ١٧٧ ب] كل واحد

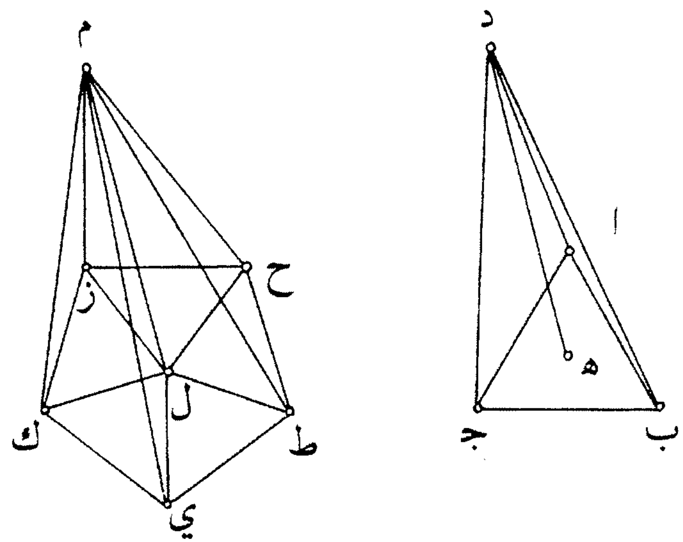
[54] الكرة: - (د، ط) || [55] ولأن (ط).

منها مثل نصف قطر الكرة فالناريات[٥٦] كلها متساوية متشابهة. وكذلك نبين في كل مجسم من المجسمات الباقية التي تحيط بها الكرة أنه ينقسم[٥٧] بأشكال ناريات[٥٨] متساويات متشابهات. وتكون الأعمدة الواقعة من رؤوسها على قواعدها متساوية لأن[٥٩] الدوائر [ط ٨٥ ب] الحادثة في الكرة من قطوع هذه القواعد لها متساوية فالأعمدة الواقعة عليها من مركز الكرة متساوية[٦٠] بما تقدم بيانه وذلك ما أردناه.

كب. كل ناريين سهماهما متساويان وهما قائمان على قاعدتيهما وقاعدة أحدهما مثلث وقاعدة الآخر مخمس متساوي الأضلاع والزوايا فإن نسبة أحدهما إلى الآخر كنسبة قاعدته إلى قاعدة الآخر. فليكن ناري قاعدته مثلث ا ب جـ المتساوي الأضلاع وسهمه د هـ وناري قاعدته مخمس ز ح ط ي ك المتساوي[٦١] الأضلاع والزوايا وسهمه [ف ١٨٧ ب] ل م وليكن د هـ ل م متساويين قائمين على قاعدتيهما. فأقول إن نسبة ناري ا ب جـ د[٦٢] إلى ناري ز ح ط ي ك م كنسبة مثلث ا ب جـ إلى مخمس ز ح ط ي ك[٦١] [د ١٦٨ ب].

برهانه إنا نصل خطوط ل ز ل ح ل ط ل ي ل ك وخطوط م ز م ح م ط م ي م ك [هـ ١٧٨ ا]. فنسبة ناري ا ب جـ د إلى ناري ز ح ل م كنسبة مثلث ا ب جـ إلى مثلث ل ز ح (هـ من يب). ونسبة ناري م ز ح ل إلى ناري ز ح ط ي ك م كنسبة مثلث ل ز ح إلى مخمس ز ح ط ي ك (ي جـ من هـ). ففي نسبة المساواة نسبة ناري ا ب جـ د إلى ناري ز ح ط ي ك م كنسبة مثلث ا ب جـ إلى مخمس ز ح ط ي ك (كب من هـ) وذلك ما أردنا أن نبين[٦٣].

[٥٦] فالنارية (د) ‖ [٥٧] ينقسم (د، ط) انقسم خ (حاشية في ط) ‖ [٥٨] نارية (د) ‖ [٥٩] + لأن (د) ‖ [٦٠] متساو (د) ‖ [٦١] المتساوي ... ز ح ط ي ك: (مكررة في د) ‖ [٦٢] ا جـ د (ط، د المرة الثانية) ‖ [٦٣] أردناه (ط).



شكل ٢٢

كجـ. نسبة مجسم المكعب إلى مجسم ذي الثماني قواعد كنسبة ضلع المثلث المتساوي الأضلاع إلى عموده. برهانه إن نسبة مجسم المكعب أعني ستة أمثال الناري الذي قاعدته قاعدة المكعب ورأسه مركز الكرة المحيطة بالمكعب (كا[٦٤] من يه) إلى ستة أمثال الناري الذي قاعدته قاعدة ذي الثماني قواعد ورأسه مركز الكرة المحيطة بهما أعني ثلاثة أرباع مجسم ذي الثماني قواعد (كا من يه)[٦٥] كنسب الناري [ف ١٨٨ ا] الذي قاعدته قاعدة المكعب ورأسه مركز الكرة إلى الناري الذي قاعدته مثلث ذي الثماني قواعد ورأسه مركز الكرة ‹وهذه النسبة كنسبة قاعدة المكعب إلى قاعدة ذي الثماني قواعد› لأن ارتفاع الناري الذي قاعدته قاعدة المكعب مثل ارتفاع الناري الذي قاعدته قاعدة ذي الثماني قواعد (جـ[٦٦] من يه). ونسبة قاعدة المكعب إلى قاعدة ذي الثماني قواعد كنسبة ثلثي ضلع المثلث المتساوي الأضلاع إلى نصف عموده (ح من يه)[٦٧]. فنسبة المكعب إلى ذي الثماني قواعد كنسبة ثلثي ضلع المثلث

[٦٤] ي (د، ط) ‖ [٦٥] ي من يه (ط)، - (د) ‖ [٦٦] ز (ط) ‖ [٦٧] ح من يه: - (د).

المتساوي الأضلاع إلى ثلثي [د ١٦٩ ا] عموده التي هي كنسبة ضلع المثلث المتساوي الأضلاع إلى عموده وذلك ما أردنا بيانه[٦٨] [ط ٨٦ ا].

كد. المجسم المتوازي السطوح الذي قاعدته [هـ ١٧٨ ب] مساوية لمخمس ذي العشرين قاعدة وارتفاعه مثل ثلثي قطر الكرة مساوٍ لمجسم ذي العشرين قاعدة. برهانه إذا أسقط من مركز الكرة عمود على مخمس ذي العشرين قاعدة وأنفذ إلى زاوية ذي العشرين قاعدة كان المنفصل من هذا الخط بين المركز والمخمس نصف ضلع مسدس الدائرة المحيطة بالمخمس والخط المنفصل منه بين سطح المخمس وزاوية ذي العشرين قاعدة مثل ضلع معشرها (ز من يد). وإذا وصل[٦٩] بين مركز الكرة وزوايا مخمس ذي العشرين قاعدة بخطوط مستقيمة انفصل[٧٠] من ذي العشرين قاعدة ناريان رأس أحدهما مركز الكرة وقاعدته المخمس المذكور وارتفاعه نصف ضلع المسدس ورأس الآخر زاوية ذي العشرين قاعدة وقاعدته المخمس المذكور وارتفاعه ضلع المعشر. وهذان الناريان مثل خمس ناريات من الناريات التي قواعدها قواعد ذي العشرين قاعدة ورؤوسها مركز الكرة فهي مثل ربع مجسم ذي العشرين قاعدة. فالمجسم المتوازي السطوح الذي قاعدته مخمس ذي العشرين قاعدة وارتفاعه مثل سدس ضلع المسدس وثلث ضلع[٧١] المعشر اللذان[٧٢] هما سدس قطر الكرة مساو لهذين الناريين أعني ربع مجسم ذي [ف ١٨٨ ب] العشرين قاعدة. فالمجسم المتوازي السطوح الذي قاعدته مثل مخمس ذي العشرين قاعدة وارتفاعه ثلثا قطر الكرة مثل مجسم ذي العشرين قاعدة.

كه. نسبة عمود مجسم ذي العشرين قاعدة إلى قطر الكرة المحيطة كنسبة نصف ضلع المعشر ونصف ضلع المسدس الواقعين فى دائرة إلى ضعف عمود مثلث ضلع مخمسها ويعني بعمود لكل شكل مجسم كل العمود الواقع من

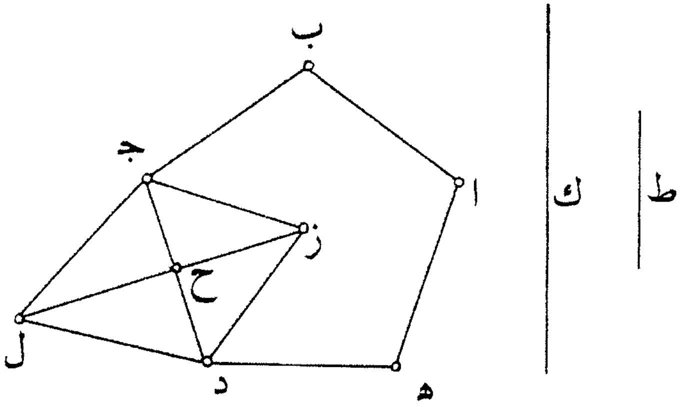
[٦٨] أردناه || [٦٩] فصل (ط) || [٧٠] الفصل (د) || [٧١] ضلع: (حاشية فى ط فقط) || [٧٢] للذان (د).

مركز الكرة على قاعدة الشكل المجسم . فليكن مخمس ذي العشرين قاعدة ا ب جـ د هـ ومركزه ز والعمود الواقع على ضلعه من المركز ز ح وليكن ط عمود ذي العشرين قاعدة [هـ ١٧٩ ا] و ك قطر الكرة المحيطة به [د ١٦٩ ب] ونعمل على د جـ مثلث د ل جـ متساوي الأضلاع ونصل ل ح فبين أنه عمود على د جـ .

فأقول إن نسبة ز ح المساوي لنصف ضلع المسدس والمعشر إلى ضعف ح ل كنسبة ط إلى ك. برهانه إن المجسم المتوازي السطوح الذي قاعدته مخمس ا ب جـ د هـ وارتفاعه مثل ثلثي ك مثل مجسم ذي العشرين قاعدة (كد[٧٣] من يه) والمجسم المتوازي السطوح الذي قاعدته مثل سطح ذي العشرين قاعدة وارتفاعه مثل ثلث ط مساو لمجسم ذي العشرين قاعدة أيضاً وذلك بين. فالمجسم المتوازي السطوح الذي قاعدته مخمس ا ب جـ د هـ وارتفاعه مثل ثلثي ك مساو للمجسم المتوازي السطوح الذي قاعدته مساوية لسطح ذي العشرين قاعدة وارتفاعه مثل ثلث ط. فنسبة مخمس ا ب جـ د هـ [ط ٨٦ ب] إلى سطح ذي العشرين قاعدة كنسبة ثلث ط إلى ثلثي ك بالتكافىء (لد من يا). ولأن مثلث ز د جـ خمس مخمس ا ب جـ د هـ فنسبة مثلث ز د جـ أي[٧٤] خمس المخمس إلى خمس سطح ذي العشرين كنسبة سطح مخمس ا ب جـ د هـ إلى سطح ذي العشرين. فنسبة[٧٥] مثلث ز د جـ إلى خمس [ف ١٨٩ ا] سطح ذي العشرين قاعدة أعني أربع مثلثات من مثلثاته (يد من هـ) كنسبة ثلث ط إلى ثلثي ك.[٧٦] فنسبة ثلث ط إلى ثلث ك كنسبة مثلث ز د جـ إلى ضعف مثلث ذي العشرين قاعدة أعني مثلث جـ ل د أعني كنسبة ز ح إلى ضعف ح ل (ا من و). ونسبة ثلث ط إلى ثلث ك كنسبة [هـ ١٧٩ ب] ط إلى ك (يد من هـ) فنسة ط إلى ك كنسبة ز ح إلى ضعف ح ل أي <نسبة>

[٧٣] كه (ط) || [٧٤] إلى (ط) || [٧٥] كنسبة (د، ط) || [٧٦] + فنسبة مثلث ز د جـ إلى أربع مثلثات من مثلثات ذي العشرين (د، ط).

عمود[٧٧] المخمس أي[٧٨] نصف ضلعي [د ١٧٠ ا] المسدس والمعشر إلى ‹ضعف› عمود مثلث ضلع[٧٩] المخمس وذلك ما أردناه.



شكل ٢٥

كو. نسبة نصف قطر الكرة المحيطة بذي الثماني قواعد إلى عمود مجسمه كنسبة عمود المثلث المتساوي الأضلاع إلى نصف ضلعه. برهانه إن عمود ذي الثماني قواعد مثل عمود المكعب (جـ من يه) ومربع نصف قطر الكرة ثلاثة أمثال مربع عمود المكعب (جـ من يد) فهو ثلاثة أمثال مربع عمود ذي الثماني قواعد. ومربع عمود المثلث المتساوي الأضلاع[٨٠] ثلاثة أمثال[٨١] مربع نصف ضلعه (آ من يه) و (د من ب). فنسبة مربع نصف قطر الكرة إلى مربع عمود ذي الثماني قواعد كنسبة مربع عمود المثلث إلى مربع نصف ضلعه. فنسبة نصف قطر الكرة إلى عمود ذي الثماني قواعد كنسبة عمود المثلث إلى نصف ضلع المثلث (كب من و) وذلك ما أردنا بيانه[٨٢].

[٧٧] العمود (ط) || [٧٨] إلى (د، ط) || [٧٩] ضلع: - (ط) || [٨٠] + إلى (د، ط) || [٨١] مثال (د) || [٨٢] أردناه (ط).

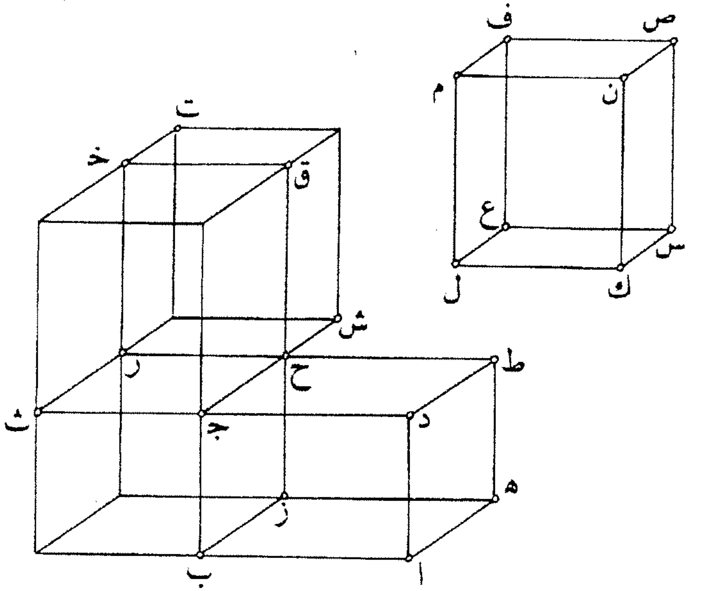
كز. نسبة عمود مجسم ذي العشرين قاعدة إلى عمود ذي الثماني قواعد كنسبة ضلعي المسدس والمعشر الواقعين في دائرة واحدة إلى ضلع مخمس الدائرة. برهانه فلأن نسبة عمود ذي العشرين قاعدة إلى قطر الكرة كنسبة نصف ضلع المعشر ونصف ضلع المسدس مجموعين [ف ١٨٩ ب] إلى ضعف عمود مثلث ذي العشرين قاعدة (كه من يه) ونسبة قطر الكرة إلى عمود ذي الثماني قواعد كنسبة ضعف [هـ ١٨٠ ا] عمود مثلث ذي العشرين قاعدة إلى نصف ضلعه (كو من يه)، ويلزم من مجموعهما أن تكون نسبة عمود مجسم ذي العشرين إلى قطر الكرة كنسبة نصف ضلعي المعشر والمسدس إلى عمود مثلث ذي العشرين ونسبة قطر الكرة إلى عمود مجسم ذي الثماني قواعد[٨٣] كنسبة ضعف عمود مثلث ذي العشرين إلى نصف ضلع مثلثه فبالمساواة نسبة عمود ذي العشرين قاعدة إلى عمود ذي الثماني قواعد كنسبة نصف ضلع المعشر ونصف ضلع المسدس إلى نصف ضلع ذي العشرين أعني كنسبة ضلع المسدس وضلع المعشر إلى ضلع المخمس وذلك ما أردنا أن نبين[٨٤] [د ١٧٠ ب].

كح. كل مجسمين متوازيي السطوح قائمي الزوايا فإن نسبة أحدهما إلى الآخر مؤلفة من نسبة قاعدته إلى قاعدة الآخر بنسبة ارتفاعه إلى ارتفاع الآخر. فليكن مجسم ا ب جـ د هـ ز ح ط متوازي السطوح قائم الزوايا وكذلك مجسم ك ل م ن س ع ف ص متوازي السطوح قائم الزوايا. فأقول إن نسبة مجسم ا ح إلى مجسم ك ف مؤلفة من [ط ١٨٧ ا] نسبة[٨٥] قاعدة هـ ح إلى قاعدة س ف وارتفاع جـ ح إلى ارتفاع م ف.

برهانه إنا نخرج خط ز ح ونفصل ح ق مثل ك ن. ونخرج خط ط ح ونفصل ح ر مثل ك ل. ونخرج خط جـ ح ونفصل ح ش مثل ك س [هـ ١٨٠ ب]. ونتمم مجسم ح ت متوازي السطوح فهو مساوٍ لمجسم ك ف لأن قاعدته مساوٍ لقاعدته وارتفاعه مساوٍ لارتفاعه. ونتمم أيضاً مجسمي ب ر ث ق متوازيي السطوح. فنسبة مجسم ا ح إلى مجسم ح ت أعني مجسم ك ف [ف

[٨٣] قواعد: - (د) || [٨٤] أردناه (ط) || [٨٥] + من نسبة (د).

١٩٠ ا] مؤلفة من نسبة مجسم ا ح إلى مجسم ب ر أعني نسبة قاعدة هـ ح إلى قاعدة ز ر ومن نسبة مجسم ب ر إلى مجسم ث ق أعني نسبة قاعدة ر ز إلى قاعدة ر ق ومن نسبة مجسم ث ق إلى مجسم ح ت أعني نسبة قاعدة ث ح إلى قاعدة ر ش. لكن النسبة المؤلفة من نسبة قاعدة ا جـ إلى قاعدة رز ونسبة قاعدة ر ز إلى قاعدة ر ق هي نسبة قاعدة ا جـ إلى قاعدة ر ق أعني قاعدة ك م. ونسبة قاعدة ث ح إلى قاعدة ر ش كنسبة جـ ح ارتفاع مجسم ا ح [د ١٧١ ا] إلى ح ش أعني ارتفاع مجسم ش خ أي ك س أي ارتفاع مجسم م س[86]. فنسبة[87] مجسم ا ح إلى مجسم ك ف مؤلفة من نسبة قاعدة هـ ح إلى قاعدة س ف وارتفاع جـ ح إلى ارتفاع م ف. وذلك ما أردنا أن نبين[88].



شكل ٢٨

كط. نسبة مجسم ذي العشرين قاعدة إلى مجسم ذي الثماني قواعد كنسبة ضلع معشر الدائرة مع ضلع مخمسها إلى ضلع مخمسها.

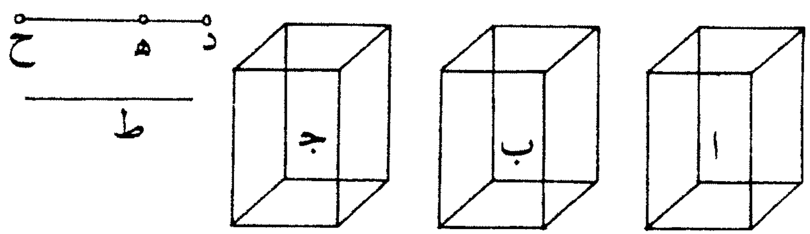
[86] م ف (ط، د) || [87] بنسبة (د) || [88] أردناه (ط).

برهانه إنا نجعل مجسم ا متوازي السطوح قاعدته مثل سطح ذي العشرين قاعدة وارتفاعه مثل ثُلث عمود ذي العشرين قاعدة فهو مثل مجسم ذي العشرين قاعدة. ونجعل قاعدة مجسم ب المتوازي السطوح مثل سطح ذي الثماني قواعد وارتفاعه مثل ثلث عمود ذي الثماني قواعد فهو مثل مجسم ذي الثماني قواعد. ونجعل قاعدة مجسم جـ المتوازي السطوح مثل فضلة سطح ذي العشرين قاعدة [ف ١٩٠ ب] على سطح ذي الثماني قواعد [هـ ١٨١ ا] وارتفاعه مثل ثلث عمود ذي العشرين قاعدة فهو مثل فضل مجسم ا على مجسم ب. وليكن دهـ ضلع [ط ٨٧ ب][89] معشر الدائـــرة و هـ ح ضلع مخمسها. فأقول إن نسبة مجسم ا إلى مجسم ب كنسبة د ح إلى ح هـ. برهانه إن نسبة مجسم جـ إلى مجسم ب مؤلفة من نسبة قاعدة جـ فضلة سطح ذي العشرين قاعدة إلى قاعدة ب أعني سطح ذي الثماني ومن نسبة ارتفاع مجسم جـ أعني ثلث عمود ذي العشرين قاعدة إلى ارتفاع مجسم ب أعني ثلث عمود ذي الثماني قواعد. وليكن ط ضلع المسدس والمعشر الواقعين في الدائرة التي ضلع مخمسها هـ ح وضلع معشرها هـ د. فنسبة فضلة سطح [د ١٧١ ب] ذي العشرين قاعدة على سطح ذي الثماني قواعد إلى سطح ذي الثماني قواعد كنسبة د هـ ضلع المعشر إلى ط ضلع المسدس والمعشر (يه من يه).

ونسبة[90] عمود ذي العشرين قاعدة إلى عمود ذي الثماني قواعد كنسبة ضلع المسدس والمعشر إلى ضلع المخمس (كز[91] من يه). فنسبة مجسم جـ إلى مجسم ب مؤلفة من نسبة د هـ إلى ط ومن نسبة ط إلى هـ ح. لكن النسبة المؤلفة من ‹نسبة› د هـ إلى ط[92] [هـ ١٨١ ب] ونسبة ط إلى هـ ح كنسبة د هـ إلى هـ ح. فنسبة [ف ١٩١ ا] مجسم جـ إلى مجسم ب كنسبة د هـ[93] إلى هـ ح. فبالتركيب نسبة مجسمي جـ و ب أعني مجسم ذي العشرين قاعدة إلى مجسم ب أعني مجسم ذي الثماني قواعد كنسبة د ح إلى ح هـ أي ضلع المخمس والمعشر جميعاً إلى ضلع المخمس. وقد استبان من هذا أن نسبة مجسم

[89] + ضلع (ط) || [90] فنسبة (ط) || [91] لح (ط) || [92] ط ز (ط) || [93] زهـ (ط).

ذي العشرين قاعدة إلى مجسم ذي الثماني قواعد كنسبة الخط القوي على الخط المقسوم بنسبة ذات وسط وطرفين وعلى قسمه الأعظم مع قسمه الأعظم إلى الخط القوي على ذلك الخط وعلى قسمه الأعظم وذلك ما أردناه.

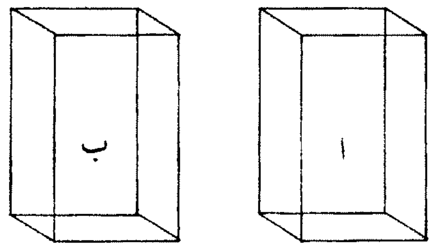


شكل ٢٩

لَ . نسبة مجسم ذي الاثنتي عشرة قاعدة إلى مجسم ذي العشرين قاعدة كنسبة الخط القوي على الخط المقسوم بنسبة ذات وسط وطرفين وعلى قسمه الأعظم إلى الخط القوي على الخط كله وعلى قسمه الأصغر.

برهانه إن العمود الواقع من مركز الكرة على مخمس ذي الاثنتي عشرة قاعدة مساو للعمود الواقع من مركزها على مثلث ذي العشرين قاعدة لما تبين من أن هذا المخمس والمثلث تحيط بهما دائرة واحدة وأن الأعمدة الواقعة من مركز الكرة على سطوح الدوائر المتساوية المرسومة على بسيطها متساوية. فليكن مجسم آ متوازي السطوح وقاعدته مثل سطح ذي الاثنتي عشرة وارتفاعه مثل ثلث عموده وليكن مجسم بَ متوازي السطوح وقاعدته مثل سطح ذي العشرين قاعدة وارتفاعه مثل ثلث عموده. فمن البين أن مجسم آ مثل [ف ١٩١ ب] مجسم ذي [ه ١٨٢ ا] الاثنتي عشرة قاعدة ومجسم بَ [ط ٨٨ ا] مثل مجسم ذي العشرين [د ١٧٢ ا] قاعدة فنسبة مجسم ذي الاثنتي عشرة قاعدة إلى مجسم ذي العشرين قاعدة كنسبة مجسم آ إلى مجسم ب (يدَ من هَ). ونسبة مجسم آ إلى مجسم بَ كنسبة قاعدة مجسم آ

أعني سطح ذي الاثنتي عشرة قاعدة إلى قاعدة مجسم بَ أعني سطح ذي العشرين قاعدة لتساوي ارتفاعيهما. فنسبة مجسم ذي الاثنتي عشرة قاعدة إلى مجسم ذي العشرين قاعدة كنسبة سطح ذي الاثنتي عشرة قاعدة إلى سطح ذي العشرين قاعدة أعني كنسبة الخط القوي على كل خط يقسم بنسبة ذات وسط وطرفين وعلى قسمه الأعظم إلى الخط القوي على الخط كله وعلى قسمه الأصغر وذلك ما أردنا بيانه[٩٤].



شكل ٣٠

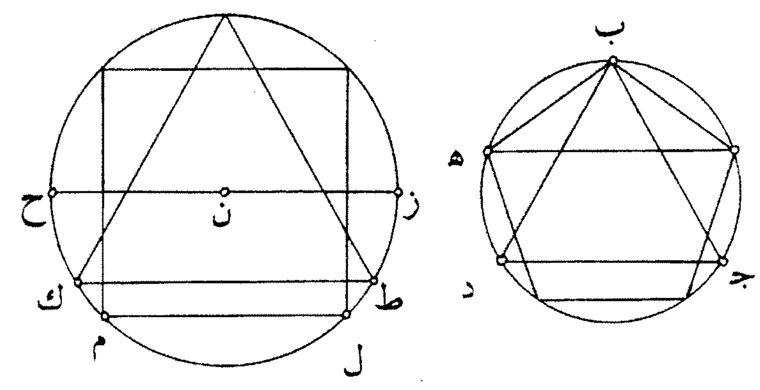
لاَ. نريد أن نجد الخطوط المتوالية على نسب أضلاع الأشكال الخمسة. فنرسم الدائرة المحيطة بمخمس ذي الاثنتي عشرة قاعدة ومثلث ذي العشرين قاعدة وليكن ضلع مخمسها ا ب وضلع مثلثها جـ د ووتر زاوية خمسيها ا ه فهو ضلع المكعب (طَ من يدَ). وليكن ز ح قوياً على ضعف ا ه وننصفه على نَ وندير على مركز نَ وببعد ن ح دائرة وليكن ط ك ضلع مثلثها و ل م ضلع مربعها. فمربع ز ح أربعة أمثال مربع ن ح ومربع ن ح ثلث مربع ط ك (يا من يجـ) فمربع ز ح مثل وثلث لمربع ط ك[٩٥]. ومربع قطر الكرة مثل ونصف[٩٦] مربع ضلع الناري (آ من يدَ) وهو ضعف مربع ضلع ذي الثماني قواعد (هَ من يدَ) فمربع ضلع الناري مثل وثلث مربع ضلع ذي الثماني قواعد. فنسبة مربع ز ح إلى مربع ط ك كنسبة مربع ضلع [ف ١٩٦ ا] الناري إلى [ه ١٨٢ ب] مربع ضلع [د ١٧٢ ب] ذي الثماني قواعد (كبَ من وَ).

[٩٤] أردناه (ط) || [٩٥] ك ط (ط) || [٩٦] ونصف: نصف (د، ط).

وأيضاً فلأن مربع ز ح مثل وثلث مربع ط ك كما بينا ومربع ز ح ضعف مربع ل م (مز من ا) فمربع ط ك مثل ونصف مربع ل م. ومربع قطر الكرة ضعف مربع ضلع ذي الثماني قواعد (هـ من يد) وثلاثة أمثال مربع ضلع المكعب (جـ من يد). فمربع ضلع ذي الثماني قواعد مثل ونصف مربع ضلع المكعب. فنسبة مربع ط ك إلى مربع ل م كنسبة مربع ضلع ذي الثماني قواعد إلى مربع ضلع المكعب. فنسبة مربع ز ح أي مربع ضلع الناري إلى مربع ط ك أي مربع ضلع ذي الثماني قواعد كنسبة مربع ط ك إلى مربع ل م أي ضلع المكعب. فنسبة ط ك إلى ل م كنسبة ضلع ذي الثماني قواعد إلى ضلع المكعب (كب من و). وأيضاً فإن مربع ز ح[97] ضعف مربع ل م (مز من ا) وضعف مربع ا هـ بالفرض فمربع ل م كمربع ا هـ فـ ل م مثل ا هـ. فنسبة ل م أي ا هـ إلى جـ د كنسبة ضلع المكعب إلى ضلع ذي العشرين قاعدة. ولأن ا ب ضلع ذي الاثنتي عشرة قاعدة و جـ د ضلع ذي العشرين [هـ ١٨٣ ا] قاعدة فنسبة جـ د إلى ا ب كنسبة ضلع ذي العشرين قاعدة إلى ضلع ذي الاثنتي عشرة قاعدة[98] [ط ٨٨ ب]. فإذن نسبة ز ح إلى ط ك كنسبة ضلع الناري إلى ضلع ذي الثماني قواعد ونسبة ط ك إلى ل م أعني اهـ كنسبة ضلع ذي الثماني قواعد إلى ضلع المكعب ونسبة [د ١٧٣ ا] ل م أي ا هـ ضلع المكعب إلى د جـ ‹كنسبة ضلع المكعب إلى ضلع ذي العشرين قاعدة ونسبة جـ د إلى ا ب كنسبة› ضلع ذي العشرين قاعدة إلى ضلع ذي الاثنتي عشرة قاعدة وذلك ما أردنا أن نبين[99].

وقد استبان من هذا أن ضلع الناري أعظم من ضلع ذي الثماني قواعد وأن ضلع ذي الثماني قواعد أعظم [ف ١٩٢ ب] من ضلع المكعب وأن ضلع المكعب أعظم من ضلع ذي العشرين قاعدة وأن ضلع ذي العشرين قاعدة أعظم من ضلع ذي الاثنتي عشرة قاعدة.

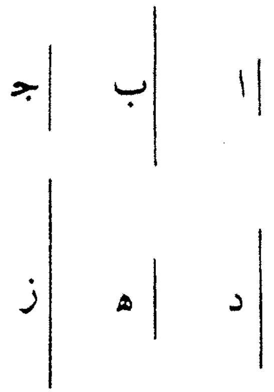
[97] ز ح: ك ط (د، ط) || [98] ذي الاثنتي عشرة قاعدة: (مطموسة في ط) || [99] أردناه (ط).



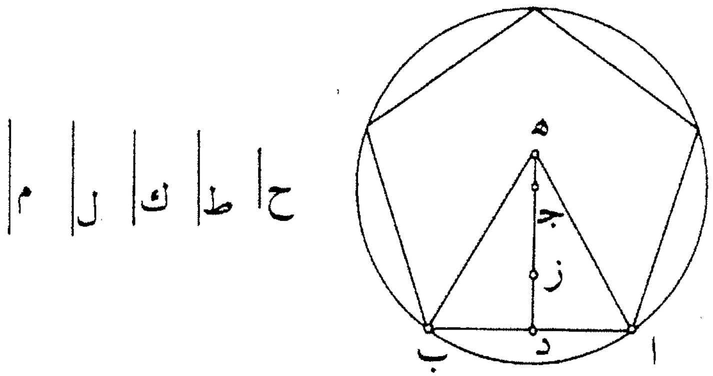
شكل ٣١

لب. نسبة عمود الناري إلى عمود المكعب كنسبة ثلث عمود المثلث المتساوي الأضلاع إلى نصف ضلعه. فليكن[100] عمود الناري ا ونصف قطر الكرة المحيطة به ب وعمود المكعب جـ. وليكن د عمود مثلث متساوي[101] الأضلاع ونصف ضلعه هـ و ز ثلاثة أمثال هـ. فـ ا سدس قطر الكرة (ا من يد) فهو ثلث ب. فنسبة ا إلى ب كنسبة هـ إلى ز. ومربع ب ثلاثة أمثال مربع جـ (جـ من يد) ومربع د ثلاثة أمثال مربع هـ (ا من يه) فنسبة مربع ب[102] إلى مربع جـ كنسبة مربع د إلى مربع هـ [هـ ١٨٣ ب] فنسبة ب إلى جـ كنسبة د إلى هـ (كب من و[103]). فبالمساواة المضطربة نسبة مربع ا عمود الناري إلى مربع جـ أعني عمود المكعب كنسبة مربع د أي عمود المثلث المتساوي الأضلاع إلى مربع ز ثلاثة أمثال نصف ضلعه. فنسبة عمود الناري إلى عمود المكعب كنسبة عمود المثلث إلى ثلاثة أمثال نصف ضلعه. فنسبة عمود الناري إلى عمود المكعب كنسبة ثلث عمود المثلث المتساوي الاضلاع إلى نصف ضلعه أعني ثلث ز وذلك ما أردنا أن نبين[104].

[100] وليكن (د، ط) || [101] المثلث المتساوي (ط) || [102] اب (ط) || [103] هـ (ط) || [104] أردناه (ط).



شكل ٣٢



شكل ٣٣

لج. نريد أن نجد الخطوط الخمسة المتوالية على نسب[١٠٥] أعمدة الأشكال الخمسة. فنرسم دائرة ضلع مخمسها ا ب ومركزها جـ. ونعمل عليه مثلثاً[١٠٦] متساوي الأضلاع وهو مثلث [د ١٧٣ ب] ا هـ ب. ونسقط عمود هـ د ونجعل د ز ثلث د هـ. وليكن ح مثل د ز و ط مثل ا د و ك مثل ط و ل مثل جـ د أي العمود المخرج [ف ١٩٣ ا] من المركز على ضلع المخمس و م مثل ل. فنسبة ح إلى ط كنسبة عمود الناري إلى عمود ذي الثماني قواعد أعني عمود المكعب

[١٠٥] نسبة (ط) || [١٠٦] مثلثاً: - (ط).

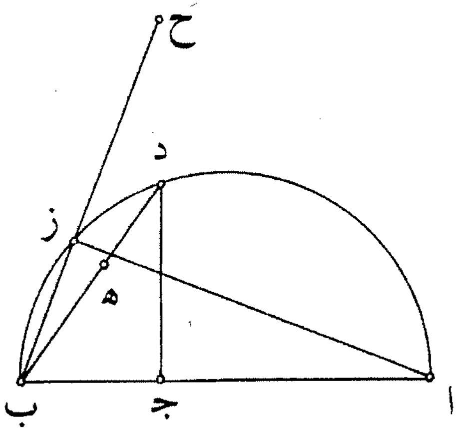
(جـ من يه). ونسبة ط إلى ك كنسبة عمود ذي الثماني قواعد[١٠٧] إلى عمود المكعب لتساويهما (جـ من يه). ونسبة ك أي نصف ضلع المخمس أعني ا د إلى ل أعني جـ د المساوي لنصف ضلع المسدس والمعشر مجموعين (ز من يه[١٠٨]) كنسبة عمود المكعب أعني عمود ذي الثماني قواعد إلى عمود ذي العشرين قاعدة (كز من يه). ونسبة ل إلى م كنسبة عمود ذي العشرين قاعدة [ط ٨٩ ا] إلى عمود ذي الاثنتي عشرة قاعدة لتساويهما (هـ من يه).

فإذن نسبة ح إلى ط كنسبة عمود الناري [هـ ١٨٤ ا] إلى عمود ذي الثماني قواعد ونسبة ط إلى ك كنسبة عمود ذي الثماني قواعد إلى عمود المكعب ونسبة ك إلى ل كنسبة عمود المكعب إلى عمود ذي العشرين قاعدة ونسبة ل إلى م كنسبة عمود ذي العشرين قاعدة إلى عمود ذي الاثنتي عشرة قاعدة. فقد وجدنا الخطوط الخمسة التي على نسب أعمدة الأجسام[١٠٩] الخمسة وذلك ما أردناه.

لد. كيف نجد خطاً مقسوماً بنسبة ذات وسط وطرفين يكون مربعه ومربع أصغر قسميه مساوياً[١١٠] لمربع خطٍ مفروضٍ. فليكن الخط المفروض ا ب وندير على ا ب نصف دائرة ا د ب ونجعل ا جـ مثلي جـ ب ونخرج عمود جـ د ونصل ب د فمربع ا ب ثلاثة أمثال مربع [د ١٧٤ ا] [ف ١٩٣ ب] ب د. فنقسم ب د بنسبة ذات وسط وطرفين على هـ وليكن قسمه الأعظم ب هـ. ونخرج وتر ب ز مثل ب هـ (ا من د) ونخرج ب ز على استقامة[١١١] ونفصل ز ح مثل ب د[١١٢]. فـ ح ب مقسوم بنسبة ذات وسط وطرفين وقسمه الأعظم ح ز (ز من يجـ). فمربعا ب ح ب ز ثلاثة أمثال مربع ز ح. فأقول إن ا ب يقوى على خطي[١١٣] ح ب ب ز.

[١٠٧] قواعد: - (د) || [١٠٨] هـ (ط) || [١٠٩] أجسام (د) || [١١٠] مساوي (د) مساو (ط) || [١١١] الاستقامة (ط) || [١١٢] ز ح مثل ب د : ب ح مثل ب هـ (د، ط) || [١١٣] خطي: (في حاشية ط فقط).

برهانه إنا[١١٤] نصل ا ز فزاوية ا ز ب قائمة (ل من جـ) فمربع ا ب أعني مربعي ا ز ز ب (مز من ا) ثلاثة أمثال مربع ح ز فمربعا ا ز[١١٥] ب ز مثل مربعي ح ب[١١٦] ز ب فنسقط مربع ب ز المشترك فيبقى مربع ح ب[١١٧] كمربع ا ز فـ ح ب[١١٨] مثل ا ز[١١٤] فـ ا ب[١١٩] يقوى على خطي ح ب ب ز وذلك ما أردنا بيانه[١٢٠].

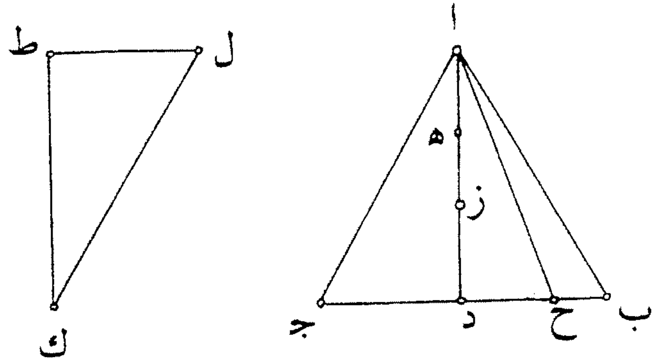


شكل ٣٤

له. نريد أن نجد الخطوط الخمسة المتوالية على نسب سطوح الأشكال الخمسة. فنعمل مثلث ا ب جـ متساوي [هـ ١٨٤ ب] الأضلاع ونخرج عمود ا د ونجعل ا هـ ثلثه. ونقسم ا د[١٢١] بنسبة ذات وسط وطرفين على ز وليكن ز د قسمه الأصغر والأطول ا ز ونفصل د ح مثل د ز ونصل ا ح فهو يقوى على ا د وعلى قسمه الأصغر أعني د ح أي د ز. ونفرض ط ك مثل ا د. ونجد الخط القوي على ا د ا ز بأن نخرج عمود ط ل ونفصله مثل ا ز ونصل ك ل فهو

[١١٤] إنا. . . مثل ا ز: إن مربع ا ب ثلاثة أمثال مربع د ب أي ح ز ومربعا ح ب ب ز ثلاثة أمثال مربع ح ز فمربعا ح ب ب ز كمربع ا ب (هـ) || [١١٥] ا ب (د، ط) || [١١٦] ح ر (د، ط) || [١١٧] ح ر (د، ط) || [١١٨] فـ ح ر (د، ط) || [١١٩] فـ ا ر (د، ط) || [١٢٠] أردنا (ط) || [١٢١] ا د : – (د).

يقوى على ط ك ط ل أعني ا د ا ز. فنسبة ك ل إلى ا ح كنسبة سطح ذي الاثنتي عشرة قاعدة إلى سطح ذي العشرين قاعدة (يو من يه) ونسبة ا د د ح كخط أي الخط[١٢٢] كله وقسمه الأصغر إلى ا د أي الخط كله كنسبة [ف ١٩٤ ا] [ط ٨٩ ب] سطح[١٢٣] ذي العشرين قاعدة إلى سطح ذي الثماني قواعد (يه من يه). ونسبة ا د إلى جـ ب كنسبة سطح ذي الثماني قواعد إلى سطح المكعب [د ١٧٤ ب] (يب من يه). ونسبة د هـ إلى د ا كنسبة سطح الناري إلى سطح ذي الثماني قواعد (يز من يه) . ونسبة ا د ‹إلى[١٢٤] جـ ب كنسبة سطح ذي الثماني قواعد[١٢٥] إلى سطح المكعب (يب من يه) فتكون بالمساواة نسبة د هـ›[١٢٤] إلى جـ ب كنسبة سطح الناري إلى سطح المكعب فبالعكس نسبة جـ ب إلى د هـ كنسبة سطح المكعب إلى سطح الناري فقد وجدنا الخطوط المتوالية على نسب السطوح الخمسة وذلك ما أردنا بيانه[١٢٦].



شكل ٣٥

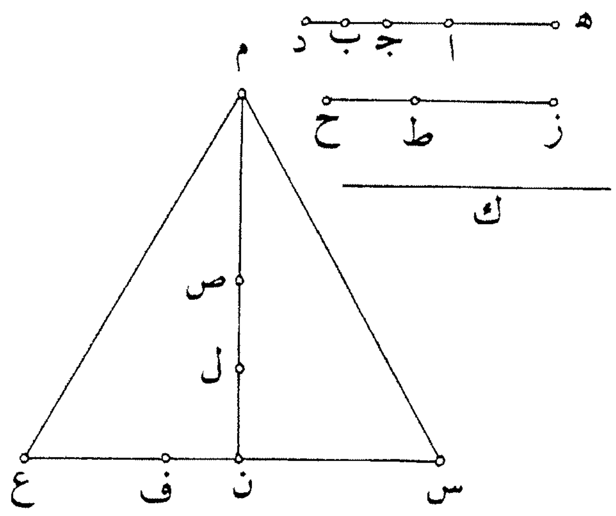
لو. نريد أن نجد الخطوط المتوالية على نسب قواعد الأجسام الخمسة. فنضع خط ا ب ونقسمه بنسبة ذات وسط وطرفين على جـ [هـ ١٨٥ ا] وقسمه الأصغر جـ ب ونخرج ا ب ونجعل ب د[١٢٧] مثل جـ ب. ونخرج د ا ونفصل ا هـ

[١٢٢] كخط أي الخط: كالخط (ط) || [١٢٣] سطح: – (ط) || [١٢٤] إلى ... د هـ (هـ): – (د ، ط) || [١٢٥] + إلى سطح ذي الثماني قواعد (هـ) || [١٢٦] أردناه (ط) || [١٢٧] ي د (د، ط).

مثل ثلثي ا د فـ ا د ثلاثة أخماس هـ د. ونجد خطاً منقسماً بنسبة ذات وسط وطرفين يقوى عليه خط د هـ وعلى قسمه الأصغر (لد من يه[128]) وليكن خط ز ح وقسمه الأصغر ط ح. ونجد خطاً يقوى على خط ح ز وعلى قسمه الأعظم ز ط وهو ك. ونضع خط ل م مثلين ونصفاً لخط ا ب ونخرج م ل ونفصل ل ن مثل ثلث ل م[129] فخط ل م ثلاثة أرباع م ن. وننصف م ن على ص ونخرج من نقطة ن عمود ن س ونفرضه في الجهتين بغير نهاية. ونعمل كل واحدة من زاويتي[130] م ثلث قائمة ونخرج الخطين[131] حتى يلتقيان العمود المخرج من نقطة ن على نقطتي س ع فمثلث م س ع متساوي الأضلاع. فنفصل ع ف ثلث س ع. فلأن نسبة سطح ذي الاثنتي عشرة قاعدة إلى سطح ذي العشرين قاعدة كنسبة خط ك إلى هـ د (يو من يه) فيكون ‹نسبة›[132] قاعدة ذي الاثنتي عشرة قاعدة أي نصف سدس سطح ذي الاثنتي عشرة قاعدة إلى قاعدة ذي العشرين قاعدة أي[133] ‹نصف عشر سطحه كنسبة نصف سدس ك إلى نصف عشر هـ د التي هي كنسبة خط ك إلى ثلاثة أمثال خط هـ د (يد من هـ) أعني كنسبة ك إلى ا د. ونسبة سطح ذي العشرين قاعدة›[134] إلى[135] سطح ذي الثماني قواعد [د ١٧٥ ا] كنسبة ا د إلى ا ب (يه من يه). فنسبة قاعدة سطح ذي العشرين [ف ١٩٤ ب] قاعدة أعني نصف عشر سطحه إلى قاعدة ذي الثماني قواعد أعني ثمن سطحه كنسبة نصف عشر ا د إلى ثمن ا ب أعني كنسبة ا د إلى مثلي ا ب ونصفه وهو م ل (يد من هـ) [هـ ١٨٥ ب]. ونسبة قاعدة ذي الثماني قواعد إلى قاعدة الناري كنسبة م ل إلى م ن (بقوة يز من يه). ونسبة قاعدة المكعب إلى قاعدة ذي الثماني قواعد كنسبة ثلثي س ع أي س ف إلى نصف م ن أي م ص (ح من يه). ونسبة قاعدة ذي الثماني قواعد إلى قاعدة الناري كنسبة نصف م ن إلى ثلثيه (ب من يه). فبالمساواة نسبة قاعدة

[128] من يه: - (ط) || [129] مثل ثلث ل م: مثل وثلث ل م (د)، مثلا وثلثا ل ل م (ط) || [130] من زاويتي: (حاشية في ط فقط) || [131] خطين (د) || [132] نسبة (هـ): - (د، ط) || [133] أي: - (د) || [134] نصف ... قاعدة (هـ): - (د، ط) || [135] إلى: - (ط).

المكعب إلى قاعدة الناري كنسبة ثلثي س ع إلى ‹ثلثي› م ن. فبالعكس نسبة قاعدة الناري إلى قاعدة المكعب كنسبة ثلثي م ن إلى ثلثي س ع[136] أعني كنسبة م ن إلى س ع. فقد وجدنا الخطوط المتوالية على نسب قواعد الأشكال الخمسة وذلك ما أردناه.

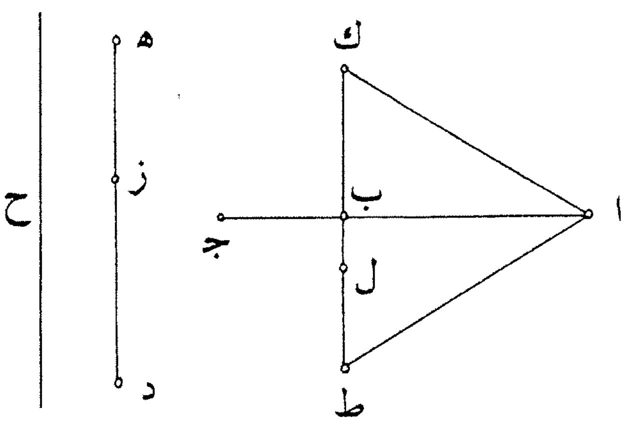


شكل ٣٦

لز. نريد أن نجد الخطوط المتوالية على نسب الأجسام الخمسة. فنفرض خط ا ب ضلع مخمس دائرة ونخرجه على [ط ١٩٠ ا] استقامته ونفصل منه ب ج مثل ضلع معشرها. ونجد خطاً منقسماً بنسبة ذات وسط وطرفين يقوى عليه وعلى قسمه الأصغر ا ج وليكن د هـ وليكن قسمه الأعظم د ز. ونجد الخط القوي على د هـ د ز وهو ح. ونخرج من نقطة ب عموداً على ا ب ونخرجه[137] في الجهتين بغير نهاية. ونعمل كل واحدة من زاويتي ا ثلث قائمة ونخرج الخطين حتى يلتقيا العمود على نقطتي ط ك فمثلث ا ط ك متساوي الأضلاع ونجعل ط ل ثلث ط ك. فنسبة ح أي القوي على د هـ [د ١٧٥ ب] وعلى

[136] م ن (د، ط) || [137] ونخرج (د).

قسمه الأعظم إلى[138] القوي [ف ١٩٥ ا] عليه وعلى [هـ ١٨٦ ا] قسمه الأصغر أي ا جـ كنسبة مجسم ذي الاثنتي عشرة قاعدة إلى مجسم ذي العشرين قاعدة (ل من يه). ونسبة ا جـ إلى ا ب كنسبة مجسم ذي العشرين قاعدة إلى مجسم ذي الثماني قواعد (كط من يه) ونسبة ط ك إلى ا ب كنسبة مجسم المكعب إلى مجسم ذي الثماني قواعد (كجـ من يه) ونسبة ا ب إلى ط ل كنسبة مجسم ذي الثماني قواعد إلى مجسم الناري (ك من يه) فبالمساواة نسبة ط ك إلى ط ل كنسبة مجسم المكعب إلى مجسم الناري. فخطوط ح ا جـ ا ب ط ك ط ل متوالية على نسب المجسمات الخمس[139] المخطوطة في كرة واحدة وذلك ما أردنا أن نبين[140].



شكل ٣٧

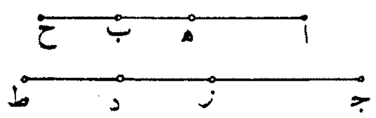
تمت المقالة الخامسة عشرة وهي آخر الكتاب. عدد أشكال الكتاب سوى المقدمات واختلاف الأوضاع ٥١٦[141].

[138] أي (د، ط) || [139] الخمسة (د) || [140] أردناه (ط) || [141] + وفرغت من نسخته ليلى الخميس وقت العشاء الأخير ببلدة مراغة عمرها الله تعالى بتأريخ شهور سنة تسع وخمسين وستمائة وأنا الراجي إلى رحمة *** الحسن بن احمد بن علي *** والحمد لله رب العالمين وصلواته على نبيه محمد وآله أجمعين (د)؛ + تحرير هذا المختصر على يد أفقر خلق الله تعالى إلى رحمته وغفرانه العبد إبراهيم بن يعقوب بن ميمون *** يوم السبت تاسع ذي الحجة سنة خفه هجرية

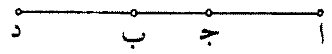
ي[142]. كل خطين مختلفين يقسم كل واحد منهما على نسبة ذات وسط وطرفين فإن نسبة أحد الخطين إلى الثاني كنسبة قسمه الأعظم إلى قسم الثاني الأعظم[143] وكنسبة القسم الأصغر إلى القسم الأصغر. مثاله خطا ا ب جـ د مختلفان وقد قسم ا ب بنسبة ذات وسط وطرفين على هـ [ف ١٥٧ ا] فكان قسمه الأعظم ا هـ وقسم جـ د بنسبة ذات وسط وطرفين على ز فكان قسمه الأعظم جـ ز. فأقول إن نسبة ا ب إلى جـ د كنسبة ا هـ إلى جـ ز وكنسبة هـ ب إلى ز د.

برهانه إنا[144] نخرج كل واحد من خطي ا ب جـ د على استقامة[145] ونجعل ب ح مثل هـ ب ود ط مثل ز د. فمربع ا هـ كضرب ا ب في ب هـ ومربع جـ ز مثل ضرب جـ د في د ز (يز من و). فنسبة مربع ا هـ إلى ضرب ا ب في هـ ب كنسبة مربع جـ ز إلى ضرب جـ د في ز د فبالتبديل نسبة مربع ا هـ إلى مربع جـ ز كنسبة ضرب ا ب في ب هـ إلى ضرب جـ د في د ز. فنسبة مربع ا هـ وأربعة أمثال ضرب ا ب في ب هـ أعني خمسة أمثال مربع ا هـ أعني[146] مربع ا ح (ح من ب) إلى مربع جـ ز وأربعة أمثال ضرب جـ د في د ز أعني خمسة أمثال مربع[146] جـ ز أعني مربع جـ ط (ح من ب) كنسبة مربع ا هـ إلى مربع جـ ز (يد من هـ). فنسبة ا ح إلى جـ ط [د ١٤١ ب] كنسبة ا هـ إلى جـ ز (كب من و). فنسبة هـ ح إلى ز ط كنسبة ا ح إلى جـ ط (يط من هـ) أعني كنسبة ا هـ إلى جـ ز. ونسبة هـ ح إلى ز ط كنسبة هـ ب إلى ز د فنسبة ا هـ إلى جـ ز كنسبة هـ ب إلى ز د. فنسبة ا ب إلى جـ د كنسبة ا هـ إلى جـ ز وكنسبة هـ ب إلى ز د (يح من هـ)[147] وذلك ما أردناه.

وعلقه من نسخة *** وأما من جهة خط الأرقام فلا تُصف. والحمد لله رب العالمين وصلى الله على محمد وعلى آله الطيبين الطاهرين ونسأل ربنا تعالى مبصر عين معترفين بالتقاصير والذنوب الخمسة العظيمة أن لا يؤاخذنا بسوء أعمالنا وأن يغفر لنا ذنوبنا *** على نفسنا إنه تعالى ولي الإجابة (ط) || [142] هذا الشكل هو آخر المقالة الرابعة عشر من الأصل لأنه محتاج إليه *** حاشية (حاشية في ط) || [143] إلى قسم الثاني الأعظم: - (ط) إلى القسم ال<أعظم> من الآخر (حاشية في ط) || [144] ان (د) || [145] الاستقامة (ط) || [146] أعني . . . مربع (حاشية في د) || [147] يجـ من هـ (د، ط).



شكل ١٠ من مقالة ١٣



شكل ١٤ من مقالة ١٣

يد[١٤٨]. إذا قسم ضلع المسدس بنسبة ذات وسط وطرفين فإن قسمه الأعظم ضلع المعشر. فليكن ا ب ضلع المسدس ونقسمه بنسبة ذات وسط وطرفين على جـ وليكن ا جـ أعظم قسميه فأقول إنه ضلع المعشر. برهانه إنا نخرج ا ب إلى د ونجعل ب د مثل ا جـ فـ ا د مقسوم بنسبة ذات وسط وطرفين وقسمه الأعظم ا ب (ز من يجـ) و ا ب ضلع المسدس و ب د أعني ا جـ ضلع المعشر (يجـ من يجـ).

[١٤٨] يد: يه (د).

Introduction to the translation.

In the translation all explanatory additions made by me are in parentheses. Angular brackets include translations of words and passages that I have added to the Arabic text in D and T in order to restore the original of al-Maghribī. In the English translation I have transcribed al-Maghribī's references to previous propositions as in the following example: VI:22 for proposition 22 of Book VI of the *Revision*. In the translation the number of the corresponding proposition in the *Elements* appears in brackets, if the number differs from the *Revision*. Thus V:14 (=El. V:15) means that proposition 14 of Book V of the *Revision* is the same as proposition 15 of Book V of the *Elements* in the editions of Heiberg and Heiberg-Stamatis and the translation of Heath. Some of the references in the text to earlier propositions in the *Revision* are incorrect. If such incorrect references occur systematically or cannot be explained as scribal errors, I have assumed that they were in the original of al-Maghribī. The correct references are to be found in a note in the translation, using the abbreviations M for al-Maghribī's *Revision* and El. for Euclid's *Elements*.

The figures are printed twice, in the Arabic text and in the English translation. Some of the three-dimensional figures may look odd to modern eyes, because they have been drawn in the way in which they occur in the manuscripts.

We can suppose that most medieval students of Book XV of the *Revision* had read the first fourteen Books. To understand the translation in all historical details one needs to have a detailed knowedge of Books I-VI, XI and XIII of the *Elements*. The modern reader who is not familiar with the *Elements* can follow practically everything if he has vol. 3 of Heath's translation of the *Elements* at hand. Then he can look up the Euclidean constructions of the regular polyhedra (pp. 467-503) and the other theorems in El. XIII if necessary.

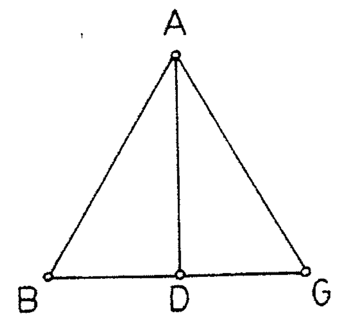
As long as the Hebrew text has not been published, it is not possible to give an adequate historical commentary on individual propositions in al-Maghribī's text. Thus my footnotes to the translation are only concerned with terminology, mathematics and references to other propositions in the *Revision* and the *Elements* (including Hypsicles' Book XIV).

Translation of Book XV of the Revision of the Elements.

The fifteenth book, on the relation between the five solids.

1. The square of the altitude of every equilateral triangle is three-fourths of the square of its side. Thus let triangle ABG be equilateral, and (let) its altitude (be) AD. I say that the square of AD is three-fourths of the square of AB.

Proof: Angles ABG and AGB are equal, the two angles D are right (angles), and line AD is common to triangles ADB, ADG. Thus lines BD and DG are equal. Hence the square of BD is one-fourth of the square of BG, that is AB, II:4. The (sum of the) squares of AD, DB is equal to the square of AB, I:47. But the square of DB is one-fourth of the square of AB. Hence, by subtraction, the square of AD is three-fourths of the square of AB. That is what we wanted to prove.



prop. 1

2. The side of the octahedron is equal to the altitude of the triangle of the pyramid inscribed in the same sphere.

Proof: The square of the diameter of the sphere is one and a half times the square of the side of the pyramid, XIV:1 (=El. XIII:13). The square of the diameter of the sphere is also twice the square of the side of the octahedron, XIV:5 (=El. XIII:14). Thus one and a half times the square of the side of the pyramid is equal to twice the square of the side of the octahedron, so three-fourths of the square of the side of the pyramid is equal to the square of the side of the octahedron. But three-fourths of the square of the side of the pyramid

is equal to the square of the altitude of its triangle[1]. Thus the square of the side of the octahedron is equal to the square of the altitude of the triangle of the pyramid. Thus the side of the octahedron is equal to the altitude of the triangle of the pyramid. That is what we wanted to prove.

It is clear from this that the triangle of the octahedron is three-fourths of the triangle of the pyramid, since the ratio of the triangle of the octahedron to the triangle of the pyramid is equal to the ratio of the squares of the sides, and the square of the side of the octahedron is three-fourths of the square of the side of the pyramid.

3. The square of the side of the cube and the triangle of the side of the octahedron constructed in the same sphere are circumscribed by the same circle. Thus let the square of the side of the cube be AB, (let) the radius of its circumscribed circle (be) GB, (let) the triangle of the octahedron (be) DEZ and (let) the radius of its circumscribed circle (be) HD. Let the diameter of the sphere be line T. Then the square of T is three times the square AB, XIV:3 (=El. XIII:15), but the square AB is twice the square of GB, I:47, so the square of T is six times the square of GB. The square of T is twice the square of DE, XIV:5 (=El.XIII:14), and the square of DE is three times the square of HD, XIII:11[1], so the square of T is six times the square of HD. Thus the square of GB is equal to the square of HD, so GB is equal to HD.

Thus the circle which circumscribes the square of the side of the cube is equal to the circle which circumscribes the triangle of the octahedron. That is what we wanted to prove.

It has become clear from this that the perpendicular falling from the centre of the sphere on the square of the side of the cube is equal to the perpendicular falling from its centre on the triangle of the octahedron, because the perpendiculars falling from the centre of the sphere on equal circles drawn on the surface of the sphere are equal[2].

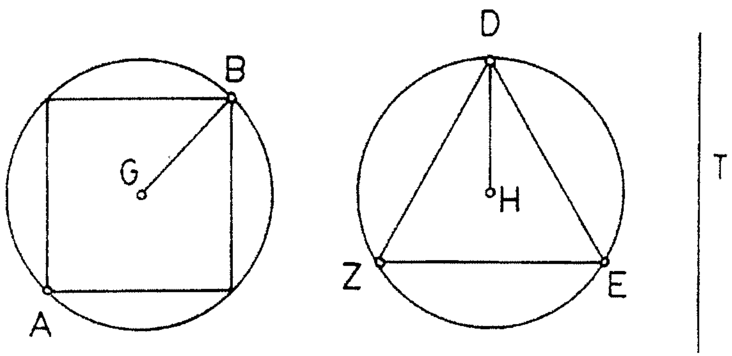
It has also become clear that the square of the side of the octahedron is one and a half times the square of the side of the cube, since the square of the side of the octahedron is three times the square of the

2.1. Proved in the preceding theorem (M XV:1).

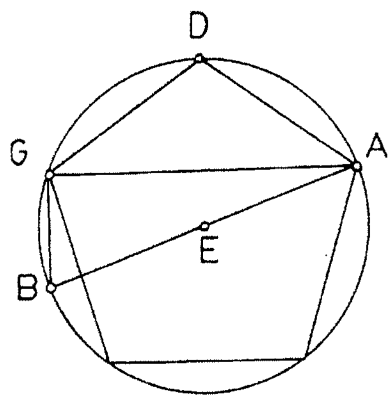
3.1. The reference should be to M XIII:12 (=El. XIII:12). Compare notes 5.8, 31.1.

3.2. This is proved in a lemma in the end of M XIII, which is a special case of Theodosius, *Spherics*, I:6.

radius of the circumscribing circle, that is (the circle) which circumscribes the square of the side of the cube, and (since) the square of the side of the cube is twice the square of the radius of this circle. Thus the square of the side of the octahedron is one and a half times the square of the side of the cube.



prop. 3



prop. 4

4. In every circle, the sum of the squares of the side of its pentagon and the chord of the angle of the pentagon is five times the square of the radius of the circle circumscribing the pentagon[1].

Thus let there be a circle with diameter AB, (let) the side of its pentagon (be) GD, (let) the chord of the angle of the pentagon (be)

4.1. This is El. XIV § 2 (the lemma in prop. 2 in Heath's translation, vol. 3, pp. 513-514).

AG, and (let) the centre of the circle (be) E. I say that the (sum of the) squares of AG, GD is five times the square of EB. Proof: We join GB. Then GB is the chord of the decagon, thus[2] the (sum of the) squares of AG, GB is equal to the square of AB, I:47, that is to say four times the square of EB. We add the square of EB, then the (sum of the) three squares of AG, GB and EB is five times the square of EB. But the (sum of the) squares of GB, EB is equal to the square of GD, XIII:13[3], so the (sum of the) squares of AG, GD is five times the square of EB. That is what we wanted to prove.

5. The triangle of the icosahedron and the pentagon of the dodecahedron circumscribed by the same sphere fall in the same circle[1].

Example: We make the pentagon of the dodecahedron pentagon ABGDE, the centre of its circumscribed circle Z, its radius ZD, the triangle of the icosahedron triangle HTK, the centre of the circle which circumscribes it L and its radius LK. I say that LK is equal to ZD.

Proof: We join BE; it is the side of the cube[2].

We make the diameter of the sphere MN, and the radius of the circle of which it has been shown[3] that the side of its pentagon is equal to the side of the triangle of the icosahedron CO. We divide MN in extreme and mean ratio at Q, let the greater part be QN. We also divide CO in extreme and mean ratio at S, let the greater part be SO. Then SO is the side of the decagon, XIV:14[4]. We also divide BE in extreme and mean ratio at point F, such that the greater part is FE; then it is equal to AE, XIII:14[5].

4.2. All manuscripts have "thus" (*fa-*), although $AG^2+GB^2=AB^2$ is not a consequence of the fact that BG is the chord of the decagon.

4.3. The reference should have been to M XIII:15 (=El..XIII:10). Compare notes 6.5, 13.3.

5.1. The same theorem is proved in essentially the same way in El. XIV § 3 (=prop. 2 in Heath's translation, vol. 3, pp. 514-515.).

5.2. As shown in the Euclidean construction of the dodecahedron (M XIV:9 =El. XIII:17).

5.3. This circle through five points of the icosahedron is used in El. XIII:16 (=M XIV:7).

5.4. The reference should be to M XIII:14 (see the appendix for the text and translation), compare notes 6.4, 13.2. The decagon is inscribed in a circle with radius CO.

5.5. The reference should be to M XIII:16 (=El. XIII:8). Compare note 6.8.

Then the ratio of MN to CO is equal to the ratio of QN to SO, XIV:10[6]. Thus the ratio of the square of MN to the square of CO is equal to the ratio of the square of QN to the square of SO, VI:22. But it has been shown that the square of MN is five times the square of OC, XV:4[7]. Thus the square of QN is five times the square of SO. Thus the (sum of the) squares of MN, NQ is five times the sum of the squares of CO, OS, that is to say the <square of the> side of the pentagon of the circle of which the radius is CO, that is the square of HT, XIII:15 (=El. XIII:10), that is to say three times the square of LK, XIII:11[8]. Thus the (sum of the) squares of MN, NQ is fifteen times the square of LK.

Again, the ratio of the square of MN to the square of BE, that is the side of the cube, is equal to the ratio of the square of QN to the square of FE[9]. But the square of MN is three times the square of BE, XIV:3 (=El. XIII:9). Thus the square of QN is three times the square of FE. Thus the (sum of the) squares of MN, NQ is three times the (sum of the) squares of BE, EF. But the (sum of the) squares of BE, EF is five times the square of DZ, XV:4. Thus the (sum of the) squares of MN, NQ is fifteen times the square of DZ. Thus the squares of LK and DZ are equal, so lines LK and ZD are equal. Thus the circles ABGDE and HTK are equal. That is what we wanted to prove.

It has become clear from this that the perpendicular falling from the centre of the sphere on the face of the dodecahedron is equal to the perpendicular falling from the centre of the sphere on the face of the icosahedron, because the perpendiculars falling from the centre of the sphere on equal circles drawn on its surface are equal[10]. That is what we wanted.

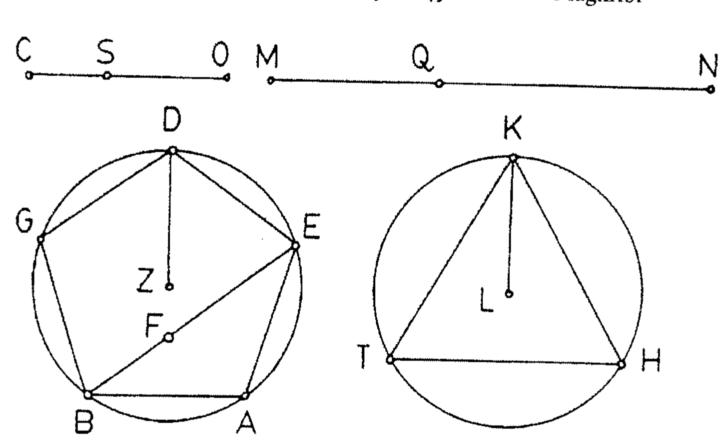
5.6. The reference should be to M XIII:10 (see the appendix for the text and translation).

5.7. This is a very interesting reference. We locate the diameter MN of the sphere in such a way that M and N are opposite points on the icosahedron. Let MP be one of the sides of the icosahedron that ends at M, and draw PN. Because P is on a circle with diameter MN, angle MPN is a right angle, so $MN^2=PN^2+PM^2$. Segment PN is a diagonal in a regular pentagon formed by sides of the icosahedron, and the radius of the circumscribing circle is equal to CO. Thus by M XV:4, $PN^2+PM^2=5CO^2$. Euclid proves $MN^2=5CO^2$ in a different way in El. XIII:16, cor.

5.8. The reference should be to M XIII:12 (=El. XIII:12).

5.9. By M XIII:10 (see the appendix).

5.10. Proved in a lemma at the end of M XIII, cf. note 3.2.



prop. 5

6. (For) every line divided in extreme and mean ratio, the ratio of the line, the square of which is equal to the square of the whole line plus the square of its greater part to the line, the square of which is equal to the square of the whole line plus the square of its lesser part is equal to the ratio of the side of the cube to the side of the icosahedron inscribed in the same sphere[1].

Example: Let AB be the radius of the circle circumscribing the pentagon of the dodecahedron and the triangle of the icosahedron[2], and let AG be the side of its triangle, and AD the side of its pentagon, and DE the chord of two-fifths of it. Then it is the side of the cube that is circumscribed by the sphere which circumscribes the dodecahedron and the icosahedron[3]. We divide AB in extreme and mean ratio at Z, let its greater part be BZ. Then BZ is the side of the decagon, XIV:14[4]. Thus the square of AD is equal to the square of AB plus the square of BZ, XIII:13[5]. Let H be the line, the square of which is equal to the square of AB plus the square of AZ.

6.1. This theorem is proved by Hypsicles in El. XIV, § 9 (=prop. 7 in Heath's translation, vol. 3, p. 518).The technical term "*qawi*", literally "equal in power to", is analogous to the Greek term "dunamenè", cf. Heath, *Euclid's Elements*, vol. 3, p. 522.

6.2. Here XV:5 is assumed.

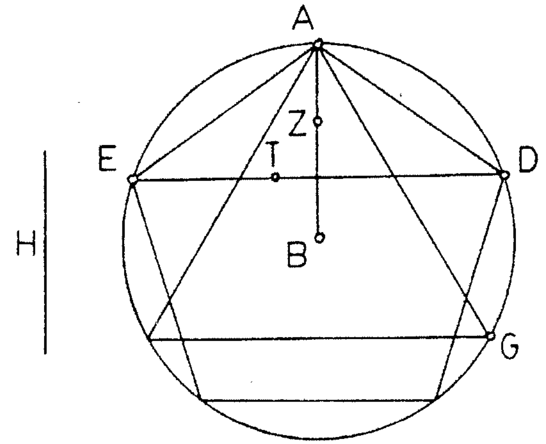
6.3. Shown in the construction of the dodecahedron in M XIV:9 (=El. XIII:17)

6.4. The reference should be to M XIII:14 (see the appendix).

6.5. The reference should be to M XIII:15 (=El. XIII:10). The expression AD *yaqwā 'alā* AB BZ, "AD is equal in power to AB, BZ" means $AD^2=AB^2+BZ^2$, compare note 6.1.

I say that the ratio of AD to H is equal to the ratio of the side of the cube to the side of the icosahedron.

Proof: The square of H, which is equal to the square of AB plus the square of AZ, is also equal to three times the square of BZ, XIII:8 (=El. XIII:4). But the square of AG is equal to three times the square of AB. Thus the ratio of the square of AG to the square of AB is equal to the ratio of the square of H to the square of BZ, XIII:14[6]. Thus the ratio of AG to AB is equal to the ratio of H to BZ, VI:22. Thus, alternately[7], the ratio of AG to H is equal to the ratio of AB to BZ. We divide DE in extreme and mean ratio at T, let the greater part be DT. Then it is equal to AD, XIII:14[8]. Thus the ratio of ED to DT, that is to say: to AD, is equal to the ratio of AB to BZ[9], that is to say: equal to the ratio of AG to H. Thus, alternately, the ratio of DE, the side of the cube, to AG, the side of the icosahedron, is equal to the ratio of AD, the square of which is equal to the square of the whole line plus the square of its greater part, to H, the square of which is equal to the square of the whole line plus the square of its lesser part. That is what we wanted to prove.



prop. 6

6.6. The reference should be to M V:14 (=El. V:15).

6.7. "Alternately" refers to the transformation of a:b=c:d into a:c=b:d (based on El. V:16).

6.8. The reference should be to M XIII:16 (=El. XIII:8).

6.9. By M XIII:10 (see the appendix).

7. In every circle, the perpendicular drawn from its centre to the side of its pentagon is equal to the sum of half of the side of its decagon and half of the side of its hexagon[1].

Thus let there be a circle with centre A, let the side of its pentagon be BG, let the side of its decagon be BD, and let the perpendicular falling from its centre on BG, the side of the pentagon, be line AE. I say that it is equal to the sum of half of AD and half of BD.

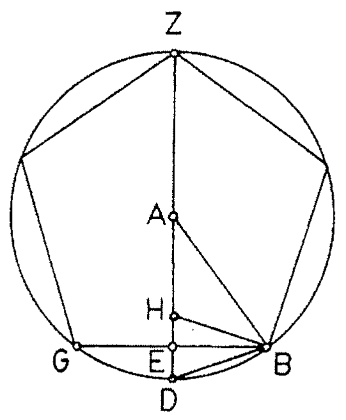
Proof: We extend perpendicular AE on both sides towards D and Z. We join AB. We cut off EH equal to ED. We join HB.

Then, since arc ZB is four times arc BD, angle ZAB is four times angle BAD. But it is twice angle BDA, I:5 and I:32. Thus angle BDA is twice angle BAD. Since angles BED, BEH are right angles, and lines EH, ED are equal, and line BE is common, so angle BDE is equal to angle BHE, I:4. Thus angle BHE is twice angle BAD. But it (angle BHE) is equal to (the sum of) angles ABH, HAB, I:32. Thus (the sum of) angles ABH, HAB is twice angle HAB, so angles ABH and HAB are equal. Thus lines AH and HB are equal, I:6. Thus line AH is equal to line BD.

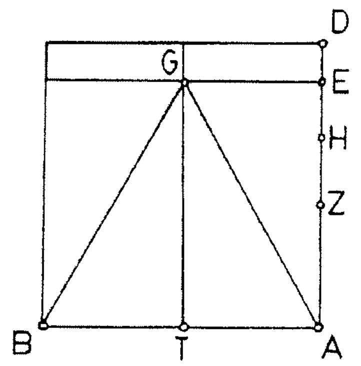
But HE is equal to ED. Thus AE is equal to ED and DB (together). We add AE, then twice AE is equal to AD and DB (together). Thus the perpendicular AE is equal to half of AD and DB (together). Thus half of it (i.e. half of 2AE), that is AE, is equal to half of AD and half of DB together[2], that is to say, half of the sides of the decagon and the hexagon. That is what we wanted to prove.

7.1. The proof is in the same elaborate style as that by Hypsicles in El. XIV § 1 (prop. 1 in Heath's translation, pp. 512-513).

7.2. The text is repetitive.



prop. 7



prop. 8

8. The ratio of the triangle of the octahedron to the square of the side of the cube is equal to the ratio of half the altitude of the triangle to two-thirds of its side.

Proof: We make the triangle of the octahedron ABG. We construct on side AB square DB, and we draw from G a line parallel to AB, which meets DA at E. We bisect AE at Z. Then AZ is half of the altitude of triangle AGB. We make AH two-thirds of AB.

Then the ratio of AZ to AD is equal to the ratio of half of rectangle EB, that is triangle AGB, I:41, to the square DB, is equal to the ratio of AZ to AD[1], VI:1. But the ratio of AD to AH is equal to the ratio of the square DB to the square of the side of the cube, because the square of the side of the octahedron is one and a half times the square of the side of the cube, XV:3, and (because) AD is one and a half times AH. Thus, ex aequali[2], the ratio of AZ, that is half the altitude of triangle ABG, to AH, two-thirds of its side, is equal to the ratio of triangle ABG, that is the triangle of the octahedron, to the square of the side of the cube. That is what we wanted to prove[3].

8.1. The last passage "is equal to the ratio of AZ to AD" occurs in all manuscripts. Al-Maghribi may have forgotten that he had already mentioned "the ratio of AZ to AD" before.

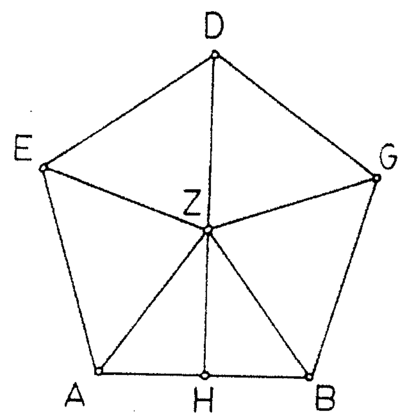
8.2. "Ex aequali" indicates an inference of the following kind: if a:b=d:e and b:c=e:f then a:c=d:f (based on El. V:22).

8.3. In all manuscripts the figure to prop. 8 includes a point T and a line GT, which are not mentioned in the text.

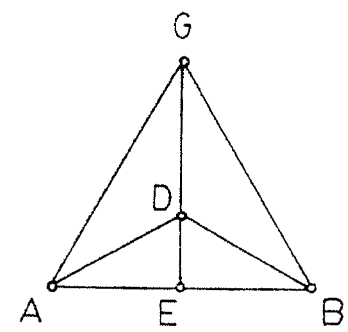
9. The product of the perpendicular drawn from the centre of the pentagon of the dodecahedron and the side of the dodecahedron is one-thirtieth of the surface of the dodecahedron[1].

Proof of this: Let the pentagon of the dodecahedron be pentagon ABGDE with centre Z, and let the perpendicular drawn from Z to the side of the pentagon be ZH. I say that the product of ZH and AB is one-thirtieth of the surface of the dodecahedron.

Proof: We join the angles of the pentagon to point Z by straight lines. Then the pentagon is divided into equal triangles. In this way each of the faces of the dodecahedron is divided into five equal triangles that are equal to triangle AZB. Thus the whole surface is divided into sixty triangles. But the product of ZH and AB is twice triangle AZB, I:41. Thus it (the product) is one-thirtieth of the surface of the dodecahedron. That is what we wanted to prove.



prop. 9



prop. 10

10. The product of the perpendicular drawn from the centre of the triangle of the icosahedron and the side of the triangle is one-thirtieth of the surface of the icosahedron[1].

Example: Let the triangle of the icosahedron be triangle ABG with centre D, and let the perpendicular drawn from the centre D to AB be

9.1. The same is proved by Hypsicles in El. XIV § 4 (=prop. 3 in Heath's translation, vol. 3, p. 515).

10.1. The same is proved by Hypsicles in El. XIV § 5, beginning (=prop. 4 in Heath's translation vol. 3, p. 515).

DE. We join D to the angles of the triangle. Then the triangle is divided into three equal triangles. In this way each of the triangles of the icosahedron is divided into three equal triangles. Thus the surface of the icosahedron is divided into sixty equal triangles. Each of them is equal to triangle ADB, and the product of DE and AB is twice triangle ABD, I:41. Thus it (the product) is one-thirtieth of the surface of the icosahedron. That is what we wanted to prove.

It has become clear from these two propositions that the ratio of the surface of the dodecahedron to the surface of the icosahedron is equal to the ratio of the product of the perpendicular drawn from the centre of the pentagon of the dodecahedron to the side of the pentagon times the side of the pentagon to the product of the perpendicular drawn from the centre of the triangle of the icosahedron to the side of the triangle times the side of the triangle[2], because the ratio of parts is equal to the ratio of equimultiples of them, V:14 (=El. V:15). That is what we wanted to prove.

11. The area of every pentagon circumscribed by a circle is equal to the product of three-fourths of the diameter of the circle and five-sixths of the chord of the angle of the pentagon.

Thus let pentagon ABGDE be (inscribed) in a circle with diameter AZH and centre Z, and let BE be the chord of the angle of the pentagon, and let five-sixths of it be BT. We bisect ZH at K. Then AK is three-fourths of the diameter. I say that the product of AK and BT is equal to pentagon ABGDE[1].

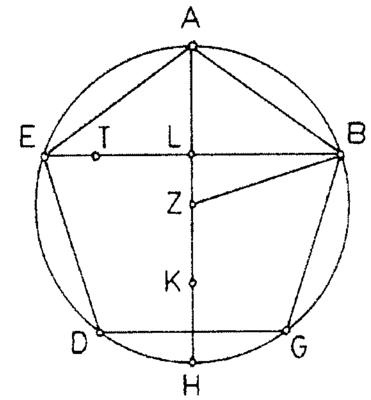
Proof: We join ZB. Then triangle AZB is one-fifth of pentagon ABGDE, as we proved more than once[2]. But BL, that is to say, half of BE, is three times TE, and AK is three times half of AZ. Thus the ratio of BL to TE is equal to the ratio of AK to half of AZ. Thus the product of BL and half of AZ , that is triangle AZB, I:41, is equal to

10.2. The same corollary is stated by Hypsicles in El. XIV § 5, end (prop. 5 in Heath's translation). Hypsicles uses this corollary in El. XIV § 6 (this is the first proof of prop. 6 in Heath's translation, vol. 3, pp. 515-516). Al-Maghribi does not use the corollary anywhere, so theorems 9 and 10 are superfluous in the present text. See § 4 of this paper.

11.1. The same theorem is proved in essentially the same way by Hypsicles in El. XIV § 7 (the preliminary to the second proof in prop. 6 in Heath's translation, vol. 3, pp. 516-517).

11.2. It is not clear to me to what "proofs" Al-Maghribi refers here. The same thing is used (but not proved) in M XV:9.

the product of AK and TE, VI:16. Thus the product of AK, that is three-fourths of the diameter of the circle, and one-sixth of the chord of the angle of the pentagon is equal to one-fifth of pentagon ABGDE. Thus the product of AK and five times TE, that is BT, is equal to pentagon ABGDE. That is what we wanted to prove.



prop. 11

12. The ratio of the surface of the cube to the surface of the octahedron is equal to the ratio of the side of an equilateral triangle to its altitude.

Proof: The ratio of the square of the side of the cube to the triangle of the octahedron is equal to the ratio of two-thirds of the side of the equilateral triangle to half of its altitude, XV:8. Thus the ratio of six times the square of the side of the cube, that is the surface of the cube, to six times the triangle of the octahedron, that is three-fourths of the surface of the octahedron, is equal to the ratio of two-thirds of the side of the equilateral triangle to half of its altitude, V:14 (=El. V:15). Thus the ratio of the surface of the cube to eight times the <triangle of the> octahedron, that is the surface of the octahedron, is equal to the ratio of two-thirds of the side of the equilateral triangle to two-thirds of its altitude, that is to say, equal to the ratio of the side of the equilateral triangle to its altitude.

It has become clear from this that the ratio of the surface of the cube to the surface of the octahedron is equal to the ratio of the square of a line to twice the equilateral triangle constructed on it, because the ratio of every line to the altitude of its triangle is equal to the ratio of the square of that line to the product of it and the altitude of its triangle according to the lemma[1], and (because) the product of the altitude and the base is twice that triangle.

By the "triangle of any line" we always mean the equilateral triangle constructed on it. Similarly the square of it, and the pentagon of it, and so on. That is what we wanted.

13. The ratio of the surface of the cube to the surface of the icosahedron is equal to the ratio of the square of the side of the pentagon of the circle of the icosahedron[1] to three and one-third times the triangle of the line, the square of which is equal to three times the square of the side of its (i.e. the circle's) decagon.

Thus let the radius of the circle of which the side of its pentagon is the side of the icosahedron be AB. We divide it in extreme and mean ratio at G, let its greater part be AG. Then AG is the side of the decagon, XIV:14[2]. Let BD be the side of the pentagon. Then the square of it (BD) is equal to the square of AB plus the square of AG, XIII:13[3]. Let line E be (the line), the square of which is equal to the square of AB plus the square of BG. Then the square of it (E) is equal to three times the square of AG, that is the side of the decagon of the circle, XIII:8 (=El. XIII:4). Let Z be the side of the cube.

I say that the ratio of the surface of the cube to the surface of the icosahedron is equal to the ratio of the square of BD to three and one-third times the triangle of line E.

Proof: The ratio of Z, that is the side of the cube, to BD, the side of the icosahedron, is equal to the ratio of BD to E, XV:6. Thus the ratio of the square of Z to the square of BD is equal to the ratio of the square of BD to the square of E, VI:22. For every pair of lines, the ratio of the square of one of them to the square of the other is equal to the ratio of the triangle of the first (line) to the triangle of the other. Thus, alternando, the ratio of the square of one of the lines to its triangle is equal to the square of the other to its triangle. Thus the ratio of the square of BD to its triangle is equal to the ratio of the

12.1. I do not know which lemma is meant here. The lemma must have been a special case of El. VI:1.

13.1. The theorem is valid for a pentagon and decagon inscribed in any circle. In the proof the circle is assumed to be a "circle of the icosahedron", that is any circle which circumscribes a regular pentagon formed by five sides of an icosahedron.

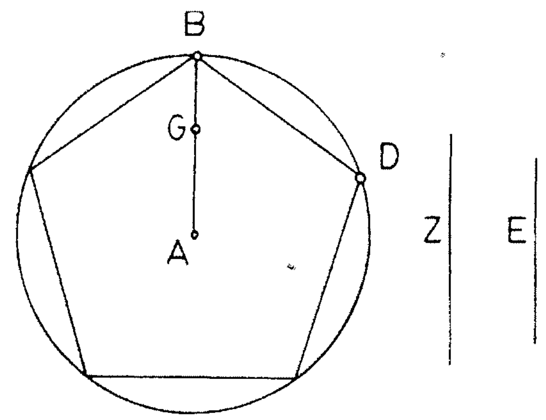
13.2. The reference should be to M XIII:14 (see the appendix).

13.3. The reference should be to M XIII:15 (=El. XIII:10).

square of E to its triangle. But the ratio of the square of Z to the square of BD was (shown to be) equal to the ratio of the square of BD to the square of E.

Thus, ex aequali, the ratio of the square of Z to the triangle of BD is equal to the ratio of the square of BD to the triangle of E. Thus the ratio of six times the square of Z, that is the surface of the cube, to six times the triangle of BD, that is three-tenths of the surface of the icosahedron, is equal to the ratio of the square of BD to the triangle of E, V:14 (=El. V:15). Thus the ratio of the surface of the cube to the surface of the icosahedron is equal to the ratio of the square of BD to three and one-third times the triangle of E. That is what we wanted to prove.

It has become clear from this that the ratio of three-fifths of the surface of the icosahedron to the surface of the cube is equal to the ratio of twice the triangle of the line, the square of which is equal to three times the square of the side of the decagon to <the square of> the side of the pentagon, after inversion[4] (of the ratio).



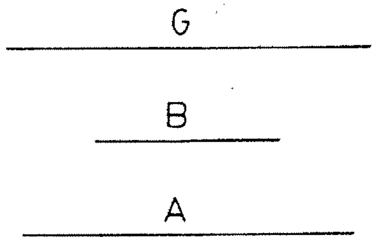
prop. 13

14. The ratio of the surface of the icosahedron to the surface of the octahedron is equal to the ratio of five times the square of the side of the decagon of a circle to the square of the side of its pentagon. Thus let line A be the side of the pentagon of the circle and line B the side

13.4. To invert a ratio a:b means to change it to b:a. If a:b=c:d according to Euclid's definition 5 of El. V (Heath vol. 2, p. 114), then b:a=d:c as an immediate consequence of that definition.

of its decagon. I say that the ratio of the surface of the icosahedron to the surface of the octahedron is equal to the ratio of five times the square of B to the square of A.

Proof: We make line G such that the square of it is equal to three times the square of B, XIII:18[1]. Then the ratio of three-fifths of the surface of the icosahedron to the surface of the cube is equal to the ratio of the surface of twice the triangle of G to the square of A, XV:13. But the ratio of the surface of the cube to the surface of the octahedron is equal to the ratio of the square of A to twice its triangle, XV:12. Thus, ex aequali, the ratio of three-fifths of the surface of the icosahedron to the surface of the octahedron is equal to the ratio of twice the triangle of G to twice the triangle of A, that is to say, equal to the ratio of the triangle of G to the triangle of A, V:14 (=El. V:15), that is to say, equal to the ratio of the square of <G> to the square of A. But the square of G is three times the square of B. Thus the ratio of three-fifths of the surface of the icosahedron to the surface of the octahedron is equal to the ratio of three times the square of B to the square of A. Thus the ratio of the surface of the icosahedron to the surface of the octahedron is equal to the ratio of five times the square of B, the side of the decagon, to the square of A, the side of the pentagon, V:14 (=El. V:15). That is what we wanted to prove.



prop. 14

15. The ratio of the excess of the surface of the icosahedron over the surface of the octahedron to the surface of the octahedron is equal to the ratio of the lesser part of any line divided in extreme and mean ratio to the whole line.

Thus let there be a circle with centre A, such that the side of its pentagon is BG, the side of its decagon is BD, and the two chords of

14.1. The reference should be to M XIII:20, where this (easy) construction is explained.

the two angles of its pentagon are BE and EG. We divide BE in extreme and mean ratio, let its greater part be BZ. I say that the ratio of the excess of the surface of the icosahedron over the surface of the octahedron to the surface of the octahedron is equal to the ratio of ZE to EB.

Proof: We draw lines AB, AG, AD. Then the two angles A are equal, III:26 (=El. III:27). Thus angle BAG is twice angle BAD, but it is also equal to twice angle BEG, III:19 (=El. III:20). Thus angle BAD is equal to angle BEG. Thus, by subtraction, the two equal angles EBG, EGB, I:5, are equal to the two equal angles ABD, ADB, I:5. Thus triangles ABD, EBG are similar, VI:4. Thus the ratio of DB to BA is equal to the ratio of GB to BE. Thus the ratio of the square of DB to the <square> of BA is equal to the ratio of the square of GB to the square of BE, VI:22, and equal to the ratio of five times the square of DB to five times the square of AB, that is to say, equal to the ratio of the square of BG to the square of BE. Thus, alternately, the ratio of five times the square of DB to the square of BG is equal to the ratio of five times the square of AB to the square of BE.

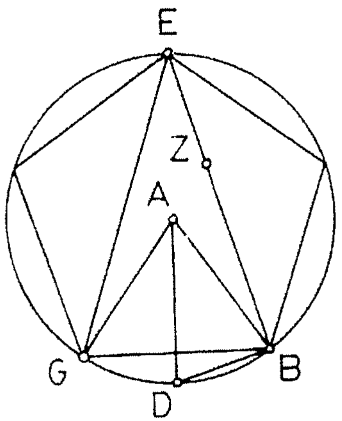
But five times the square of AB, that is to say the (sum of the) squares of EB, BZ, XV:4, to the square of BE[1], and the ratio of five times the square of DB to the square of BG is equal to the ratio of the surface of the icosahedron to the surface of the octahedron, XV:14. Thus the ratio of the surface of the icosahedron to the surface of the octahedron is equal to the ratio of the (sum of the) squares of EB, BZ to the square of BE. Thus, separando[2], the ratio of the excess of the surface of the icosahedron over the surface of the octahedron to the surface of the octahedron is equal to the ratio of the square of BZ to the square of BE. Thus, by inversion[3], the ratio of the surface of the octahedron to the excess of the surface of the icosahedron over it is equal to the ratio of the square of EB to the square of BZ, i.e. the product of BE and EZ, that is to say, the ratio of BE to EZ. Thus, by inversion[3], the ratio of the excess of the surface of the icosahedron over the surface of the octahedron <to the surface of the octahedron> is equal to the ratio of ZE to EB. That is what we wanted.

15.1. The sentence is ungrammatical. I have not emended the text, because the whole passage is confused.

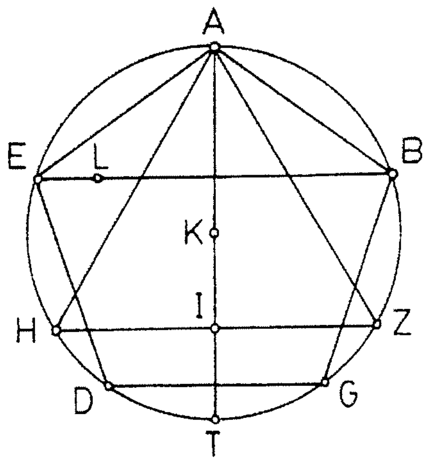
15.2. Separando means: by changing a:b=c:d (with a>b, c>d) to a-b:b=c-d:d (based on El. V:17)

15.3. The double inversion (compare note 13.4) is superfluous.

It has become clear from this that, componendo[4], the ratio of the surface of the icosahedron to the surface of the octahedron is equal to the ratio of any line divided in extreme and mean ratio plus its lesser part to the whole line.



prop. 15



prop. 16

16. The ratio of the surface of the dodecahedron to the surface of the icosahedron is equal to the ratio of the side of the cube to the side of the icosahedron[1].

Thus let the pentagon of the dodecahedron be pentagon ABGDE, and let the triangle of the icosahedron be triangle AZH, and let one circle with diameter AT and centre K circumscribe them, XV:5, and let it (AT) intersect ZH at I. Then KI is one-fourth of the diameter[2]. We join BE, it is the side of the cube, XIV:9 (=El. XIII:17).

I say that the ratio of the surface of the dodecahedron to the surface of the icosahedron is equal to the ratio of BE to ZH.

15.4. Componendo means: by changing a:b=c:d to a+b:b=c+d:d (based on El. V:18).

16.1. Hypsicles proves this theorem in essentially the same way in El. XIV § 8 (which corresponds to the alternative proof of prop. 6 in Heath's translation, vol. 3, p. 517). The dodecahedron and icosahedron are supposed to be inscribed in the same sphere.

16.2. Al-Maghribi proves this trivial fact in a separate proposition M XIII:18 (not in the Greek).

Proof: We make BL five-sixths of BE. Then the product of AI and BL is equal to pentagon ABGDE[3], and the product of AI and ZI is equal to triangle AZH. Thus the ratio of pentagon ABGDE to triangle AZH is equal to the ratio of BL to ZI, VI:1.

But pentagon ABGDE is one-twelfth of the surface of the dodecahedron, and triangle AZH is one-twentieth of the surface of the icosahedron. Thus the ratio of the surface of the dodecahedron to the surface of the icosahedron is equal to the ratio of twelve times BL to twenty times ZI, that is to say, equal to the ratio of ten times BE to ten times ZH, that is to say, equal to the ratio of BE to ZH. That is what we wanted to prove.

It has become clear from this and from XV:6[4] that the ratio of the surface of the dodecahedron to the surface of the icosahedron is equal to the ratio of the (i.e. any) line, the square of which is equal to the square of a line divided in extreme and mean ratio plus the square of its greater part to the line, the square of which is equal to the square of the whole (divided) line plus the square of its lesser part. The reason[5] is that this ratio is equal to the ratio of the side of the pentagon of the (i.e. any) circle to the line, the square of which is equal to three times the square of the side of its decagon[6].

17. The surface of the octahedron is one and a half times the surface of the tetrahedron inscribed in the same sphere.

Proof: Every two triangles of the octahedron are one-fourth of its surface, and every triangle of the tetrahedron is one-fourth of its

16.3. By M XV:11.

16.4. The Oxford, Utrecht and Aya Sofya manuscripts have a strange corruption here: "by 6 and 15". Only the Mihrishah ms. has "by 6 of 15", i.e. by XV:6. One wonders what was in the original text of Al-Maghribi. In notes 16.5 and 16.6 it is shown that al-Maghribi derived the corollary in a very strange way.

16.5. The corollary is an immediate consequence of M XV:6, so the further motivation is unnecessary.

16.6. Al-Maghribī proves this in a special proposition M XIII:21. However, if one wishes to deduce the corollary of M XV:16 from M XIII:21, one also has to know that the diagonal of the pentagon and the side of an equilateral triangle are in the same ratio as the side of the pentagon and "the line, the square of which is equal to three times the square of the side of the decagon" (all figures being inscribed in the same circle). Al-Maghribī does not prove this anywhere, but it can be proved using BE:BA=BG:BD in M XV:15 (compare note 16.4) and the fact that the square of the side of the equilateral triangle is three times the square of its circumcircle (proved in M XIII:12 =El. XIII:12).

surface. Thus the ratio of every two triangles of the octahedron to a triangle of the tetrahedron is equal to the ratio of the surface of the octahedron to the surface[1] of the tetrahedron, V:14 (=El. V:15). But every two triangles of the octahedron are one and a half times every triangle of the tetrahedron, by prop.[2] XV:2. Thus the surface of the octahedron is one and a half times the surface of the tetrahedron inscribed in the same sphere. That is what we wanted to prove.

18. Every parallelepipedal solid with base[1] equal to twice the face[1] of a pyramid and height equal to the diameter of the sphere circumscribing the pyramid is nine times the pyramid.

Proof: The parallelepipedal solid with base equal to the face of the pyramid and height equal to its height is three times it (the pyramid), XII:6 (=El. XII:7). Thus the parallelepipedal solid with base twice the face of the pyramid and height equal to its height is six times it (the pyramid). Thus the parallelepipedal solid with base twice the face of the pyramid and height one and a half times its height is nine times it (the pyramid). But one and a half times the height of the pyramid is one time the diameter of the circumscribing sphere, because the height of the pyramid is two thirds of the diameter of the sphere, prop. XIV:1 (=El. XIII:13). Thus the parallelepipedal solid with base twice the face of the pyramid and height equal to the diameter of the circumscribing sphere is nine times the pyramid.

It follows from this that the (parallelepipedal) solid with base twice the face of the pyramid and height one-ninth of the diameter of the (circumscribing) sphere is equal to the pyramidal solid[2].

19. Every parallelepipedal solid with base equal to the square of the side of the octahedron and height equal to the diameter of the (circumscribing) sphere is three times the octahedron.

17.1. The Utrecht and Oxford manuscripts have "one of the triangles" instead of "the surface". The text in the Mihrishah ms. is correct (The whole passage is missing in the Ayasofya ms.)

17.2. The abbreviation "prop" serves as a translation of the abbreviation *sh* in the manuscripts (for *shakl*, meaning proposition).

18.1. The text uses the same word (*qā'ida*) for "base " (of a parallelepiped) and "face" (of a regular polyhedron).

18.2. The text sometimes uses the terms "pyramidal solid, octahedral solid" etc., mainly in cases where we would use the terms "volume of a pyramid, volume of an octahedron" etc.

Proof: The parallelepipedal solid with base the square of the side of the octahedron and height equal to the radius of the sphere circumscribing the octahedron is three times the pyramid[1] with base the square of the side of the octahedron and height equal to the radius of the sphere, that is to say, half of the octahedral solid. Thus the parallelepipedal solid with base the square of the side of the octahedron and height equal to the diameter of the sphere is six times half of the octahedral solid. Thus it (the parallelepiped) is three times it (the octahedron). It follows from this that the (parallelepipedal) solid with base the square of the side of the octahedron and height one-third of the diameter of the (circumscribing) sphere is equal to the octahedral solid.

It has become clear from this that the parallelepipedal solid with base three times the square of the side of the octahedron and height equal to one-ninth of the diameter of the sphere is equal to the octahedral solid.

20. The ratio of the tetrahedral solid to the octahedral solid is equal to the ratio of the side of the (i.e. any) equilateral triangle to three times its altitude.

Proof: We make AB the side of the tetrahedral solid. We draw from point B perpendicular BG, and we cut it off equal to the altitude of the triangle of the pyramid. Then it (BG) is equal to the side of the octahedron, XV:2. We extend AB in a straight line, and we cut off from it BD (equal to) three times BG. We complete the rectangles AG, GD.

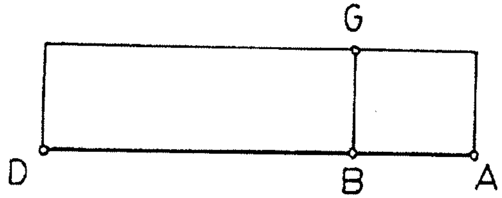
Then rectangle AG is twice the triangle of the pyramid, and BD is three times the side of the octahedron. Thus the parallelepipedal solid with base AG and height one-ninth of the diameter of the (circumscribing) sphere is equal to the pyramidal solid, XV:19[1], and the parallelepipedal solid with base GD and height one-ninth of the diameter of the sphere is equal to the octahedral solid, XV:19. The ratio of the parallelepipedal solid with base AG and height one-ninth of the diameter of the sphere to the parallelepipedal solid with base GD and height one-ninth of the diameter of the sphere is equal to the ratio of AG to GD, that is to say, AB to BD. But BD is three times

19.1 The Arabic word is *makhrūt*, (literally: "cone-shaped") which can be used for a cone as well as for a (not necessarily regular) pyramid.

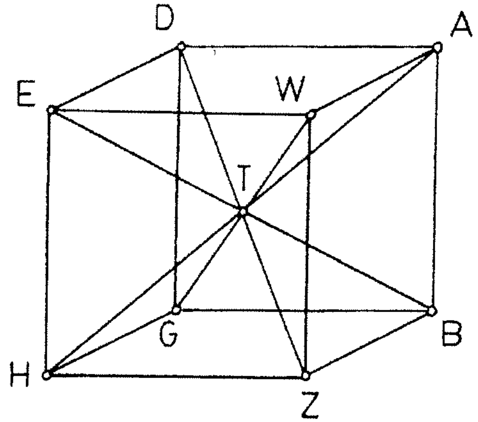
20.1 The reference should be to M XV:18.

the side of the octahedron. Thus the ratio of the pyramidal solid to the octahedral solid is equal to the ratio of the side of the triangle of the pyramid to three times its altitude.

It has become clear from this that the ratio of the pyramidal solid to the octahedral solid is equal to the ratio of the side of the pyramid to three times the side of the octahedron.



prop. 20



prop. 21

21. (For) every equilateral and equiangular solid circumscribed by a sphere, if its solid angles and the centre of the sphere are joined by straight lines, it (the solid) is divided into equal and similar pyramidal[1] figures, the number of which is equal to the number of faces of the

21.1. In propositions 21-23 the "pyramidal figures" and "pyramids" are not necessarily regular, even though the text uses the term "*nārī*", which usually means "regular pyramid".

solid. The perpendiculars falling from the centre of the sphere onto the faces of these planes[2] are equal.

Example: Let there be a cube ABGDEWZH with centre T. We join the straight lines TA, TB, TG, TD, TE, TW, TZ, TH. Then they are equal because they are radii of the sphere circumscribing the cube. But they are sides of the pyramids whose bases are the faces of the cube and whose vertices are the centre of the sphere. Thus the cube has been divided into six pyramids, the number of which is equal to the number of its six faces. These pyramids have equal bases, and the lines drawn from T, that is to say from their vertices, to the angles of their bases are equal, because each one of them is equal to the radius of the sphere. Thus the pyramids are all equal and similar. In this way it can be shown that each of the remaining solids circumscribed by the sphere is divided into equal and similar pyramidal figures. The perpendiculars falling from their vertices to their bases are equal, because the circles which are produced on the sphere by the intersections of these bases are equal. Thus the perpendiculars falling onto them from the centre of the sphere are equal, by what was shown previously[3]. That is what we wanted.

22. (For) every two pyramids whose axes are equal and perpendicular to their bases, such that the base of one of them is a triangle and the base of the other one is an equilateral and equiangular pentagon, the ratio between them is equal to the ratio between their bases[1].

Thus let there be a pyramid with base the equilateral triangle ABG and axis DE, and a pyramid with base the equilateral and equiangular pentagon ZHTIK and axis LM. Let DE, LM be equal and perpendicular to their bases. I say that the ratio of pyramid ABGD to pyramid ZHTIKM is equal to the ratio of triangle ABG to pentagon ZHTIK.

Proof: We join lines LZ, LH, LT, LI, LK and lines MZ, MH, MT, MI, MK[2]. Then the ratio of pyramid ABGD to pyramid ZHLM is

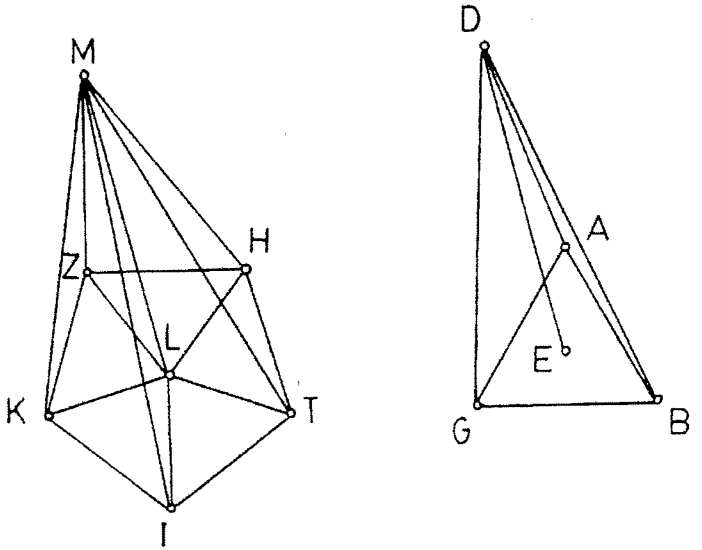
21.2. It would have been better to say "planes of the faces".

21.3. In the lemma at the end of M XIII, cf. note 3.2.

22.1. This theorem is in fact a special case of El. XII:6, which does not occur in al-Maghribī's Revision. In El. XII:6 Euclid proves that pyramids with polygonal bases are to one another as the bases. M XV:22 is not used in the text (compare notes 23.1 and 30.4).

22.2. Lines MZ, MH, MT, MI and MK do not have to be joined, because they are sides of the pyramid ZHTIKM.

equal to the ratio of triangle ABG to triangle LZH, XII:5. But the ratio of pyramid MZHL to pyramid ZHTIKM is equal to the ratio of triangle LZH to pentagon ZHTIK, V:13[3]. Thus, ex aequali, the ratio of pyramid ABGD to pyramid ZHTIKM is equal to the ratio of triangle ABG to pentagon ZHTIK, V:22. That is what we wanted to prove.



prop. 22

23. The ratio of the solid of the cube to the octahedral solid is equal to the ratio of the side of the (i.e. any) equilateral triangle to its altitude.

Proof: The ratio of the solid of the cube, that is six times the pyramid with base the face of the cube and vertex the centre of the sphere circumscribing the cube, XV:21, to six times the pyramid with base the face of the octahedron and vertex the centre of the circumscribing sphere, that is three-fourths of the solid of the octahedron, XV:21, is equal to the ratio of the pyramid with base the face of the cube and vertex the centre of the sphere, to the pyramid with base the triangle of the octahedron and vertex the centre of the sphere. <But this ratio is equal to the ratio of the face of the cube to the face of the octahedron,>[1] because the height of the pyramid with

22.3. The reference should be to M V:14 (=El. V:15).

23.1 I have added the passage in angular brackets in order to make mathematical sense. Note that an analogon of M XV:22 is used here, but not M XV:22 itself.

base the face of the cube is equal to the height of the pyramid with base the face of the octahedron, XV:3. But the ratio of the face of the cube to the face of the octahedron is equal to the ratio of two-thirds of the side of the equilateral triangle to half of its altitude, XV:8. Thus the ratio of the cube to the octahedron is equal to the ratio of two-thirds of the side of the equilateral triangle to two-thirds of its altitude, which is the ratio of the side of the equilateral triangle to its altitude. That is what we wanted to prove.

24. The parallelepipedal solid, the base of which is equal to the pentagon of the icosahedron, and the height of which is equal to two-thirds of the diameter of the sphere, is equal to the icosahedral solid.

Proof: If a perpendicular is drawn from the centre of the sphere to the pentagon of the icosahedron, and if it is extended to the angle (i.e. angular point) of the icosahedron, the (part) cut off from this line between the centre and the pentagon is (equal to) half of the side of the hexagon[1] of the circle circumscribing the pentagon. But the line cut off from it between the plane of the pentagon and the angle of the icosahedron is equal to the side of its decagon[1], XIV:7 (=El. XIII:16). If the centre of the sphere is joined to the angles (i.e. angular points) of the pentagon of the icosahedron by straight lines, two pyramids are separated from the icosahedron; the vertex of one of them is the centre of the sphere, and its base is the above-mentioned pentagon, and its height is half of the side of the hexagon, and the vertex of the other one is the angle of the icosahedron, its base is the above-mentioned pentagon, and its height is the side of the decagon. These two pyramids (together) are equal to five of the pyramids, the bases of which are the faces of the icosahedron, and the vertices of which are the centre of the sphere. Thus they (the five pyramids) are equal to one-fourth of the icosahedral solid[2]. Thus the parallelepipedal solid, the base of which is equal to the pentagon of the icosahedron, and the height of which is one-sixth of the side of the hexagon plus one-third of the side of the decagon, which are (together) equal to one-sixth of the

24.1. The fact that these segments are equal to the side of the regular hexagon or and the side of the regular decagon inscribed in the circle (by M XIV:7 =El. XIII:16) is irrelevant. One only needs to know that the sum of the two segments is equal to the radius of the sphere (which is trivial).

24.2. By M XV:21.

diameter of the sphere[3], is equal to these two pyramids, that is one-fourth of the icosahedral solid.

Thus the parallelepipedal solid, the base of which is equal to the pentagon of the icosahedron, and the height of which is two-thirds of the diameter of the sphere, is equal to the icosahedral solid.

25. The ratio of the perpendicular of the icosahedral solid to the diameter of the circumscribing sphere is equal to the ratio of half of the side of the decagon plus half of the side of the hexagon inscribed in a circle to twice the altitude of the triangle of the side of its pentagon. By the "perpendicular of a solid figure" we mean any perpendicular drawn from the centre of the sphere to a face of the solid figure[1].

Thus let the pentagon of the icosahedron be ABGDE, let its centre be Z and let the perpendicular drawn to its side from the centre be ZH. Let T be the perpendicular of the icosahedron, and K the diameter of the circumscribing sphere. We construct on DG the equilateral triangle DLG. We join LH; it is clearly perpendicular to DG.

I say that the ratio of ZH, which is equal to half of the side of the hexagon plus (half of) the side of the decagon[2], to twice HL is equal to the ratio of T to K.

Proof: The parallelepipedal solid, the base of which is (equal to) pentagon ABGDE, and the height of which is equal to two-thirds of K is equal to the icosahedral solid, XV:24. But the parallelepipedal solid, the base of which is equal to the surface of the icosahedron and the height of which is equal to one-third of T, is also equal to the icosahedral solid. That is clear[3].

Thus the parallelepipedal solid, the base of which is (equal to) pentagon ABGDE and the height of which is equal to two-thirds of K, is equal to the parallelepipedal solid, the base of which is equal to the surface of the icosahedron, and the height of which is equal to one-third of T. Thus the ratio of pentagon ABGDE to the surface of the icosahedron is equal to the ratio of one-third of T to two-thirds of K, by reciprocality, XI:34. But since triangle ZDG is one-fifth of

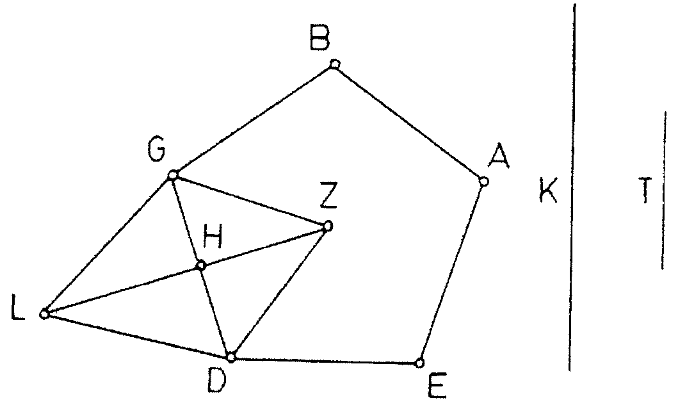
24.3. Because the sum of the segments in note 24.1 is equal to the radius of the sphere.

25.1. The definition makes sense for regular solids only.

25.2. By M XV:7.

25.3. By M XV:21.

pentagon ABGDE, the ratio of triangle ZDG, that is, one-fifth of the pentagon, to one-fifth of the surface of the icosahedron, is equal to the ratio of the surface of pentagon ABGDE to the surface of the icosahedron. Thus the ratio of triangle ZDG to one-fifth of the surface of the icosahedron, that is four of its triangles, V:14 (=El. V:15)[4], is equal to the ratio of one-third of T to two-thirds of K[5]. Thus the ratio of one-third of T to one-third of K is equal to the ratio of triangle ZDG to twice the triangle of the icosahedron, that is triangle GLD, that is to say, equal to the ratio of ZH to twice HL, VI:1. But the ratio of one-third of T to one-third of K is equal to the ratio of T to K, V:14 (=El. V:15). Thus the ratio of T to K is equal to the ratio of ZH to twice HL, that is the <ratio of the> perpendicular of the pentagon, i.e. half of the (sum of the) sides of the hexagon and the decagon, to <twice> the altitude of the triangle of the side of the pentagon. That is what we wanted.



prop. 25

26. The ratio of the radius of the sphere circumscribing the octahedron to the perpendicular of its solid (i.e. the octahedron) is equal to the ratio of the altitude of the (i.e. any) equilateral triangle to half of its side.

25.4. The reference is misplaced. It should have been at the end of the preceding sentence.

25.5. Here I have excised a passage "Thus the ratio of triangle ZDG to four triangles of the icosahedron" in all MSS.

Proof: The perpendicular of the octahedron is equal to the perpendicular of the cube, XV:3. But the square of the radius of the sphere is three times the square of the perpendicular of the cube, XIV:3[1]. Therefore it is three times the square of the perpendicular of the octahedron. But the square of the altitude of the equilateral triangle is three times the square of half of its side, XV:1 and II:4. Therefore the ratio of the square of the radius of the sphere to the square of the perpendicular of the octahedron is equal to the ratio of the square of the altitude of the (equilateral) triangle to the square of half of its side. Therefore the ratio of the radius of the sphere to the perpendicular of the octahedron is equal to the ratio of the altitude of the triangle to half of the side of the triangle, VI:22. That is what we wanted.

27. The ratio of the perpendicular of the icosahedral solid to the perpendicular of the octahedron is equal to the ratio of the sum of the sides of the hexagon and the decagon inscribed in one circle to the side of the pentagon of the (i.e. this) circle.

Proof: Since the ratio of the perpendicular of the icosahedron to the diameter of the sphere is equal to the ratio of half of the side of the decagon and half of the side of the hexagon taken together to twice the altitude of the triangle of the icosahedron[1], XV:25, and (since) the ratio of the diameter of the sphere to the perpendicular of the octahedron is equal to the ratio of twice the altitude of the triangle of the icosahedron to half of its side, XV:26, and it follows[2], by the sum of them[3], that

26.1. Euclid proves in El. XIII:15 that the square of the diameter of the sphere is three times the square of the side of the cube. Al-Maghribī concludes in a corollary to M XIV:3 (=El. XIII:15) that the square of the radius is three times the square of the perpendicular from the centre to any face of the cube.

27.1. It is tacitly assumed here that the above-mentioned decagon and hexagon are inscribed in a circle through five angular points of the icosahedron.

27.2. The passage "and it follows ... triangle" is mathematically superfluous, because it repeats what has already been said. However, it is not the exact repetition of the preceding passage, so it cannot be a scribal error.

27.3. "By the sum of them" is my translation of the Arabic *bi-majmū'himā*. What is necessary here is not addition but multiplication of two ratios. The Euclidean terms for the "composition" of ratios (derived from suntithèmi), can have an additive and a multiplicative meaning (see Heath's translation of the *Elements*, vol. 2, p. 135). We could therefore explain the nonsensical "sum" in our text by assuming that proposition 27 is of Greek origin, that a technical term like "sunthenti" in the Greek was mistranslated into Arabic in the additive meaning, and that the passage following the sum ("that the ratio ... ex aequali") was added either by Al-Maghribī or by a predecessor who tried to make sense of the text.

the ratio of the perpendicular of the icosahedral solid to the diameter of the sphere is equal to the ratio of half of the sum of the sides of the decagon and the hexagon to <twice> the altitude of the triangle of the icosahedron, and the ratio of the diameter of the sphere to the perpendicular of the octahedral solid is equal to the ratio of twice the altitude of the triangle of the icosahedron to half of the side of its triangle. Hence, ex aequali, the ratio of the perpendicular of the icosahedron to the perpendicular of the octahedron is equal to the ratio of half of the side of the decagon plus half of the side of the hexagon to half of the side of the icosahedron, that is to say, equal to the ratio of the side of the hexagon plus the side of the decagon to the side of the pentagon. That is what we wanted to prove.

28. The ratio of every two parallelepipedal rectangular solids is compounded of the ratio of their bases and the ratio of their heights.

Thus let solid ABGDEZHT be parallelepipedal and rectangular, and let similarly solid KLMNSOFC be parallelepipedal and rectangular. I say that the ratio of solid AH to solid KF is compounded of the ratio of base EH to base SF and (of the ratio of) height GH to height MF.[1]

Proof: We extend line ZH, and we cut off HQ equal to KN. We extend line TH and we cut off HR equal to KL. We extend line GH, and we cut off HJ equal to KS.

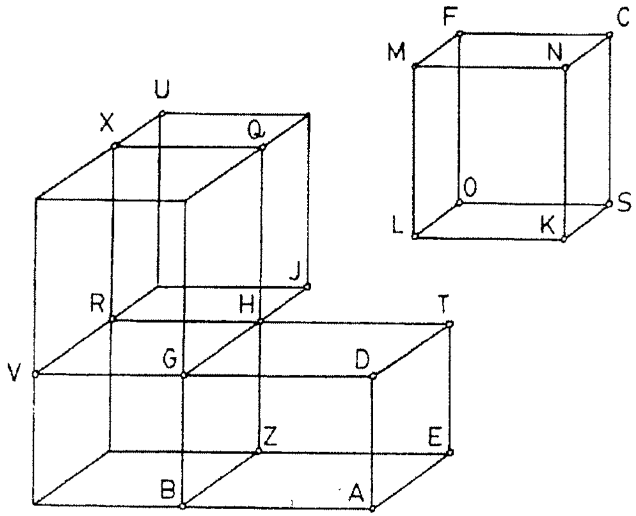
We complete the parallelepipedal solid HU. Then it is equal to solid KF, because the bases and heights are equal. We also complete the parallelepipedal solids BR, VQ.

Then the ratio of solid AH to solid HU, that is, solid KF, is compounded of the ratio of solid AH to solid BR, that is the ratio of base EH to base ZR, and of the ratio of solid BR to solid VQ, that is, the ratio of base RZ to base RQ, and of the ratio of solid VQ to solid HU, that is the ratio of base VH to base RJ.

But the ratio compounded of the ratio of base AG to base RZ and the ratio of base RZ to base RQ is the ratio of base AG to base RQ, that is, base KM. And the ratio of base VH to base RJ is equal to the ratio of GH, the height of solid AH, to HJ, the height of solid JX, that is KS, that is the height of solid MS. Thus the ratio of solid AH to

28.1. The proof resembles that of El. XI:33.

solid KF is compounded of the ratio of base EH to base SF and (the ratio of) height GH to height MF. That is what we wanted to prove.



prop. 28

29. The ratio of the icosahedral solid to the octahedral solid is equal to the ratio of the side of the decagon of a circle plus the side of its pentagon to the side of its pentagon[1].

Proof: We make the parallelepipedal solid A such that its base is equal to the surface of the icosahedron and its height is equal to one-third of the perpendicular of the icosahedron. Then it is equal to the icosahedral solid. We make the base of the parallelepipedal solid B equal to the surface of the octahedron, and its height equal to one-third of the perpendicular of the octahedron. Then it is equal to the octahedral solid. We make the base of the parallelepipedal solid G equal to the excess of the surface of the icosahedron over the surface of the octahedron, and its height equal to one-third of the perpendicular of the icosahedron. Then it is equal to the excess of solid A over solid B.[2]

Let DE be the side of the decagon of the (i.e. any) circle, and let EH be the side of its pentagon. I say that the ratio of solid A to solid B is equal to the ratio of DH to HE.[3]

Proof: The ratio of solid G to solid B is compounded of the ratio of the base of G, the excess of the surface of the icosahedron (over the surface of the octahedron) to the base of B, that is the surface of the octahedron, and of the ratio of the height of solid G, that is one-third of the perpendicular of the icosahedron, to the height of solid B, that is one-third of the perpendicular of the octahedron[4]. Let T be the side of the hexagon plus (the side of) the decagon inscribed in the circle of which EH is the side of the pentagon and ED is the side of the decagon. Then the ratio of the excess of the surface of the icosahedron over the surface of the octahedron to the surface of the octahedron is equal to the ratio of DE, the side of the decagon, to T, the side of the hexagon plus the (side of the) decagon, XV:15[5].

But the ratio of the perpendicular of the icosahedron to the perpendicular of the octahedron is equal to the ratio of the side of the hexagon plus (the side of the) decagon to the side of the pentagon, XV:27. Therefore the ratio of solid G to solid B is compounded of the ratio of DE to T and the ratio of T to EH. But the ratio compounded of <the ratio of> DE to T and of the ratio of T to EH is equal to the ratio of DE to EH. Therefore the ratio of solid G to solid B is equal to the ratio of DE to EH.[6] Therefore, componendo[7], the ratio of the (sum

29.1. The theorem is incorrect; the ratio between the volumes of the icosahedron and the octahedron is equal to the ratio of the side of the pentagon of a circle to the side of its decagon.

29.2. The heights of A and G are equal, but the height of B is not equal to these, so $G \neq A - B$.

29.3. Actually A:B=EH:DE.

29.4. By M XV:28.

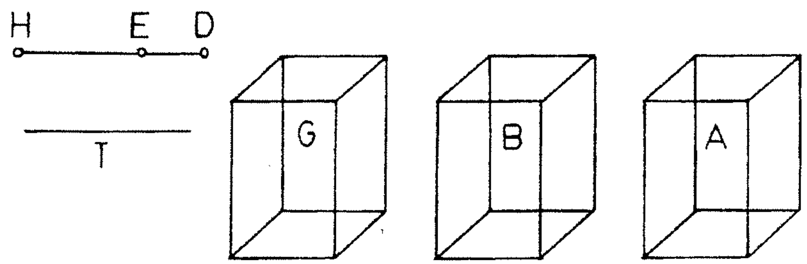
29.5. M XIII:14 is also used here.

29.6. The result G:B=DE:EH is correct.

29.7. Here the text assumes the incorrect relation G=A-B. The ratio A:B can be found in a correct way as follows. We have $A:G=s_{20}:s_{20}-s_8$. Using the notation and the figure of M XV:15, $s_{20}:s_8=EB^2+BZ^2:EB^2$, so $s_{20}:s_{20}-s_8 = EB^2+BZ^2:BZ^2 = EB^2+BG^2:BG^2 = BA^2+BD^2:BD^2 = BG^2:BD^2 = c(5)^2:c(10)^2$, where c(n) stands for the side of a regular n-gon inscribed in a fixed circle. One can rephrase this argument by putting EB=c(2.5), BZ=BG=c(5), BA=c(6), BD=c(10); the essential step is c(2.5):c(6)=c(5):c(10). Now, by composition of ratios, $A:B=(A:G)\cdot(G:B)=(c(5)^2:c(10)^2)\cdot(c(10):c(5))=c(5):c(10)$.

Thus the proof in the text can be corrected using the same auxiliary solid G as in the text, by means of multiplication rather than addition of ratios, and the resulting correct theorem, stated in note 29.1, resembles the incorrect theorem in the text. These coincidences may be accidental, but it is also possible that the theorem was stated and proved correctly (but concisely) in an earlier source, and that the errors are the result of editorial changes made somewhere in the process of transmission, before the proposition was adopted by al-Maghribī. One is reminded of the confusion of) the two solids G and B, that is to say, the icosahedral solid, to solid B, that is the octahedral solid, is equal to the ratio of DH to HE, that is, the side of the pentagon and the decagon together to the side of the pentagon.

It has become clear from this that the ratio of the icosahedral solid to the octahedral solid is equal to the ratio of the line, the square of which is equal to the square of the (i.e. any) line divided in extreme and mean ratio plus the square of its greater part, plus the greater part, to the line, the square of which is equal to the square of that line plus the square of its greater part.[8] That is what we wanted.



prop. 29

30. The ratio of the dodecahedral solid to the icosahedral solid is equal to the ratio of the line, the square of which is equal to the square of the (i.e.any) line, divided in extreme and mean ratio, plus the square of its greater part, to the line, the square of which is equal to the square of the whole (divided) line plus the square of its lesser part[1].

Proof: The perpendicular falling from the centre of the sphere on the pentagon of the dodecahedron is equal to the perpendicular falling from its centre on the triangle of the icosahedron, since it has been shown that this pentagon and (this) triangle are circumscribed by the same circle, and that the perpendiculars falling from the centre of the

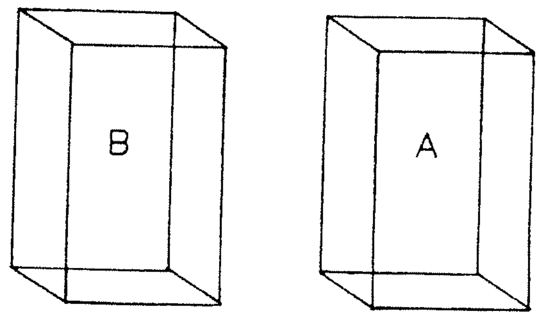
between addition and multiplication in proposition 27 (note 27.3).

29.8. It is mathematically correct to say that "the ratio of the icosahedral solid to the octahedral solid is equal to the ratio of the line, the square of which is equal to the square of the line divided in extreme and mean ratio plus the square of its greater part, to the greater part".

30.1. Hypsicles states this theorem in his summary of results in El. XIV § 12 ((4) in Heath's translation, vol. 3, p. 519).

sphere on the planes of equal circles drawn on its surface are equal[2]. Thus let solid A be a parallelepiped with base equal to the surface of the dodecahedron and height equal to one-third of its perpendicular, and let solid B be a parallelepiped with base equal to the surface of the icosahedron and height equal to one-third of its perpendicular. Then it is clear that solid A is equal to the dodecahedral solid, and that solid B is equal to the icosahedral solid. Thus the ratio of the dodecahedral solid to the icosahedral solid is equal to the ratio of solid A to solid B, V:14.[3] But the ratio of solid A to solid B is equal to the ratio of the base of solid A, that is the surface of the dodecahedron, to the base of solid B, that is the surface of the icosahedron, because of the equality of their heights[4].

Thus the ratio of the dodecahedral solid to the icosahedral solid is equal to the ratio of the surface of the dodecahedron to the surface of the icosahedron, that is to say, equal to the ratio of the line, the square of which is equal to the square of any line divided in extreme and mean ratio plus the square of its greater part, to the line, the square of which is equal to the square of the whole line to the square of its lesser part[5]. That is what we wanted to prove.



prop. 30

30.2. This is the corollary to M XV:5.

30.3. The reference is pointless, because the theorem a:b = na:nb (M V:14 =El. V:15) is quoted here for the case n = 1.

30.4. By M XI:33 (=El.XI:32). One could also prove the theorem M XV:30 without parallelepipeds and M XI:33, but with pyramids and M XV:22. This is essentially Hypsicles' method for proving that the volumes of the icosahedron and the dodecahedron are in the same ratio as their surfaces in El. XIV § 10 (tr. Heath, vol. 3, p. 518).

30.5. By M XV:16, corollary.

31. We want to find lines proportional to the sides of the five figures[1]. Thus we draw the circle circumscribing the pentagon of the dodecahedron and the triangle of the icosahedron[2]. Let the side of its pentagon be AB, (let) the side of its triangle (be) GD, and (let) the chord of two-fifths of it be AE. Then it (AE) is the side of the cube, XIV:9 (=El. XIII:17).

Let the square of ZH be equal to twice the square of AE. We bisect it (ZH) at N, and we draw with centre N and radius NH a circle, let TK be the side of its triangle, and LM the side of its square. Then the square of ZH is four times the square of NH. But the square of NH is one-third of the square of TK, (by prop.) XIII:11[3]. Thus the square of ZH is one and a third times the square of TK.

But the square of the diameter of the sphere is one and a half times the square of the side of the pyramid, XIV:1 (=El. XIII:13), and it is twice the square of the side of the octahedron, XIV:5 (=El. XIII:14). Thus the square of the side of the pyramid is one and a third times the square of the side of the octahedron. Thus the ratio of the square of ZH to the square of TK is equal to the ratio of the square of the side of the pyramid to the square of the side of the octahedron, VI:22[4]. Again, since the square of ZH is one and a third times the square of TK, as we have shown, and (since) the square of ZH is twice the square of LM, I:47, the square of TK is one and a half times the square of LM. But the square of the diameter of the sphere is twice the square of the side of the octahedron, XIV:5 (=El. XIII:14), and three times the square of the side of the cube, XIV:3 (=El. XIII:15). Thus the square of the side of the octahedron is one and a half times the square of the side of the cube. Thus the ratio of the square of TK to the square of LM is equal to the ratio of the square of the side of the octahedron to the square of the side of the cube. Thus the ratio of the square of ZH, that is the square of the side of the pyramid, to the square of TK, that is the square of the side of the octahedron, is equal to the ratio of the square of TK to the square of

31.1. The same problem is solved in a different way in El. XIII:18 (this proposition does not occur in al-Maghribī's *Revision*).

31.2. By M XV:5.

31.3. The reference should have been to M XIII:12 (=El. XIII:12).

31.4. By VI:22 ZH:TK is equal to the ratio between the sides of the cube and the octahedron. It is curious that Al-Maghribī does not state this conclusion.

LM, that is the side of the cube[5]. Thus the ratio of TK to LM is equal to the ratio of the side of the octahedron to the side of the cube, VI:22. Again, the square of ZH is twice the square of LM, I:47, and (also) twice the square of AE by assumption. Thus the square of LM is equal to the square of AE, so LM is equal to AE[6]. Thus the ratio of LM, that is AE, to GD is equal to the ratio of the side of the cube to the side of the icosahedron.

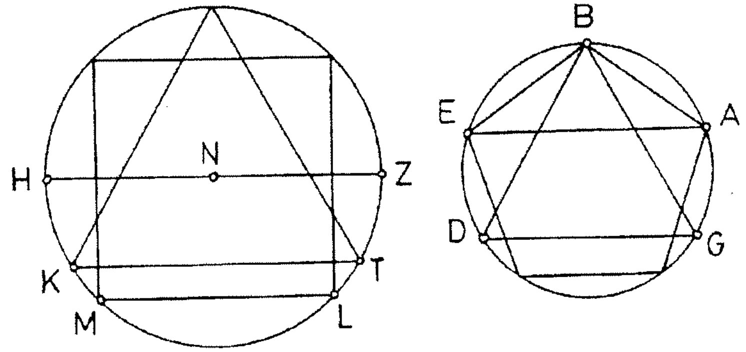
Since AB is the side of the dodecahedron, and GD is the side of the icosahedron, the ratio of GD to AB is equal to the ratio of the side of the icosahedron to the side of the dodecahedron.

Therefore the ratio of ZH to TK is equal to the ratio of the side of the pyramid to the side of the octahedron; the ratio of TK to LM, that is AE, is equal to the ratio of the side of the octahedron to the side of the cube; the ratio of LM, that is AE, the side of the cube, to DG <is equal to the ratio of the side of the cube to the side of the icosahedron; and the ratio of GD to AB is equal to the ratio of> the side of the icosahedron to the side of the dodecahedron. That is what we wanted to prove.

It has become clear from this that the side of the pyramid is greater than the side of the octahedron, and that the side of the octahedron is greater than the side of the cube, and that the side of the cube is greater than the side of the icosahedron, and that the side of the icosahedron is greater than the side of the dodecahedron.

31.5. This sentence is false. In fact $ZH^2:TK^2 \neq TK^2:LM^2$, and that ZH, TK and LM are the sides of the pyramid, octahedron and cube has not yet been proved. The text is much clearer if the false sentence is deleted.

31.6. Thus LM is the side of the cube. If M XV:1,2,3 had been used, the first part of the proof in the text could have been shorter. The faces of the cube and the octahedron have the same circumscribed circle by M XV:3, so KT is the side of the inscribed octahedron. Because $TK^2 = 3ZN^2 = (3/4)ZH^2$, ZH is the side of the inscribed pyramid by M XV:1,2.



prop. 31

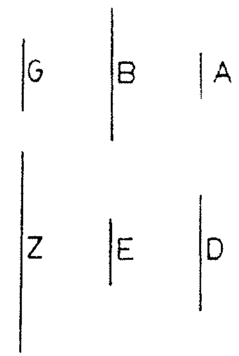
32. The ratio of the perpendicular of the pyramid to the perpendicular of the cube is equal to the ratio of one-third of the altitude of the (i.e. any) equilateral triangle to half of its side. Thus let the perpendicular of the pyramid be A, (let) the radius of the sphere circumscribing it (be) B, and (let) the perpendicular of the cube (be) G. Let D be the altitude of an equilateral triangle, (let) half of its side (be) E, and (let) Z (be) three times E.

Then A is one-sixth of the diameter of the sphere, XIV:1 (=El. XIII:13), so it is one-third of B. Thus the ratio of A to B is equal to the ratio of E to Z. But the square of B is three times the square of G, XIV:3[1], and the square of D is three times the square of E, XV:1. Thus the ratio of the square of B to the square of G is equal to the ratio of the square of D to the square of E. Thus the ratio of B to G is equal to the ratio of D to E, VI:22. Thus, perturbando[2], the ratio of the square of A, the perpendicular of the pyramid, to the square of G, that is the perpendicular of the cube, is equal to the square of D, that is the altitude of the equilateral triangle, to the square of Z, three times half of its side. Thus the ratio of the perpendicular of the pyramid to the perpendicular of the cube is equal to the altitude of the triangle to three times half of its side. Thus the ratio of the perpendicular of the pyramid to the perpendicular of the cube is equal to the ratio of one-third of the altitude of the equilateral triangle

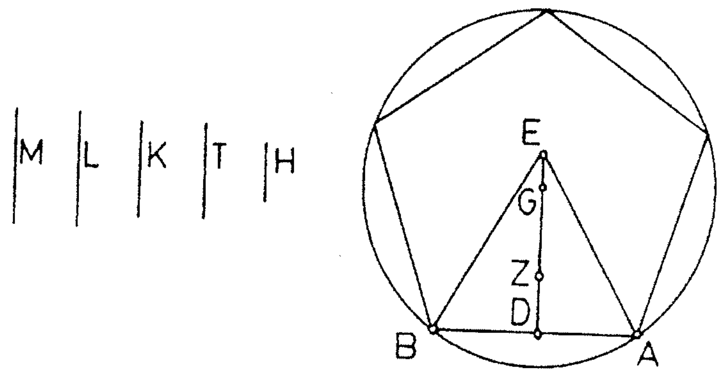
32.1. By M XIV:3, corollary (see note 26.1).

32.2. The term "perturbando" indicates a reasoning of the following form: if A:B=E:Z and B:G=D:E, then A:G=D:Z (based on El. V:23). Therefore the squares in the sentence after the footnote could be removed.

to half of its side, that is, one-third of Z. That is what we wanted to prove.



prop. 32



prop. 33

33. We want to find the five lines proportional to the perpendiculars of the five figures.

Thus we draw a circle such that the side of its pentagon is AB and its centre is G. We construct on it (AB) an equilateral triangle, namely triangle AEB. We drop perpendicular ED, and we make DZ one-third of DE. Let H be equal to DZ, and (let) T (be) equal to AD, and (let) K (be) equal to T, and (let) L (be) equal to GD, that is the perpendicular drawn from the centre to the side of the pentagon, and (let) M (be) equal to L.

Then the ratio of H to T is equal to the ratio of the perpendicular of the pyramid to the perpendicular of the octahedron, that is the perpendicular of the cube, XV:3[1]. The ratio of T to K is equal to the ratio of the perpendicular of the octahedron to the perpendicular of the cube, because of their equality, XV:3. The ratio of K, that is half of the side of the pentagon, that is AD, to L, that is GD, which is equal to half of the sum of the side of the hexagon and the (side of the) decagon, XV:7, is equal to the ratio of the perpendicular of the cube, that is the perpendicular of the octahedron, to the perpendicular of the icosahedron, XV:27. The ratio of L to M is equal to the ratio of the perpendicular of the icosahedron to the perpendicular of the dodecahedron, because of their equality, XV:5.

Therefore the ratio of H to T is equal to the ratio of the perpendicular of the pyramid to the perpendicular of the octahedron, the ratio of T to K is equal to the ratio of the perpendicular of the octahedron to the perpendicular of the cube, the ratio of K to L is equal to the ratio of the perpendicular of the cube to the perpendicular of the icosahedron, and the ratio of L to M is equal to the ratio of the perpendicular of the icosahedron to the perpendicular of the dodecahedron. Thus we have found the five lines proportional to the perpendiculars of the five figures. That is what we wanted.

34. How do we find a line divided in extreme and mean ratio, such that the square of it plus the square of its lesser part is equal to the square of an assumed line?

Thus let the assumed line be AB. We construct on AB semicircle ADB. We make AG twice GB. We draw perpendicular GD, and we join BD. Then the square of AB is three times the square of BD[1]. Then we divide BD in extreme and mean ratio at E, let the greater part be BE. We draw chord BZ equal to BE, IV:1. We extend BZ in a straight line, and we cut off ZH equal to BD. Then HB is divided in extreme and mean ratio, and its greater part is HZ, XIII:7 (=El. XIII:5). Thus the squares of BH, BZ (taken together) are three times the square of ZH[2].

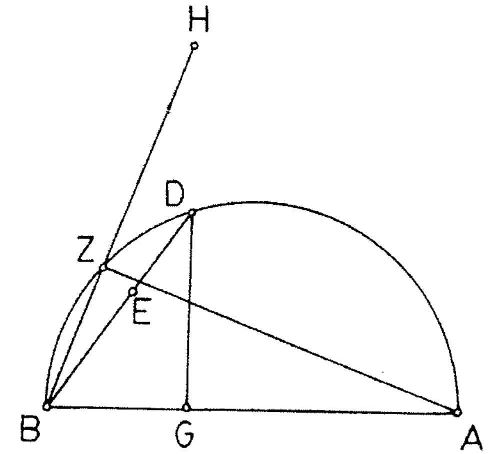
33.1. The ratio H:T is equal to the ratio between the perpendiculars of the pyramid and the cube by M XV:32.

34.1. Because D is on the circle, $BD^2=BG^2+GD^2=BG^2+BG \cdot GA=BG \cdot BA$.

34.2. Here the Mihrishah manuscript contains a correct reference to M XIII:8 (=El. XIII:4).

I say that the square of AB is equal to the sum of the squares of lines HB, BZ.

Proof[3]: We join AZ. Then angle AZB is (a) right (angle), III:30 (=El. III:31). Thus the square of AB, that is, the (sum of the) squares of AZ, ZB, I:47, is three times the square of HZ. Thus the (sum of the) squares of AZ, BZ is equal to the (sum of the) squares of HB, ZB. We subtract the common square of BZ, then the square of HB is equal to the square of AZ. Thus HB is equal to AZ. Thus the square of AB is equal to the sum of the squares of lines HB, BZ. That is what we wanted to prove.



prop. 34

35. We want to find the five lines proportional to the surfaces of the five figures.

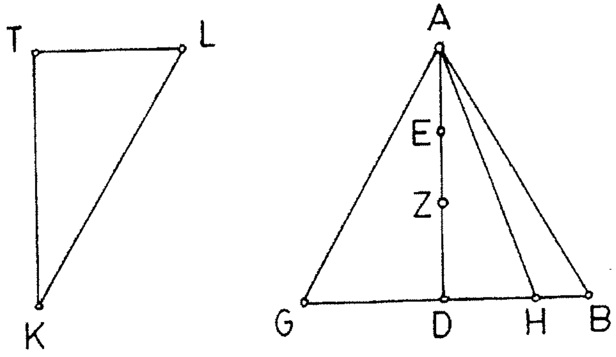
Thus we construct the equilateral triangle ABG. We draw the altitude AD, and we make AE one-third of it. We divide AD in extreme and mean ratio at Z, let ZD be the lesser part, and AZ the longer part. We cut off DH equal to DZ, and we join AH, then its square is equal to the square of AD plus the square of its lesser part, that is to say DH, that is DZ. We assume TK equal to AD. We find the line such that the square of it is equal to the sum of the squares of AD, AZ by drawing perpendicular TL, cutting it off equal to AZ, and joining

34.3. The Mihrishah manuscript gives a different proof: "The square of AB is three times the square of DB, that is HZ, and the (sum of the) squares of HB, BZ are three times the square of HZ. Thus the (sum of the) squares of HB, BZ is equal to the square of AB." This proof does not use the (inessential) fact that Z is on the circle.

KL; then the square of it is equal to the sum of the squares of TK, TL, that is to say, of AD, AZ.

Then the ratio of KL to AH is equal to the ratio of the surface of the dodecahedron to the surface of the icosahedron, XV:16. The ratio of AD (plus) DH as one line, that is the whole line plus its lesser part, to AD, the whole line, is equal to the ratio of the surface of the icosahedron to the surface of the octahedron, XV:15. The ratio of AD to GB is equal to the ratio of the surface of the octahedron to the surface of the cube, XV:12.

The ratio of DE to DA is equal to the ratio of the surface of the pyramid to the surface of the octahedron, XV:17. The ratio of AD to <GB is equal to the ratio of the surface of the octahedron to the surface of the cube, XV:12, so, ex aequali, the ratio of DE to> GB is equal to the ratio of the surface of the pyramid to the surface of the cube; thus, invertendo, the ratio of GB to DE is equal to the ratio of the surface of the cube to the surface of the pyramid. Thus we have found the lines[1] proportional to the ratios of the five surfaces. That is what we wanted to prove.



prop. 35

36. We want to find the lines proportional to the faces of the five solids.

35.1. It is not correct to say that we have found the five lines. We have KL:AH = $s_{12}:s_{20}$, (AD+DH):AD:GB:DE=$s_{20}:s_8:s_6:s_4$, and AD+DH≠AH, so that six lines are involved. The analogous problem for the areas of the faces is solved correctly in proposition 36, compare note 36.3.

Thus we put down line AB, and we divide it in extreme and mean ratio at G, such that its lesser part is GB. We extend AB, and we make BD equal to GB. We extend DA, and we cut off AE equal to two-thirds of AD. Then AD is three-fifths of ED.

We find a line divided in extreme and mean ratio, such that the square of ED is equal to the square of it plus the square of its lesser part, XV:34, let it be line ZH, and let its lesser part be TH. We find a line such that the square of it is equal to the square of line HZ plus the square of its greater part ZT, namely K. We put down line LM (equal to) two and a half times line AB. We extend ML, and we cut off LN equal to one-third times LM. Then line LM is three-fourths of MN. We bisect MN at C. We draw from point N perpendicular NS, and we assume it (extended) indefinitely on both sides. We make each of the angles (at) M one-third of a right angle, and we extend the two lines till they meet the perpendicular drawn from point N at points S, O. Then triangle MSO is equilateral. We cut off OF equal to one-third of SO.

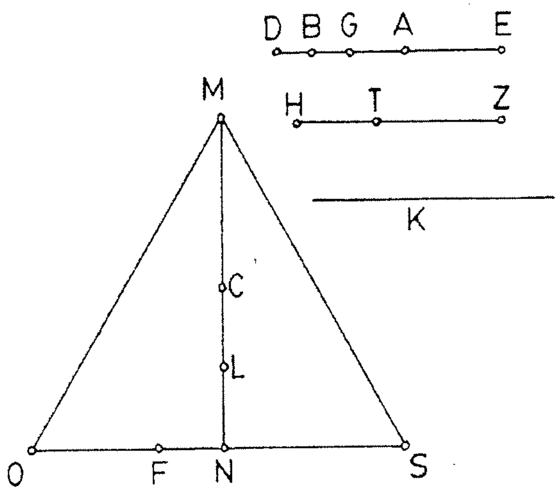
Then, since the ratio of the surface of the dodecahedron to the surface of the icosahedron is equal to the ratio of line K to ED, XV:16, the <ratio of the> face of the dodecahedron, that is one-twelfth of the surface of the dodecahedron, to the face of the icosahedron, that is <one-twentieth of its surface, is equal to the ratio of one-twelfth of K to one-twentieth of ED, which (ratio) is equal to the ratio of line K to three-fifths of line ED, V:14, that is, equal to the ratio of K to AD. The ratio of the surface of the icosahedron>[1] to the surface of the octahedron is equal to the ratio of AD to AB, XV:15. So the ratio of the face of the icosahedron, that is one-twentieth of its surface, to the face of the octahedron, that is one-eighth of its surface, is equal to the ratio of one-twentieth of AD to one-eighth of AB, that is to say, equal to the ratio of AD to two times AB plus half of it, that is ML, V:14. The ratio of the face of the octahedron to the face of the pyramid is equal to the ratio of ML to MN, by the potentiality of XV:17[2].

The ratio of the face of the cube to the face of the octahedron is equal to the ratio of two-thirds of SO, that is SF, to half of MN, that

36.1. The passage "one-twentieth of ... icosahedron" is only found in the Mihrishah ms.

36.2. A reference to M XV:2, dealing with the ratio of the faces, would have been more appropriate. Note that the text refers to XV:2 a few lines later.

is MC, XV:8. The ratio of the face of the octahedron to the face of the pyramid is equal to the ratio of half of MN to two-thirds of it, XV:2. Therefore, ex aequali, the ratio of the face of the cube to the face of the pyramid is equal to the ratio of two-thirds of SO to <two-thirds> of MN. Thus, invertendo, the ratio of the face of the pyramid to the face of the cube is equal to the ratio of two-thirds of MN to two-thirds of SO, that is to say, equal to the ratio of MN to the ratio of SO. Thus we have found the five lines proportional to the faces of the five figures. That is what we wanted[3].



prop. 36

37. We want to find the lines proportional to the five solids.

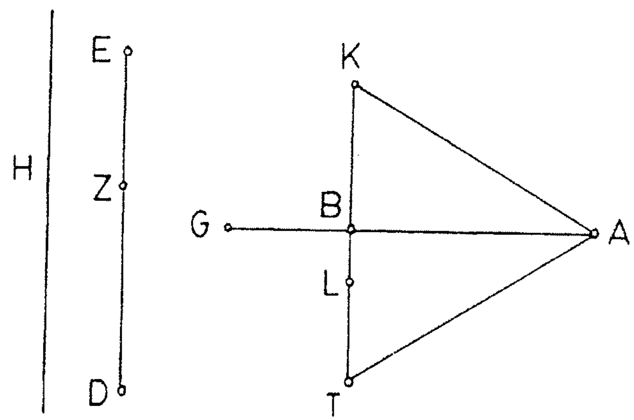
Thus we assume line AB, the side of the pentagon of a circle, and we extend it in a straight line, and we cut off from it BG equal to the side of its decagon. We find a line divided in extreme and mean ratio such that the square of AG is equal to the square of it plus the square of its lesser part[1]; let it be DE, and let its greater part be DZ. We find the line such that the square of it is equal to the sum of the squares of DE, DZ, namely H. We draw from point B a perpendicular to AB,

36.3. We have K:AD:ML:MN:SO = $f_{12}:f_{20}:f_8:f_4:f_6$, and therefore the following correct solution of the problem of prop. 35 (cf. note 35.1) K:ED:MC:MN/3:NS $=s_{12}:s_{20}:s_8:s_4:s_6$.

37.1. By M XV:34.

and we extend it indefinitely on both sides. We make each of the angles A one-third of a right angle, and we extend the two lines until they meet the perpendicular at points T, K. Then triangle ATK is equilateral. We make TL one-third of TK.

Then the ratio of H, that is (the line), the square of which is equal to the square of DE plus the square of its greater part, to the (line), the square of which is equal to the square of it (DE) plus the square of its lesser part (EZ), that is AG, is equal to the ratio of the dodecahedral solid to the icosahedral solid, XV:30. The ratio of AG to AB is equal to the ratio of the icosahedral solid to the octahedral solid, XV:29[2]. The ratio of TK to AB is equal to the ratio of the solid of the cube to the octahedral solid, XV:23. The ratio of AB to TL is equal to the ratio of the octahedral solid to the solid of the pyramid, XV:20. Thus, ex aequali, the ratio of TK to TL is equal to the ratio of the solid of the cube to the solid of the pyramid. Thus lines H, AG, AB, TK, TL are proportional to the five solids inscribed in one sphere. That is what we wanted to prove.



prop. 37

End of the fifteenth treatise; this is the end of the book.

37.2. Here the text uses the incorrect theorem M XV:29. To make the text mathematically correct, one has to change the second sentence of the proposition as follows: "Thus we assume line AG, the side of the pentagon of a circle, and we cut off from it AB equal to the side of its decagon."

The number of propositions in the book, except the lemmas and the different cases, is 516.

Appendix. Propositions XIII:10, 14 of the Revision of the Elements.

10. (For) every pair of different lines, each of which is divided in extreme and mean ratio, the ratio of one of the lines to the second (line) is equal to the ratio of its greater part to the greater part of the second (line), and equal to the ratio of the lesser part (of the first line) to the lesser part (of the second line)[1].

Example: Lines AB and GD are different. AB has been divided in extreme and mean ratio at E, and its greater part is AE. GD has been divided in extreme and mean ratio at Z, and its greater part is GZ. I say that the ratio of AB to GD is equal to the ratio of AE to GZ, and equal to the ratio of EB to ZD.

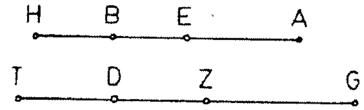
Proof: We extend each of the lines AB, GD in a straight line, and we make BH equal to EB and DT equal to ZD. Then the square of AE is equal to the product of AB and BE, and the square of GZ is equal to the product of GD and DZ, VI:17. Thus the ratio of the square of AE to the product of AB and EB is equal to the ratio of the square of GZ to the product of GD and ZD. Thus, alternately, the ratio of the square of AE to the square of GZ is equal to the ratio of the product of AB and BE to the product of GD and DZ. Thus the ratio of the square of AE plus four times the product of AB and BE, that is five times the square of AE, that is the square of AH, II:8, to the square of GZ plus four times the product of GD and DZ, that is five times the square of GZ, that is the square of GT, II:8[2], is equal to the ratio of the square of AE to the square of GZ, V:14 (=El. V:15). Thus the ratio of AH to GT is equal to the ratio of AE to GZ, VI:22. Thus the ratio of EH to ZT is equal to the ratio of AH to GT, V:19, that is to say, equal to the ratio of AE to GZ. But the ratio of EH to ZT is equal to the ratio of EB to ZD. Thus the ratio of AE to GZ is equal to the ratio of EB to ZD. Thus the ratio of AB to GD is equal to

[1] This is a form of the "ratio lemma" in Herz-Fischler, "Theorem XIV,**", see his pp. 7, 19-20.

[2] By El. II:8 $AE^2+4AB.BE = AH^2$ and $GZ^2+4GD.DZ=GT^2$ because EB=BH and ZD=DT.

the ratio of AE to GZ, and equal to the ratio of EB to ZD, V:18. That is what we wanted.

Book XIII, prop. 14



Book XIII, prop. 10

14. If the side of the hexagon is divided in extreme and mean ratio, its greater part is the side of the decagon[3]. Thus let AB be the side of the hexagon. We divide it in extreme and mean ratio at G, let AG be the greater part. I say that it is the side of the decagon. Proof: We extend AB towards D, and we make BD equal to AG. Then AD is divided in extreme and mean ratio, such that its greater part is AB, XIII:7 (=El. XIII:5). But AB is the side of the hexagon. Therefore BD, that is to say AG, is the side of the decagon, XIII:13 (=El. XIII:9).

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[3] For the history of this theorem see Herz-Fischler, "Theorem XIV,**" (Herz-Fischler discusses al-Maghribi on pp. 20-22).

In the Oxford manuscript, M XIII:14 has the proposition number 15, and M XIII:15 (=El. XIII:10) is proposition no. 14. The arrangement in the Utrecht ms. is correct, as is shown by the references to this proposition in Book XV of the *Revision*.

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