

Desargues' *Brouillon Project* and the *Conics* of Apollonius

by

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1. Introduction and summary

The *Brouillon Project d'une atteinte aux evenemens des rencontres du cone avec un plan* (Rough draft for an essay on the results of taking plane sections of a cone¹) by Girard Desargues (1591–1661) is one of the most fascinating works of pure geometry that were written in the seventeenth century. The title *Brouillon Project* will be abbreviated to *BrP* from now on. Desargues printed fifty copies of the *BrP* in 1639, but the work had little influence at the time, and by about 1680 it had completely disappeared. However, a critique of the *BrP*, written in 1640 by Jean de Beaugrand, survived.² In 1845 a manuscript copy of the *BrP* (copied in 1679 by La Hire from the printed version) was rediscovered by the French geometer Michel Chasles, and the text was published in 1864 by Poudra. The most recent edition of the *BrP* by Taton (1951) is based on one of the 50 copies of the original printed version of 1639, found in the Bibliothèque Nationale in Paris. References such as [T, 123] and [FG, 62] will be to page 123 of the Taton edition and to page 62 of the English translation by Field and Gray (1987).

In the *BrP* Desargues presents a theory of conic sections in a way that was fundamentally new in his time. The *BrP* contains many arguments on rectangles, squares, and proportions, based on the *Elements* of Euclid, but Desargues introduces a radical innovation, namely points and lines at infinity, as well as a number of new concepts, such as the equivalent of poles and polars with respect to

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a conic. He is especially interested in properties of figures that are projectively invariant. This terminology is modern, for no equivalents of the words projection and invariant occur in the *BrP*, and Desargues does not use mappings, at least not explicitly. Even so, Desargues was fully aware of the principle of projective invariance, and he emphasizes that once one has proved a property of one section of a cone, one can often prove a similar property for other sections of the cone [T 147, FG 110].

Most historians of mathematics who have studied the *BrP* have described its contents from a modern point of view, putting emphasis on the concepts and methods that were original in Desargues' work. The purpose of this paper is to study the relation between the *BrP* and the standard work on conic sections at the time of Desargues, the *Conics* of Apollonius (ca. 200 B.C.). The first four books of the *Conics* had been published in a Latin translation by Commandino in 1566.

Desargues certainly knew the *Conics*, as is shown by the following passage in the *BrP*:

There are propositions proved here, or consequences following from them, which combine together several of the propositions of the conics of Apollonius (sic), even from the end of the third book. [T 178, FG 141].³

In the *BrP*, Desargues does not presuppose that readers have any knowledge of the *Conics*, and he clearly did not want to imitate Apollonius. Jean de Beaugrand says that Desargues had told him that the *BrP* was far superior to the works of Apollonius.⁴ (Beaugrand says that this is the reason why he wrote his critique.) In a letter to Mersenne, Desargues even claims that he owes all his knowledge about conic sections to his own contemplations [T 83].

In this paper it will be shown that there is nevertheless a strong historical relationship between the *BrP* and the *Conics*.

Thanks to his points and lines at infinity, Desargues was able to derive in his *BrP* most of the Apollonian theory of diameters and ordinates in a much easier way than Apollonius, who used a large part of Book I of the *Conics* for that purpose. I summarize the relevant parts of the *Conics* in section 2, and in section 3 I describe Desargues' simplification. In section 3 I also discuss similarities in

terminology, and I list a number of implicit references to Apollonius in the *BrP*. These references show that Desargues liked to compare his own theory with that of Apollonius.

Surprisingly, in the modern historical literature I have not found a clear description of the way in which Desargues simplified the theory of Apollonius.⁵ This may be due to the fact that the *Conics* deals with a rather obscure subject. In the nineteenth century, its contents were often misunderstood, and the leading French historian of geometry Michel Chasles believed that Apollonius had only intersected the (oblique) cone with planes in special positions.⁶ Therefore Chasles believed that Desargues' main contribution was the fact that he had intersected the cone with arbitrary planes.⁷ Later authors have shown that this had in fact been done by Apollonius,⁸ but traces of the old confusion can still be found,⁹ and so far there does not seem to be any reasonably detailed description of Desargues' actual relation to Apollonius.

Sections 4 and 5 of this paper are about the parts of the *BrP* on foci. These parts of the *BrP* have received little attention in the modern literature. In section 4 I show that Desargues introduced the foci in essentially the same way as Apollonius (without saying so). His proofs are more elegant than, but nevertheless inspired by Apollonius' proofs in the *Conics*. Thus Desargues took not only most of his initial motivation from the *Conics* but also more raw material than has hitherto been realized.

From a letter to Mersenne we know that Desargues had also been working on an essentially new theory of the foci [T 84, quoted in Section 5]. In section 5 I argue that the *BrP* probably contains a trace of this research in the form of a long and mysterious proposition at the end. The fact that Desargues introduced the foci in almost the same way as in the *Conics* shows that he had not yet developed his last proposition into what he considered a satisfactory treatment of foci.

Not all the material in the *BrP* is mathematically or historically related to the *Conics*. In a paper in this issue of *Centaurus*, Kirsti Andersen studies the connection between points at infinity in the *BrP* and Desargues' work on perspective. Some of Desargues' inspiration may also have come from the *Mathematical Collection* of Pappus of Alexandria (ca. A. D. 250); for further details see section

3 below. Desargues also alludes to work by his contemporaries on the determination of maxima, minima, and tangents.¹⁰ Above all we should bear in mind that Desargues was a profoundly original mathematician, who transformed the mathematics of his predecessors into something completely new. Thus there are theorems in the *BrP* which fall totally outside the scope of the works of Desargues' predecessors.¹¹ By exhibiting the relations between the *BrP* and the *Conics*, I do not intend to detract from Desargues' originality. On the contrary, I believe that insight in the relations between Desargues and his sources can lead to even greater admiration for his mathematical genius.

Desargues' *BrP* was the first work on what is now called projective geometry, and therefore he could not assume that his readers knew anything about the subject. In this paper I do not presuppose that readers are familiar with modern projective geometry either, except in a few passages which are not essential to the main argument. A beautiful introduction to modern projective geometry can be found in Coxeter's *Introduction to Geometry*.

2. Summary of Apollonius' theory of diameters and ordinates

In this section I summarize the parts of Book I and II of the *Conics* that are necessary for understanding how Desargues simplified the theory of Apollonius. The Greek text of the *Conics* was edited by Heiberg, there is a literal French translation by Ver Eecke, and there are transcriptions (in modernized notation) into English by Heath and into German by Czwalina.

Apollonius introduces the cone as follows. (Figure 1) Consider a circle Γ and a point A not in the plane of Γ . A conic surface is the collection of the straight lines through A and the points of Γ . The straight lines are indefinitely extended on both sides, and therefore the conic surface consists of two parts, which are joined by the point A . The cone itself is the solid contained between A , the surface and the plane of Γ . The circle Γ is called the base of the cone. The straight line through A and the centre M of Γ is called the axis of the cone. The cone is called a right cone if the axis is perpendicular to the plane of Γ , otherwise it is called an oblique cone.

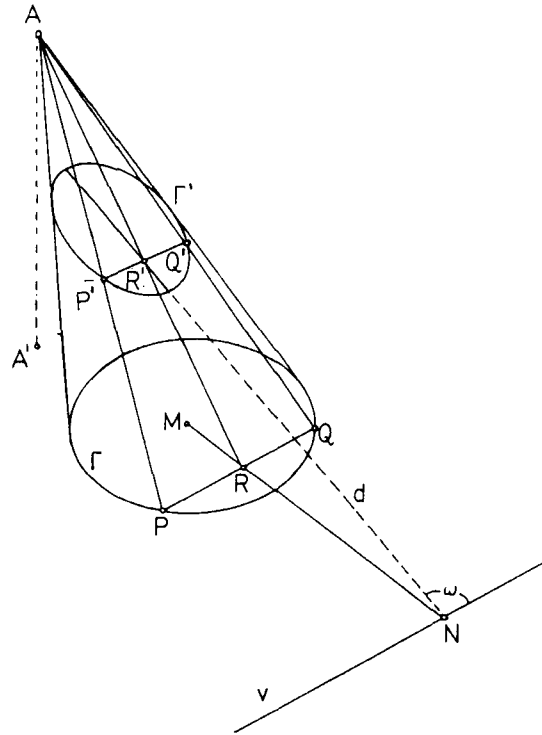


Figure 1.

Consider the intersection Γ' of a plane not through A and the conic surface. If the plane of Γ' is parallel to the plane of Γ , Γ' is also a circle. Thus there is one series of parallel planes which intersect the cone in circles. For the oblique cone, Apollonius shows in *Conics* 1:5 that there is exactly one other series of parallel planes that intersect the conic surface in circles. All other planes intersect the conic surface in a curve that is not a circle. Apollonius calls the intersection of such a plane with one part of the conic surface a conic section. He considers the two branches of a modern hyperbola as two separate conic sections, which he calls “opposite branches”. The fact that his terminology differs from modern terminology has no further consequences, because Apollonius generalizes almost all concepts and theorems on conic sections (in his sense of the word) to the two opposite branches.

The concepts of diameter and ordinate play a central role in the theory of Apollonius. We call a *chord* of a curve a straight segment joining two points on the curve. A *diameter* of a curve is a straight line which bisects all chords of a curve having a certain fixed direction (Figure 2). The halves of the chords are called the *ordinates* corresponding to the diameter. The constant angle ω between a diameter and its ordinates is called the *angle of ordinates*. If ω is a right angle, the diameter is called an *axis* of the curve. Because Apollonius wants to use similar concepts for the two opposite branches, he defines a *diameter of a pair of curves* as a straight line which bisects all segments with an endpoint on each curve and having a given direction, and so on.

For the circle one can prove on the basis of Euclid's *Elements* III:3 that every line through the centre is a diameter in the sense of Apollonius, and that the ordinates are perpendicular to the diameter to which they correspond. Thus ω is always a right angle, and every diameter is an axis.

Apollonius proves as follows that every conic section Γ' which is not a circle has one diameter d (*Conics* 1:7, Figure 1): Let the plane of Γ' meet the base plane, that is the plane of Γ , at line v . Drop perpendicular MN from the centre M of the circle onto v . Let plane AMN meet the plane of Γ' in line d . Now consider in Γ' a chord $P'Q'$ parallel to v , and suppose that $P'Q'$ meets d at R' . Draw AP' , AQ' , AR' and extend them, if necessary, to meet the base plane at P , Q and R . Then PQ is a chord of the circle Γ , and PQ

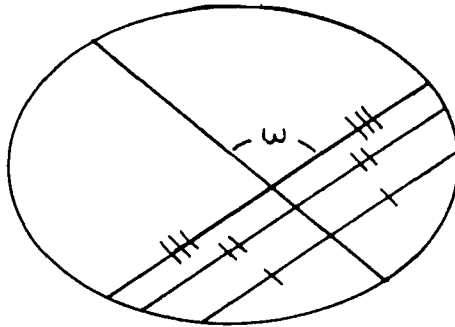


Figure 2.

intersects the diameter MN at R . Line PQ is parallel to v and hence perpendicular to MN . Therefore $PR = RQ$. Since $PQ \parallel v \parallel P'Q'$, we have $PR:RQ = P'R':R'Q'$, so $P'R' = R'Q'$. Thus d is a diameter of the conic section Γ' , the so-called *principal diameter* or *original diameter*. This concept will play an important role in the sequel.

If the cone is a right cone, plane AMN is always perpendicular to the base plane, so ω is a right angle. If the cone is oblique, the centre M of the circle is different from the perpendicular projection A' of the apex on the base plane. Clearly ω is a right angle if and only if A' , M and N are on a straight line.

Apollonius often calls the intersection of plane AMN with the cone the “triangle through the axis”, the axis of the cone being AM . Chasles thought that the term “triangle through the axis” was only used for the intersection of plane AMA' with the cone. He believed that Apollonius only intersected the cone with planes perpendicular to AMA' , in which case ω is a right angle and the principal diameter is an axis. However, Apollonius explicitly says in *Conics* 1:7 that ω need not be a right angle.

In *Conics* 1:11–14 Apollonius proves the so-called fundamental property of each type of conic section. In all three cases this property involves a segment p that is called the *latus rectum* (erect side). Apollonius draws the *latus rectum* as a segment perpendicular to the diameter and beginning at a point of intersection of the diameter with the conic. The length of the *latus rectum* depends on the cone and the intersecting plane in a way which does not concern us here.

First suppose that the diameter d intersects the conic surface in only one point B . Then every ordinate PR and its abscissa RB satisfy

$$PR^2 = p \cdot RB . \quad (2.1)$$

The conic is called a parabola.

Now suppose that d intersects the conic surface at two points B and D , but that segment BD is outside the surface. Then for every ordinate PR and corresponding abscissa RB

$$PR^2 = p \cdot RB + (p/BD) \cdot RB^2 . \quad (2.2)$$

The conic surface intersects the plane in what Apollonius considers to be two curves, namely the two “opposite branches”. He calls each curve a hyperbola.

If d intersects the conic surface at two points B and D on the same side of A , he proves

$$PR^2 = p \cdot RB - (p/BD) \cdot RB^2 . \quad (2.3)$$

Apollonius calls the conic section an ellipse (Figure 3).

In this description I have used modern notation. For later reference I also render (2.3) in the manner of Apollonius. In *Conics* I:13 he says that the square of the ordinate (PR) is equal to rectangle $BRST$ in Figure 3, that is the rectangle “applied” to the *latus rectum* $p = BU$, with width RB , and “deficient by” a rectangle $TSYU$ similar and similarly situated to the *figure* of the ellipse, that is the rectangle contained by the *latus transversum* (“transverse side”) BD and the *latus rectum* (“erect side”) BU .

Apollonius then proves a great number of theorems on the basis of (2.1), (2.2) and (2.3) only, without any stereometric considerations. I only render those that are important in connection with Desargues, and I do not discuss the proofs, which are in general complicated, and which are based on preliminary propositions that will not be mentioned here. For brevity I only discuss the ellipse. Apollonius

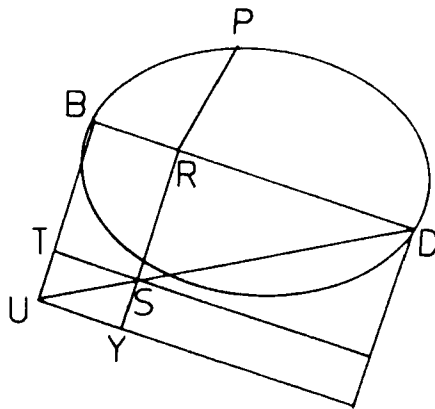


Figure 3.

proves analogous theorems for the parabola, the hyperbola and even the circle.

Consider an ellipse with the principal diameter d as constructed above. Let d intersect the ellipse at B and D , and let C be the midpoint of BD . Point C is called the centre of the ellipse. Then we have the following.

1. (*Conics I:15*, Figure 4) Let $B'C$, CD' be the ordinates of BD through C . Then $B'CD'$ is also a diameter of the ellipse, and its ordinates are parallel to BD . Line $B'D'$ is called the diameter conjugate to BD .

The ordinates corresponding to $B'D'$ also satisfy an "equation" like (2.3). Define the new *latus rectum* p' by $p \cdot p' = BD \cdot B'D'$. If $P'R'$ is an ordinate corresponding to $B'D'$, then $P'R'^2 = p' \cdot R'B' - (p'/B'D') \cdot R'B'^2$.

2. (*Conics I: 17, 32*, Figure 4) The line through B parallel to the ordinates of BD does not meet the ellipse in another point. Every other straight line through B meets the ellipse again. This is to say that the ellipse has a tangent at B parallel to the ordinates of BD . Hence the ellipse also has tangents at D and (by *Conics I:15*, cf. 1. above) at B' and D' .

3. (*Conics I: 34, 36*, Figure 5) Let B_1 be a point on the ellipse different from B, D, B', D' , consider a straight line through B_1 which intersects BD at H , and let B_1K be the ordinate corresponding to BD . If $BH:HD = BK:KD$, B_1H does not intersect the ellipse in another point, and conversely. Thus the ellipse has a tangent at B_1 .

4. (*Conics I: 47*) Let B_1C meet the ellipse again at D_1 . Then B_1D_1

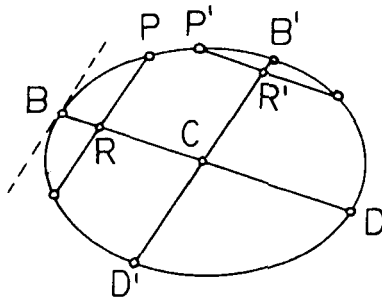


Figure 4.

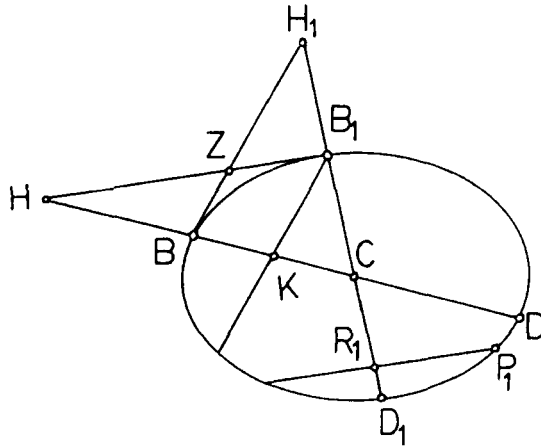


Figure 5.

is a diameter of the ellipse, and its ordinates are parallel to the tangent at B_1 . Thus all straight lines through the centre C of the ellipse are diameters.

5. (*Conics I: 50*) The ordinates corresponding to B_1D_1 satisfy an equation which resembles (2.3), as follows. Apollonius defines the new *latus rectum* p_1 by $p_1/2B_1H = ZB_1/B_1H_1$ in Figure 5, where Z is the intersection of the tangents BH_1 and B_1H . He proves for every ordinate P_1R_1 corresponding to the diameter B_1D_1

$$P_1R_1^2 = p_1 \cdot R_1B_1 - (p_1/B_1D_1) \cdot R_1B_1^2 .$$

6. Using 5, he is able to show that every ellipse has an axis. He then shows that any ellipse can be obtained as the intersection of the plane of the paper with a right cone (*Conics I: 54–56*).

Similarly, Apollonius proves that all parallels to the principal diameter of a parabola are also diameters of the parabola (*Conics I: 46*), that the ordinates satisfy a relation like (2.1) (*Conics I: 49*), and that the parabola has one axis (*Conics I: 53*). For the hyperbola, Apollonius defines the centre as the midpoint of BD , where B and D are the points of intersection of the principal diameter with the hyperbola and the opposite branch. He proves that every line through the centre that intersects the hyperbola is a diameter of the

hyperbola (*Conics* 1:47), and that the corresponding ordinates satisfy an equation like (2.2) (*Conics* I:50).

Apollonius introduces the asymptotes of the hyperbola in *Conics* II:1–19, and in *Conics* II:20 he proves among other things that every line through the centre of a hyperbola which does not intersect the hyperbola and which is not an asymptote is a diameter of the opposite branches.

Apollonius' theory of diameters and ordinates is ingenious but also curious. At the beginning he proves that every conic section has one principal diameter, and after many cumbersome planimetric arguments he concludes that there are infinitely many diameters. He points out (in *Conics* 1:51) that (2.1), (2.2) and (2.3) hold not only for the principal diameters but also for all other diameters, so all other diameters have exactly the same properties as the principal diameter. He then shows that any conic section (i.e. any intersection of an oblique cone with the plane) can also be obtained as the intersection of a right cone with the plane. Then the "principal diameter" is an axis, and thus the same conic section can have different "principal diameters" (in fact, it is easy to see that every diameter can be a principal diameter). Now the concept of "principal diameter", which seemed so important at the beginning, loses its meaning completely, and it is natural to ask whether the principal diameter cannot be dispensed with altogether. In other words, one would like to have a stereometric construction like Figure 1, in which all diameters appear at the same time.

There is one difficulty here. In Figure 1 the principal diameter BD was obtained as the intersection of the plane of Γ' with the plane through the apex A and a diameter MN of the base circle Γ . One can express this in another way by means of the modern concept of projection. For any point X in the base plane, call X' the intersection of line AX with the plane of Γ' . The mapping which sends X to X' is called a *central projection* with *centre* A , and we will call X' the *projection* of X (where it is understood that the centre of projection is the apex of the cone). In this modern terminology, the diameter d of the conic section Γ' is the projection of the diameter MN of the circle Γ . However, the centre C of Γ' is generally not the projection of the centre M of the circle, and therefore the other diameters of Γ' are not the projections of the other diameters of the

circle. Thus we cannot obtain the other diameters of Γ' in Figure 1 in the same way as the principal diameter.

Desargues found a way around this difficulty. In his letter of April 4, 1638 to Mersenne, he says that he had found a general theory of conic sections

without using for that any of the triangles through the axis, and without making a distinction between a principal diameter and the others [i.e. the other diameters] ... [T 84].

This theory is found in the *BrP* and we will now turn to it.

3. Desargues' simplification of the Apollonian theory

This section begins with a simplified description of the main ideas behind Desargues' generalization of the theory of Apollonius, and is followed by some brief comments on the actual proofs in the *BrP*. I then discuss the implicit references to the *Conics* in the *BrP*.

In the *BrP*, Desargues proves generalizations of the following theorems of Apollonius:

- Every line through the centre of an ellipse is a diameter of the ellipse.
- Every line parallel to the principal diameter of a parabola is a diameter of the parabola.
- Every line through the centre of a hyperbola, except the asymptotes of the hyperbola, is a diameter of the (single-branch) hyperbola or a diameter of the opposite branches.

Desargues' generalization involves the introduction of points and lines at infinity. He does not give any justification for their existence, and his description is not very clear. From what he says, one can conclude that he assumed the following properties [T 100–101, FG 70–71]:

- On every straight line, there is exactly one point at infinity. This point lies at infinite distance in both directions. (This would no doubt seem absurd to his contemporaries.)
- Two parallel straight lines have a common point at infinity.
- If two straight lines are not parallel, their points at infinity are

different. The points at infinity of all the straight lines in one plane lie on a straight line, the line at infinity.

– Two parallel planes have the same line at infinity.

In the rest of this paper I will call points and straight lines which are not at infinity “ordinary” points and lines. A “line” is always a straight line.

Desargues considers the two “opposite branches” of Apollonius as one hyperbola, which is divided at infinite distance into two parts [cf. T 137, FG 102]. Now the above-mentioned theorems of Apollonius can be summarized as follows: For every conic section there exists a point (the centre) such that all lines through this point are diameters of the conic section (except possibly the asymptotes of a hyperbola).¹²

The points and lines at infinity made it possible for Desargues to generalize the Apollonian concepts of diameter and ordinates. I begin by quoting two theorems of Apollonius, to which Desargues does not refer anywhere, but which may have been among the sources of inspiration for his generalization. My presentation of the theorems and my notation in Figures 6 and 7 suggest the generalization by Desargues, and they are therefore historically misleading. The obscure notation of Apollonius made the generalization more difficult for Desargues.

The two theorems in the *Conics* are the following:

a. *Conics* III:37 (Figure 6): Let PA , PB be two tangents to a conic section, and draw a line through P which intersects AB at Q and the conic at X and Y . Then $PX:PY=QX:QY$.

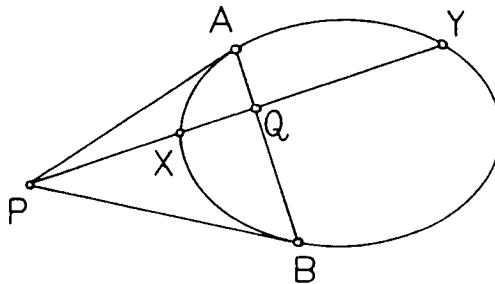


Figure 6.

b. *Conics* III:38 (Figure 7): Let CA, CB be two tangents to a conic section, let P be the midpoint of AB , and draw a line $YPXQ$ which intersects the conic at Y and X and the line through C parallel to AB at Q . Then $PX:PY=QX:QY$.

We will compare Figures 6 and 7 with Figure 8, which displays a conic section, a diameter d and a series of parallel chords $XY, X'Y'$ etc., which are bisected by the diameter at Q, Q' etc; thus $QX = QY$ etc. For Desargues, all ordinates $XQ, X'Q'$ etc. pass through a fixed point P at infinity.

In order to establish a connection between all straight lines $PXQY$ and $YPXQ$ through a fixed point P in Figures 6, 7, and 8, Desargues introduced the following terminology for configurations of points on a straight line. Two couples of points X, Y and P, Q on a line are called *four points in involution* if (a) P, Q, X, Y are four different ordinary points and $PX:PY=QX:QY$ or if (b) Q, X, Y are three different ordinary points, P is at infinity and $QX=QY$.¹³ Note that if the two points X and Y are fixed, Q is uniquely determined by P ; if P tends to infinity, $PX:PY$ tends to 1:1, so (b) is a limiting case of (a). For Apollonius, the situations in Figures 6 and 7 are essentially different from the situation in Figure 8, but the point P at infinity in Figure 8 made it possible for Desargues to see an analogy.

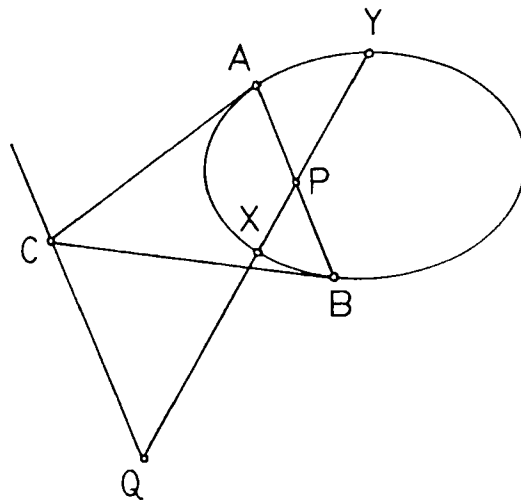


Figure 7.

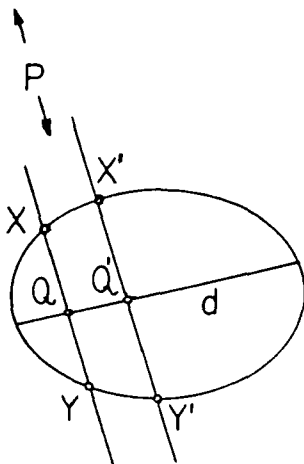


Figure 8.

Now consider a conic section, a fixed point P , a fixed line l , and arbitrary lines through P which intersect the conic at X and Y and l at Q . If for any such line, P, Q and X, Y are four points in involution, Desargues calls P the *butt*, l the corresponding *transversal*, and the lines PQ the *ordinates* [T 138-9, FG 103]. In Figure 6, P is the *butt* of the *transversal* AB , in Figure 7, P is the *butt* of the *transversal* QC and in Figure 8, P (at infinity) is the *butt* of the *transversal* d . The words *butt* and *transversal* correspond to the modern *pole* and *polar*, and from now on, I will use these more familiar equivalents. In modern mathematics there is no special term for the *ordinates* $PXQY, QXPY$ in Figures 6 and 7. Desargues does not say why he used this term, but because his *ordinates* are generalizations of the *ordinates* of Apollonius (the lines XQ, QY in Figure 8), his choice of terminology was probably motivated by the *Conics*.¹⁴

The two theorems *Conics* III:37, 38 suggest¹⁵ that every ordinary point P which does not lie on the conic and which is not the centre of it, is the pole of exactly one polar, and that every line which is not a tangent to the conic and which does not pass through its centre is the polar of exactly one pole P . Figure 8 suggests that a point P at infinity is a pole for a polar which is a diameter of Apollonius, that is to say, a line through the centre of the conic. If

P is the centre of the conic, we consider a straight line through P , which intersects the conic at X and Y , and we call Q the point at infinity on this line (compare Figure 7). Since $PX = PY$, the pairs X, Y and P, Q are four points in involution, so the centre is seen to be the pole of the line at infinity. Passage no. 2 quoted at the end of this section shows that Desargues was fully aware of this interpretation. Finally one may ask what happens with points P on the conic section. Desargues considers such a point P and the tangent to the conic at P to be a pole and the corresponding polar [T 141]. Now every point is the pole of a polar and every line is the polar of a pole.

The two theorems *Conics* III:37, 38 may have helped to inspire Desargues, but he could not assume them to be known, because he wanted to set up the theory independently of Apollonius.

Apollonius proved that

(A) for every conic section there is a point (the centre) such that all lines through that point are diameters of the conic.

As we will see, Desargues derives (A) from two other theorems:

(A₁) For every conic section, every point of the plane (including points at infinity) is the pole of a line, and every straight line (including the line at infinity) is the polar of a point.

(A₂) If a point P lies on a line q , the pole of q lies on the polar of P .¹⁶

The proof of (A) from (A₁) and (A₂) is as follows. Define the centre as the pole of the line at infinity by (A₁). Let d be a line through the centre. Line d is the polar of a pole D by (A₁), and D lies on the line at infinity by (A₂). Therefore d is an Apollonian diameter, as a consequence of the definition of pole and polar, and the corresponding ordinates are the parallels which meet at the infinitely distant point D , q.e.d.

We now turn to the proofs of (A₁) and (A₂). These two theorems can easily be proved for a circle Γ on the basis of the *Elements* of Euclid, in a much simpler way than Desargues proved them. Let M be the centre of a circle and choose a point $P \neq M$, outside or inside the circle (Figure 9). Draw MP and construct H on MP or on MP extended on the side of P such that $MP \cdot MH = r^2$, where r is the radius of the circle. Draw the perpendicular to HM through H . If PM intersects the circle at K and N , one can conclude $PK:PN =$

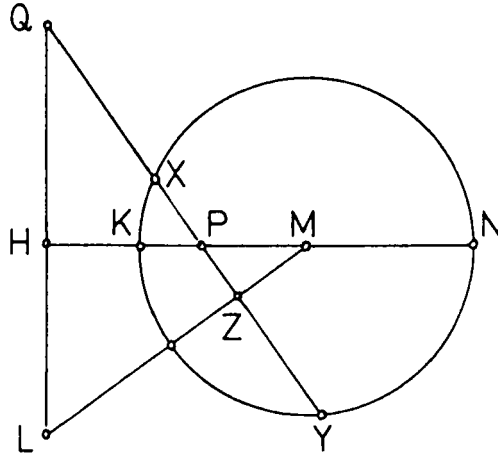


Figure 9.

$HK:HN$ from $MP \cdot MH = r^2 = MK^2$ by an argument involving proportions which was common in Greek mathematics (such an argument is also used, for example, in the proof of *Conics* I:37). Draw a line through P which intersects the perpendicular at Q and the circle at X and Y . Pappus of Alexandria proves in Book VII of the *Collection* that $PX:PY = QX:QY$, for P outside the circle in Prop. 154 (ed. Jones, p. 284) and for P inside the circle in Prop. 161 (ed. Jones p. 290). The proofs are based on the *Elements* III:35. Thus P is the pole of line HQ . By similar reasoning we show that any ordinary line which does not pass through the centre and which is not a tangent to the circle is the polar of a point. By the *Elements* III:3, any diameter of the circle bisects the chords perpendicular to it. Therefore any diameter is a polar of the infinitely distant common point of the rectilinear extensions of these chords, and vice versa. The centre of the circle is trivially seen to be the pole of the line at infinity. Finally we can define any point P on the circle to be the pole of the tangent to the circle at P . Thus (A_1) is proved. To prove (A_2) , we suppose that point P lies on a straight line q . Drop perpendicular MZ to q and extend it, if necessary, to meet the polar of P at L (compare Figure 9). Triangles MPZ and MLH have equal angles, so that $MP:MZ = ML:MH$, whence $MZ \cdot ML = MP \cdot MH =$

r^2 . Therefore L is the pole of q , which had to be proved. Desargues derived (A_1) and (A_2) from a more exciting and more general theorem, as we will see below.

We will now show that in modern terms pole and polar are projectively invariant concepts. In order to make this clear, we suppose that planes V and V' intersect a cone with apex T in a circle Γ and a conic section Γ' (Figure 10). For any point P in V , call the *projection* of P the intersection P' of TP with V' , as in Section 2. Then the projection of a line p in V is a line p' , which is the intersection of V' and the plane through T and p . The projection of Γ is of course Γ' . Now suppose that P is the pole of line p with respect to Γ . Let $PXQY$ be a straight line which intersects Γ at X and Y and p at Q . The projection of $PXQY$ is a straight line $P'Q'X'Y'$ which intersects Γ' at X' and Y' and p' at Q' . Because P is the pole of p , the pairs P, Q , and X, Y are four points in involution. By a

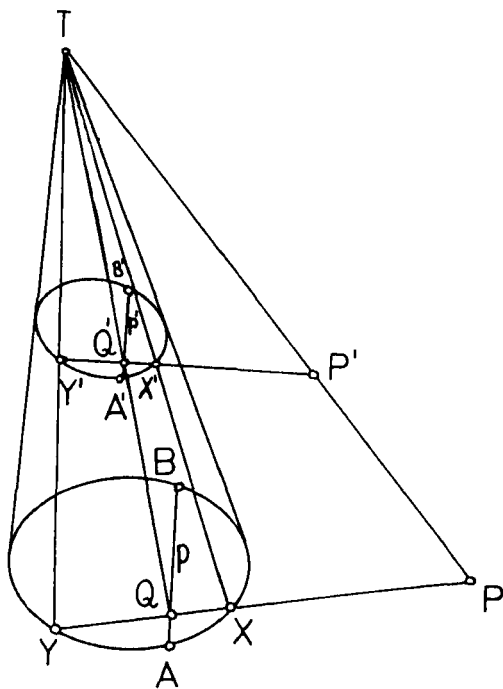


Figure 10.

theorem in elementary geometry, which Desargues proves in his *BrP*, it follows that P' , Q' and X' , Y' are also four points in involution.¹⁷ Therefore the projection P' is also the pole of line p' with respect to Γ' . Thus theorems (A_1) and (A_2) , which have been proved for the circle Γ , are also valid for the conic section Γ' .

At the end of Section 2 it was explained that (A) could not immediately be generalized from a circle to a conic section, because the projection of the centre of the circle at the base of the cone is generally not the centre of the conic section. Desargues overcame this difficulty by replacing the Apollonian diameter (which is not a projectively invariant concept) by something more general which is projectively invariant.

This concludes my description of the background to Desargues' simplification. I now make some remarks on the structure of the *BrP* and Desargues' actual proofs.

Desargues' theory of configurations of points on a straight line is more general than is suggested by the preceding description. For later use I summarize part of his theory and compare it with the modern concept of involution.

Desargues calls a configuration consisting of a point A and pairs of points B, H ; C, G ; D, F , etc. a *tree* if $AB \cdot AH = AC \cdot AG = AD \cdot AF$, etc. and if A is always between the points of each pair, or is never between the points of each pair (Figure 11). He calls A the *stump* of the tree, and the other points *knots*. He proves that

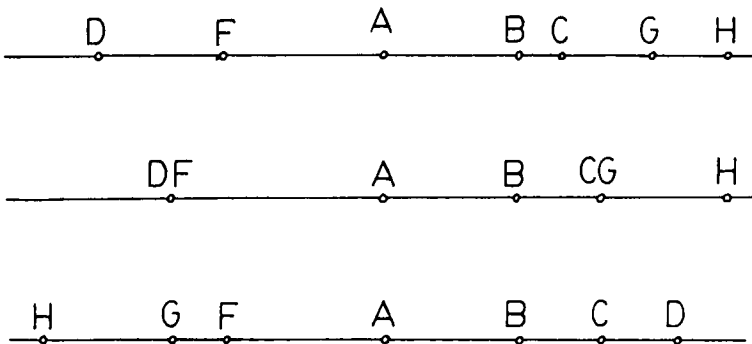


Figure 11.

$$(GD \cdot GF)/(CD \cdot CF) = AG/AC \quad (3.1)$$

and $(GD \cdot GF)/(CD \cdot CF) = (GB \cdot GH)/(CB \cdot CH) . \quad (3.2)$

He calls two pairs of points $B, H; C, G$ *mixed* (meslez) if one point of each pair is between the two points of the other pair. He calls the two pairs *unmixed* (demeslez) if this is not the case, i.e. if no point of one pair lies between the points of the other pair, or if both points of one pair lie between the points of the other pair.

He then considers six points (more precisely: three pairs of points) $B, H; C, G; D, F$ on a straight line. If these six points satisfy (3.2) and are located in such a way that either all pairs are mixed (Figure 11, bottom) or all pairs are unmixed (Figure 11, top), then Desargues says that these six points *form an involution*. [T 110, FG 77]

Thus every three couples of points on a tree form an involution. Desargues also proves the converse: if six points $B, H; C, G; D, F$ form an involution, there is a point A such that $A; B, H; C, G; D, F$ form a tree, and this *stump* A (and hence the tree) is uniquely determined.

Four points in involution are defined as six points in involution of which two pairs coincide, that is to say the case where $C = G$ and $D = F$ (Figure 11, middle). Because $(HC \cdot HG)/(BC \cdot BG) = (HD \cdot HF)/(BD \cdot BF)$, we have $HC^2/BC^2 = HD^2/BD^2$, so that $HC/BC = HD/BD$, as above.

Desargues also discusses the point at infinity in connection with trees. In the tree of Figure 11, the stump A and the point at infinity are a pair of knots just like an ordinary pair B, H [T 115, FG 81]. Although he does not explicitly say so, he clearly implies that the point at infinity and five ordinary points can be six points in involution (compare [T 119–120, FG 84–85]). This means that if we write J for the point at infinity, the three pairs $C, G; D, F$ and A, J are six points in involution if (3.1) is satisfied and if A is between both pairs of points C, G and D, F or not between any of these pairs. Note that (3.1) is a limiting case of (3.2) if B tends to A and H to infinity.

In modern projective geometry, an involution is a mapping σ of a straight line (including the point at infinity) onto itself which is a projectivity and its own inverse. Although the modern concept

is different from Desargues' definition, every tree of Desargues determines exactly one involution σ in the modern sense. For the tree with stump A and knots B, H ; etc. as in Figure 11, the corresponding modern involution is defined by $\sigma(A)=J, \sigma(J)=A, \sigma(B)=H, \sigma(H)=B, \sigma(C)=G, \sigma(G)=C$, where J is the point at infinity. For any point X on the line, $\sigma(X)=Y$, for the point Y on the line such that $AX \cdot AY=AB \cdot AH$ and such that A is between X and Y if and only if A is between B and H . Conversely, if we have an involution σ on a straight line, with point at infinity J , we can construct a tree in the sense of Desargues by choosing $A=\sigma(J)$, and different ordinary points $B, C, H=\sigma(B), G=\sigma(C)$, etc. The proof of the equivalence is a simple exercise in projective geometry.

Six points $B, H; C, G; D, F$ on a straight line are in involution in the sense of Desargues if and only if there is a modern involution σ of the straight line such that $\sigma(B)=H, \sigma(C)=G$ and $\sigma(D)=F$. Four points $B, H; C, D$ are in involution if and only if there is a modern involution τ with fixed points C and D such that $\tau(B)=H$. More details on the relation between Desargues' involutions and the modern concept can be found in Lenger.

We now return to the discussion of the *BrP*. After studying points on the straight line, Desargues deals with straight lines in the plane. He proves the following theorem (Figure 12). If the six points $B, H; C, G$ and D, F are in involution, if K is a point not on the same line, and if BK, CK, DK, FK, GK and HK intersect another line at points b, c, d, f, g and h , then the six points $b, h; c, g$ and d, f are also in involution [T 126–130, GF 92–95]. Any of the points or lines involved may be at infinite distance. Thus if six points are in involution, their projections on another straight line are also in involution.

Desargues now turns to conic sections. He considers the cylinder as a cone with apex at infinity, and for him the cone and the cylinder are two "subgenera" of one "genus", the *roll*. He therefore talks about the *section of a roll* instead of a conic section. He then defines butt (pole), transversal (polar), ordinates, and a number of other concepts that we need not discuss here. The following theorem plays a crucial role (Figure 13). Let quadrangle $DCBE$ be inscribed in a conic section, and let a straight line intersect the conic at L, M and the three pairs of opposite sides of the quadrangle (in Desargues'

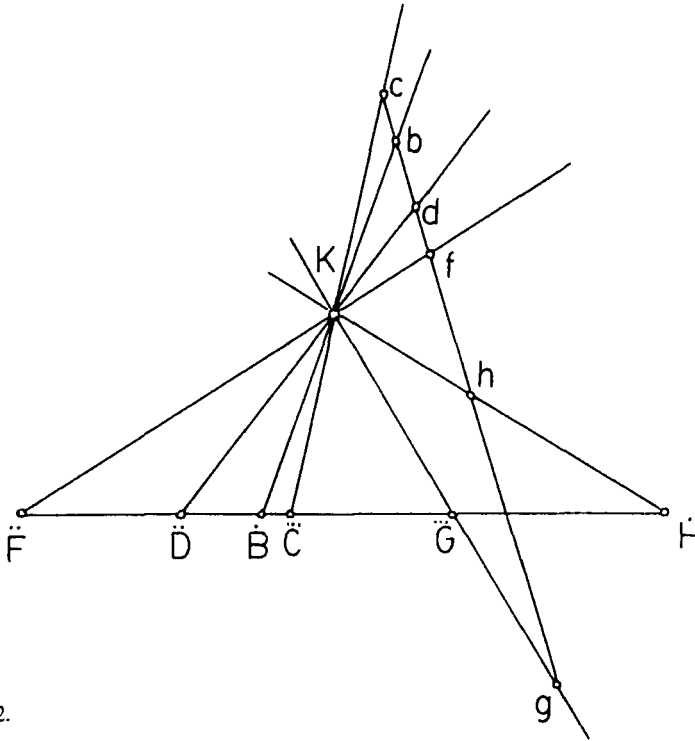


Figure 12.

words: the three “pairs of marker lines”) in $P, Q; I, K; G, H$. Then

$$\begin{aligned} (QI \cdot QK) / (PI \cdot PK) &= (QG \cdot QH) / (PG \cdot PH) \\ &= (QL \cdot QM) / (PL \cdot PM) . \end{aligned}$$

Desargues says that $Q, P; I, K; G, H; L, M;$ are “pairs of knots of an involution” [T 143, FG 106]. In modern terms, there is an involution σ such that $\sigma(P)=Q, \sigma(I)=K, \sigma(G)=H, \sigma(L)=M$. Desargues first proves the theorem for a circle, using proportions and the theorem of Menelaus. Because he has proved that the projections of six points in involution are also six points in involution, the theorem is valid for any conic section.

Historians of mathematics, beginning with Jean de Beaugrand in 1640, have argued that Desargues found some inspiration in Book VII of Pappus of Alexandria’s *Mathematical Collection* [cf. T 143

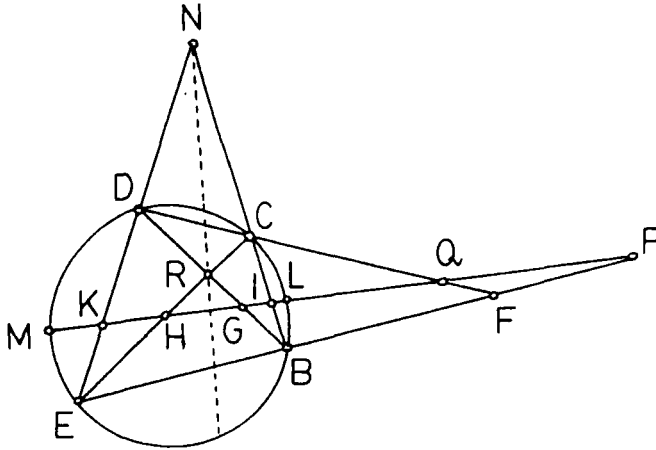


Figure 13.

note 55]. According to the converse of proposition 130 of that work (ed. Jones, p. 264) we have

$$(QI \cdot QK)/(PI \cdot PK) = (QG \cdot QH)/(PG \cdot PH)$$

in a figure which one obtains from Figure 13 by deleting L , M and the conic. It is however a large step from the the *Collection*, which involves straight lines only, to Desargues' on conics.

Desargues deduces the following corollary:

(Figure 13, the notation is mine) Let a quadrangle $BEDC$ be inscribed in a conic, and let DE and BC intersect at N , DB and EC at R and DC and EB at F . Then F is the pole of RN . It follows (by a change of notation) that R is also the pole of NF , and N is also the pole of FR . It is now easy to see that any point not on the conic section is the pole of a line, and that any line which is not tangent to the conic section, is the polar of a point (theorem A_1 , page 16). We also conclude that if F is any point, and N is a point on its polar, then F is on the polar of N . Therefore (A_2) is proved [T 152–154, FG 115–117].

Desargues proves more than is necessary for generalizing the Apollonian theory of diameters and ordinates. In particular, the

theory of six points in involution does not seem to have been directly motivated by the *Conics*. Nevertheless, parts of the *BrP* can be considered to be a reaction to the *Conics* of Apollonius. The following five quotations from the *BrP* show that Desargues liked to compare his terminology, theorems and methods with those of Apollonius.

In passages nos. 1 and 2 Desargues criticizes Apollonian terms:

[1] The most remarkable properties of the sections of a roll are common to all types, and the names ellipse, parabola and hyperbola have been given them only on account of matters extraneous to them and to their nature. [T 138, FG 102]

[2] Since in a plane the point called the centre of a section of a roll is only one case among the innumerable butts of ordinances of straight lines, we must never speak of the centre of section of a roll. [T 140, FG 104]

In the following three examples, Desargues mentions a “special case” of a theorem, without giving further information. These special cases are all theorems in Apollonius’ *Conics*.

[3] After proving that every point is the pole of a line, and every line is the polar of a point, Desargues says:

from which it follows that in the plane of a general section of a roll any straight line is, with respect to the section, a transversal (polar) of straight lines ordinate to some butt [i.e. pole], a diametral, otherwise a diametransversal, being only one case of this. And that any point is, with respect to this section, the butt of some straight lines, ordinates of a transversal, the butt of the diametrals being only one case of this. [T 154, FG 117].

This generalizes *Conics* I:46–47, discussed in Section 2. The diametral and diametransversal are defined as follows.

Every straight line which bisects a figure is called a diametral of the figure, and a diametransversal with respect to its ordinates.

The diametransversal of Desargues is therefore the same as the Apollonian diameter.

[4] At the end of his discussion of transversals and butts (polars and poles), Desargues presents the following sketch of a stereometric construction of the centre and all diameters of a conic section (Figure 14, inspired by [T, p. 138, fig. 13]):

But here we have a proposition which as it were summarises all the preceding.

Given the magnitude and position of a general section of a roll with curved edge E, D, B, C forming the base or basis of a general roll, whose vertex is also given in position, and let another plane be given, in a general position, to cut this roll, and let 4, 5, the axle of the ordinance of this cutting plane with the plane of the base or basis, also be given in position, then the figure resulting from this construction in the cutting plane is given in species and in position, each of its diametrals with its distinguishing conjugates and axes, as also each kind of ordinates and tangents to the figure, and the nature of each, their ordinances and tangents to the figure, with all possible differences, all these are given as to type and in position.

For if through the vertex of the roll we draw a plane parallel to the plane of the section, this plane through the vertex gives in the plane of the base of the roll a straight line NH parallel to the straight line 4, 5, which line NH is a transversal of an ordinance of straight lines ML, BC, TV , whose butt, F is given in position.

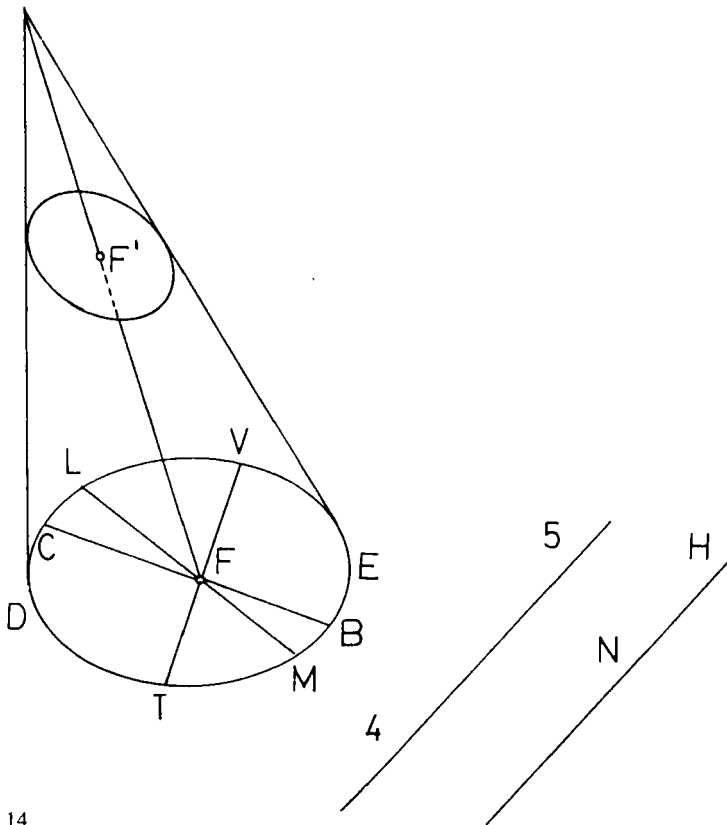


Figure 14.

And the straight line drawn through the vertex of the roll and this butt F is the axle of an ordinance of planes which generate the diametrals of the figure which this construction produces in the plane of section, of which the straight line which passes through the vertex and the butt of the diametrals of the basis of the role is only one case. [T 158, FG 120].

I now render the meaning of this quotation in a more familiar language. Desargues considers a generalized cone with base a conic section Γ . He considers another plane that intersects the cone in a curve Γ' (He does not prove that Γ' is what he calls a “section of a roll”, that is a conic section). Let the plane of Γ' meet the base plane in line 4, 5. The plane through the apex of the cone parallel to the plane of Γ' meets the base plane in a line NH parallel to 4, 5. Line NH is the polar of a point F with respect to Γ . The straight line through the apex and F meets the plane of Γ' in a point F' which is the centre of Γ' . He stresses that the planes through the apex and F intersect the plane of Γ' in the diameters of Γ' . The plane through the apex, F and the centre of Γ (called the butt of the diametrals of the base, i.e. the point common to the diameters of Γ) is only one case of this. This is precisely the plane which produces the principal diameter in the construction of Apollonius in *Conics* I:7 (where Γ is a circle, note that in Figure 1 the pole of v is a point on line MN).

I note that it was not really necessary for Desargues to give this construction, because he had already proved that any conic section has a centre and diameters. It was of course attractive for Desargues to construct the centre F' in the stereometric figure because Apollonius had not done this.

[5] In the Apollonian theory of conic sections, the “equations” (2.1), (2.2) and (2.3) between ordinates and abscissas with respect to a diameter play an important role. Needless to say, Desargues also had to do something similar, preferably more general. From his theorem on the inscribed quadrangle in the conic section he draws the following consequence (Figure 15, adapted from Taton’s Fig. 14, p. 142): Let L and M be two points on a conic section, and let LM be intersected by two parallel chords BC , DE at I , K . Choose a point S on KD such that $KS \cdot KM = KD \cdot KE$. Through M draw a line parallel to the two chords DE , BC and let LS meet this parallel at T and line IB at R . Desargues proves that $IR \cdot IM = IC \cdot IB$, so

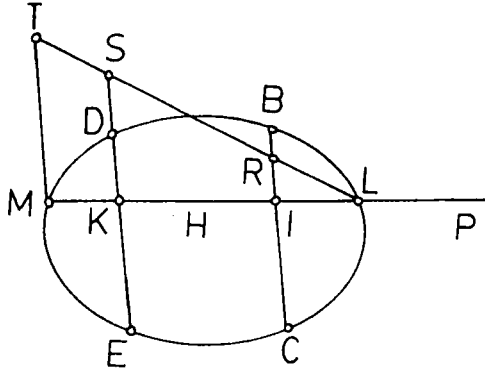


Figure 15.

the definition of T is independent of K . He also proves $KL \cdot KM / KE \cdot KD = ML / MT$. Therefore, if LM is a diameter and the chords are ordinates, $KL \cdot KM / KE^2 = ML / MT$. This is a form of the Apollonian “equation” for the hyperbola and ellipse, proved in *Conics* I:21, and MT plays the role of the Apollonian *latus rectum*. In the following quotation, reference is also made to two points P and H on LM extended. For the botanical terms “trunk, branch and shoot” one can substitute “line”. Desargues says:

Thus when the trunk PH , say, is a diametral of the figure, and the two branches ED , CB are its ordinates, the shoot MT , say, is clearly the line called *erect side*¹⁸, *parameter* or *coadjutor*, and which is only a [special] case of a [special] case of a [special] case, and moreover, its origin [i.e. construction] can be seen [T 164, FG 125].

The term *erect side* is Apollonian, and *parameter* is an alternative term introduced by Mydorge in 1631 (but also derived from the Greek text in the *Conics*, cf. Heath, *Treatise on conic sections*, p. 9).

Desargues claims that his approach is much more general than that of Apollonius, and he therefore calls the Apollonian concept a “special case of a special case of a special case”. One obtains the Apollonian *latus rectum* in the case where LM is a diameter of the conic section, and where at the same time the direction of ED , CB is that of the ordinates. So “a special case of a special case” is justified, but to call the Apollonian *latus rectum* a “special case of a special case of a special case” is an exaggeration. Desargues does

not state that the Apollonian *latus rectum* is perpendicular to LM , and he does not discuss the case of the parabola.

One may well ask why Desargues did not mention the *Conics* or Apollonius explicitly in these five quotations. Perhaps he was afraid to criticize the work of a geometer who was so admired by many other geometers of the early 17th century.

4. The introduction of foci by Desargues

In the last part of the *Brouillon Project* there is a brief section where Desargues introduces the foci of the ellipse and the hyperbola and discusses some of their properties [T 164–168 line 4, FG 126–129]. This section is based on a very long theorem on a circle, which Desargues proved earlier [T 148–152, FG 110–115]. We will discuss the relation between these passages and the *Conics* of Apollonius. The relation between the foci and the earlier theorem on the circle is not obvious, and the first person to note the connection was Zeuthen in 1903.¹⁹ Zeuthen believed that Desargues' treatment was completely independent of Apollonius, but I will show that this was not the case.

Desargues introduces the foci as follows. Consider an ellipse or hyperbola with centre Γ , diameter CE , and tangents CD , EB , which are intersected by a third tangent LBD , which touches the conic at L (Figure 16 adapted from Figure VIII of Desargues [T, fig. 18]). Let EF be the *latus rectum* corresponding to diameter CE . Desargues chooses F on BE extended, whereas in the Apollonian theory the *latus rectum* is always perpendicular to the diameter. The difference is of no consequence here. He now proves $EB \cdot CD = (1/4) CE \cdot EF$, which is exactly the same as *Conics* III:42 (compare [FG, 57]). Desargues' proof is phrased in his new terminology, and it is not completely identical to that of Apollonius, but there are marked similarities.

Desargues then assumes that CE is the “great axis of the figure” (le grand des essieux de la figure). Actually, CE is the major axis of the ellipse and the transverse axis of the hyperbola. Note that Desargues has not proved the existence of axes, unlike Apollonius.²⁰ Desargues then draws the circle with diameter BD . This circle

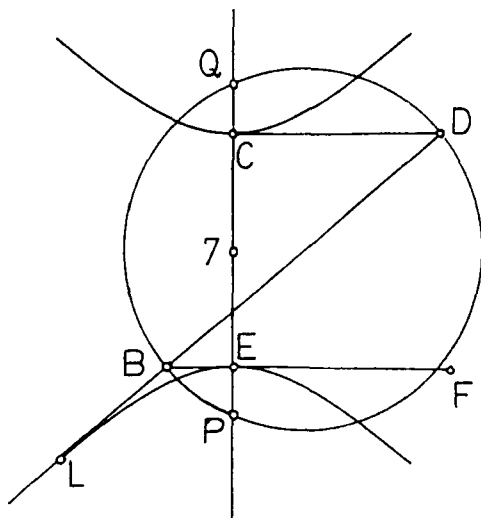


Figure 16.

intersects CE at two points P and Q , which Desargues calls the foci. He deduces several properties of the foci from his theorem on the circle which we will discuss presently. At first sight, Desargues' definition seems to be totally different from that of Apollonius, who defines the foci as points P_1 and Q_1 on the axis such that $CP_1 \cdot P_1E = CQ_1 \cdot Q_1E = (1/4) CE \cdot EF$. However, Apollonius proves immediately that P_1 and Q_1 are on the circle with diameter DE , and Desargues proves (twice, see below) that $CP \cdot PE = CQ \cdot QE = (1/4) CE \cdot EF$.

Desargues' theorem on the circle, on which his further discussion of foci is based, is as follows, in his own confusing notation. Figure 17 is identical to Desargues' Figure VI [T fig. 15], but the broken and dotted lines have been added by me.²¹

Consider a circle with centre 7 and diameter CE . A and I are points on the diameter "coupled with E and C in involution", that is to say, such that $AC : AE = IC : IE$, and L is an arbitrary point on the circumference of the circle. Draw lines LI , LA and extend them, if necessary, to meet the circle at M , S . Draw CP , $7B$ and ER perpendicular to AL and CN , $7D$, EO perpendicular to LS . Desargues proves

and he notes $ER \cdot CP = LP \cdot LR = MP \cdot MR$ by (4.1) and (4.2). He draws YIG perpendicular to CE , and he shows

$$\angle MIE = \angle EIS \text{ and } \angle LIY = \angle YIM, \quad (4.4)$$

and also $MI = IS$, so

$$LI + IM = LS. \quad (4.5)$$

It is difficult to see why Desargues would want to prove curious properties such as (4.3) and (4.5). The motivation is to be found in the application of his lemma to foci. Desargues mentions the connection so vaguely that his first editor, Poudra, did not even realize that the lemma has something to do with foci.²² Consider the ellipse with major axis PR , and passing through I . Then IZ is an ordinate, and because $CI:IE = CA:AE$ we have $PZ:ZR = PA:AR$, so AI is a tangent to the ellipse at I . The lines PC , RE are also tangents, so L and M are the foci according to Desargues' definition. By (4.1) and (4.5) we have $LI + IM = PR$, so the sum of the focal distances is equal to the major axis; the identity (4.4) implies that the tangent and the normal at I bisect the angles contained by the lines joining the foci and I . In order to find the motivation for (4.3), note that RQ is the Apollonian *latus rectum*. Desargues introduces this *latus rectum* not according to his general theorem discussed in passage no. 5 in section 3 of this paper, but in a very orthodox Apollonian way (see section 2); ZI^2 is the square on the ordinate, and $ZK \cdot ZR$ is precisely the Apollonian rectangle applied to ZR , deficient by a rectangle similar and similarly situated to the figure contained by PR and QR (compare the broken lines in the figure). Therefore (4.3) proves a special case of *Conics* III:42. Desargues proves the general case ($BE \cdot CD = (1/4) CE \cdot EF$ in Figure 16 for an arbitrary diameter EC) a little later, so (4.3) is superfluous.

In *Conics* III:45–52 Apollonius proves the same focal properties as Desargues, and he uses the same circle. The proofs of Desargues are a little more elegant than those of Apollonius; in particular, in the proof for the ellipse, Apollonius does not use a point corresponding to S in Figure 17, and his proof of $LI + IM = PR$ is therefore clumsier than that of Desargues. There is another interesting differ-

ence. Apollonius always tries to give one proof applicable to two different figures, one for the hyperbola and one for the ellipse. In his lemma on the circle, Desargues gives one figure (Figure 17) but two different proofs, one for the ellipse, and one for the hyperbola. Figure 17 can be applied to a hyperbola²³ with transverse axis NO and foci L, S . This hyperbola passes through A and $CIEA$ is a tangent to it. After his discussion of (4.2), Desargues distinguishes two cases; he gives the details of (4.3), (4.4), and (4.5) only for the case of the ellipse. For the hyperbola, one has to work out the other case. Points N and O play a role in the proofs of both cases, so Desargues tried to combine the two cases wherever he could. (The focus of the parabola is not mentioned either in the *Conics* or in the *BrP*.)

On the whole, the propositions of Desargues that we have discussed in this section are inspired by Apollonian concepts and methods, and the differences with Apollonius are of minor importance. Desargues' propositions on foci are of interest to us, because they show that he had studied the *Conics* very thoroughly. His proofs do not contain new elements from a methodological point of view. This is to say that they could also have been given by a Greek geometer.

5. The last proposition in the Brouillon Project

The *Brouillon Project* ends with a complicated and obscure proposition which has not received much attention in the modern literature.²⁴ This proposition is of interest here, because it is concerned with foci. I will first give an introductory explanation of my own.

Consider a conic section (for example an ellipse, Figure 18) with foci X and Q and centre F , and let G_1 be a point on the conic. Join G_1X and G_1Q and draw the tangent G_1Z_1 and the normal G_1R_1 . Then (a) G_1Z_1 is perpendicular to G_1R_1 , and we know (by a theorem of Apollonius or from 4.4) that (b) G_1R_1 bisects the angle between G_1X and G_1Q . Desargues showed at the beginning of the *BrP* that the consequence of (a) and (b) is that the two pairs Z_1, R_1 and X, Q are *four points in involution* [T 132, FG 97]. If we repeat the same

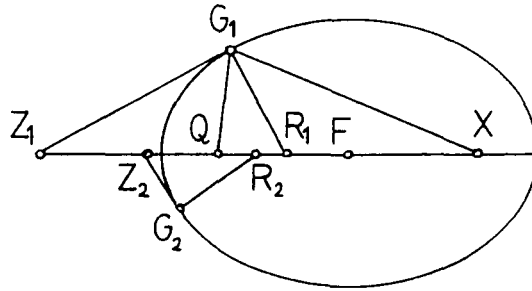


Figure 18.

argument for other points G_2 etc. on the conic, we obtain in the terminology of Desargues a *tree*, of which F is the *stump*, the pairs Z_1, R_1 and Z_2, R_2 etc. are corresponding *knots*, and the points X and Q (which correspond to themselves) are *double mean knots*. In modern terms: there is an involution φ of the major axis of the ellipse, such that X and Q are fixed points and $\varphi(Z_i) = R_i$ ($i = 1, 2, \dots$). The centre of the involution φ i.e. the point which is mapped to the point at infinity, is F .

I will argue below that the last proposition in the *BrP* is related to an attempt to construct the whole theory of foci on the basis of this idea, independently of the *Conics*. Because Desargues' treatment is confused and incomplete, it will be worth our while to first describe how one could give a complete introduction of the foci along the same lines. This will make it easier to analyse the last proposition in the *BrP* and the consequences that Desargues draws.

A complete theory of foci could consist of three parts:

1. We first have to construct the axes of the conic, so that we can choose for every point G_1 on the conic the corresponding intersections Z_1, R_1 of the axes with the tangent and the normal.

2. We then have to prove that there is an involution φ with centre F such that $\varphi(Z_1) = R_1$, $\varphi(Z_2) = R_2$ etc. In the terminology of Desargues, we have to prove that the pairs Z_1, R_1 ; Z_2, R_2 etc. form a *tree*. Then we can define the foci X and Q as the fixed points of φ , in the words of Desargues as the *double mean knots* of this tree.

3. We will then be able to prove (4.4) and use this to derive the other familiar properties of the foci.

The third part does not present any problems, so we will turn to the first two parts. My notation will be adapted to that of Desargues in the proposition to be discussed below. Consider a cone with apex θ which intersects the base plane in a circle Γ and which is cut by a plane not parallel to the base plane in a conic section Γ' , as in Figure 19. For a point such as X in the plane of Γ , we denote the intersection of θX and the plane of Γ' as X' . Let the plane through θ parallel to the plane of Γ' intersect the base plane in line p . Let F be the pole of p with respect to Γ . Then F' is the centre of Γ' by the theorems of Section 3.

Choose an arbitrary point A on p . The polar of A passes through F . Let this polar meet p at D , and let AF and DF meet Γ at J and H as in Figure 19. Lines $F'J'$ and $F'H'$ are two conjugate diameters of Γ' . The points A' and D' are at infinity, so $\theta A // F'J'$, $\theta D // H'F'$. The line $F'J'$ is an axis of Γ' if angle $H'F'J'$ is a right angle, that is to say if angle $A\theta D$ is a right angle.

Therefore, the axes of the conic section Γ' can be found if we can find points A and D on p such that (1) the polar of A with respect to Γ passes through D and such that (2) angle $A\theta D$ is a right angle.

We can rephrase this problem. Drop the perpendicular θC onto p . Let γ be the centre of Γ (Desargues denotes the centre this way), and let γF meet p at T . Then (2) is equivalent to $AC \cdot CD = C\theta^2$; thus points C and θ determine an involution σ of p such that $\sigma(A) = D$. In the terminology of Desargues, A and D are a pair of knots on a *tree* with *stump* C . Because γ is the centre of Γ , the pole of γT is a point at infinity. Therefore T is the centre of the involution π which maps any point on p on the intersection of the polar of this point with p (Desargues discusses this involution in [T 157–158, FG 120]). Because $\pi(A) = D$, $AT \cdot TD$ is a given quantity, which depends on Γ and p only. Therefore the axes of the conic section Γ' can be found if we can solve the following problem: Let points C and T on a line p be given. To construct points A and D on p so that $AC \cdot CD$ and $AT \cdot TD$ are given quantities. This problem can be solved by means of ruler and compass. (We will not discuss the details here).

Before introducing the foci we make a preliminary remark. Consider an arbitrary point G' on Γ' , draw the tangent $G'Z'$ and the normal $G'R'$, which intersect the axis $F'J'$ at Z' and R' . Points G' , Z' and R' are projections of points G , Z and R in the base plane,

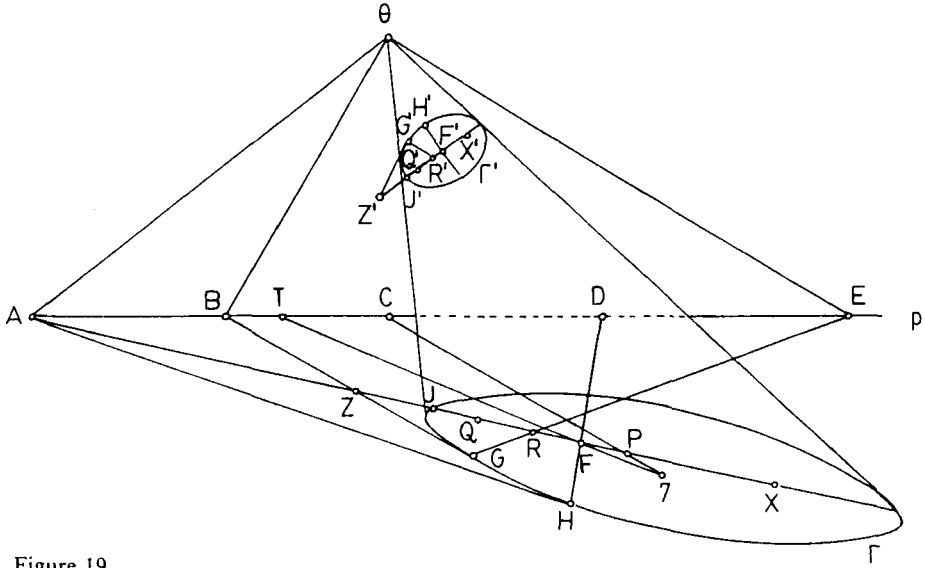


Figure 19.

and GZ is tangent to Γ . Extend GZ and GR to meet p at B and E . Then $\theta B \parallel G'Z'$ and $\theta E \parallel G'R'$, because the plane through θ and p is parallel to the plane of Γ' . Hence angle $B\theta E$ is a right angle, so $BC \cdot CE = C\theta^2 = AC \cdot CD$, that is to say $\sigma(B) = E$, where σ is the involution defined above.

In order to introduce the foci of Γ' , we have to show that there is an involution φ' on $F'J'$ with centre F' such that $\varphi'(Z') = (R')$ for all pairs Z', R' . We can do this by showing that there is an involution φ on line FJ in the base plane such that $\varphi(Z) = R$ for all pairs Z, R and such that $\varphi(F) = A$. The foci X' and Q' can then be introduced as the projections of the fixed points X and Q of this involution φ . (It is of course possible that φ has no fixed points. In this case we have to study a similar involution on line FH , which is projected onto the other axis of the conic section Γ' .) In order to prove the existence of φ , we can use the involutions σ and π on line p .

The preceding introduction may serve as a background to Desargues' proposition, but there are important differences. In his proposition, Desargues is only concerned with the base plane of the cone,

and he does not mention the cone or its apex θ . He proves a converse of what my introduction may suggest, for he assumes the involution φ on AF with fixed points X and Q (which will later turn out to be the projections of the foci) and he proves the existence of the involution σ and its centre C . He also proves the non-obvious fact that the straight line through the centres of σ and φ passes through the centre of Γ . In his proposition Γ is not a circle but a general conic section (“section of a roll”), but this difference is inessential. I now quote his proposition, with some explanatory additions of my own in parentheses. Most of the botanical terminology in the quotation can simply be ignored.

Desargues says (Figure 20, adapted from Taton, Fig. 19):

When in the plane of a general section of a roll $5Y8GH$, on the general line AF , (which is) one of the ordinates of a transversal AV (line l), the transversal point A is paired with F , the butt of the ordinates, in involution with any other two points X, Q , which are taken as two double mean knots of the involution (that is, the involution φ), (and if) each of the pairs of branches springing from this trunk XQ which pass through one of these pairs of extreme knots, such as FH, AH , and RG, ZG , and are ordinate to the butts H and G , on the edge of the figure, in such a way that one of the two touches the figure, such as HA and ZGB , I say that each of such pairs of branches arranged in this way gives on the transversal VA one of the pairs of knots DA, EB

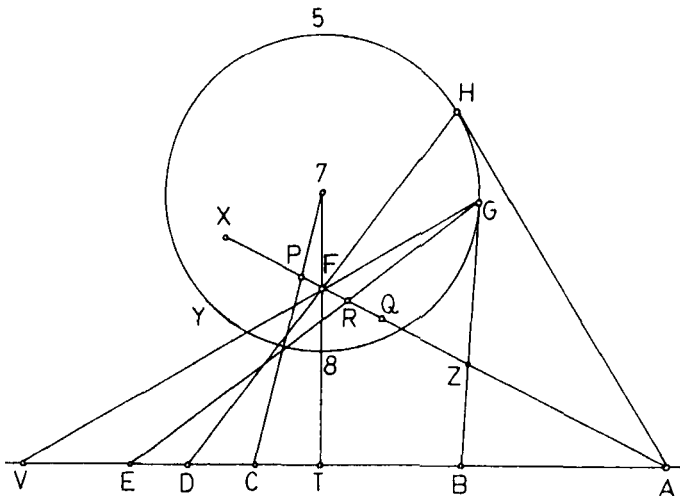


Figure 20.

belonging to a single tree (i.e. the tree determining the involution σ) whose stump, C , lies on the same straight line with γ and P , the two stumps of the trunk 578 of the same ordinates to the butt F , which line [78] is a diametral of the figure and of the other trunk AF . [T170–171, FG 133]

This means the following in the modern language of involutions, as explained in Section 3 above. Let Γ be a conic section and l an arbitrary line, and let F be the pole of l . Choose a point A on l and let φ be an involution on AF with two invariant points X and Q , and such that $\varphi(A) = F$. For any point B on l , we define a point $\sigma(B)$ on l as follows:

Draw the tangent through B to Γ . Call G the point of contact of this tangent and Z the intersection of BG and AF . Put $\varphi(Z) = R$. Point $\sigma(B)$ is defined as the intersection of GR and l . (Note that by this definition $\sigma(A) = D$ in Figure 20, and that HF is the polar of A and GF is the polar of Z).

Theorem: The mapping σ is an involution. Its centre is the intersection of γP and l , where γ is the centre of the conic Γ and P the centre of φ .

Desargues' proof is a very complicated exercise in similar triangles and proportions, and it is the most difficult part of the *BrP* from a technical point of view.²⁵ Thus one may well ask why Desargues took all this trouble. We first look at his own conclusion. I divide it into three parts, numbered [1], [2] and [3]: [T 175, FG 138]

[1] Now, from what has been proved above, it follows that if the general section of a roll 58Y is made the basis or base of a cone whose apex is at a distance from the stump C , measured normal to the transversal AV , equal to one of the mean limbs of the tree on it resulting from the construction, and in a plane parallel to another plane which cuts the cone, the two straight lines drawn through the apex of the cone and each of the points X and Q , when they lie away from the edge of the figure on the concave side, give, in the figure which results on the cutting plane in this position, the two points called navels, burning points, otherwise foci of the figure.

[2] So that if for the basis of a cone we are given a general section of a roll with curved edge, the section being given in position, and in its plane a straight line such as AV as a transversal, and the angle the plane of the section makes with the plane defined by the vertex and this transversal, and in it [the transversal] either [2a] the stump of the tree resulting of the construction as in our example, the point C , or alternatively [2b] two pairs of knots of the tree, or [2c] a point such as P not lying on the transversal, or alternatively [2d] one of the points such as X and Q , or alternatively [2e] two of the pairs of knots of the tree, such as XQ .

[then] The vertex of this cone is given in position, and the cone is given in kind and position, the figure of the section given by the position of the cutting plane is given in kind and position, ..., the butt of the ordinance of its diametrals (i.e. the centre) and the focal points, each of these is given in its derivation, kind and position.

[3] That if the vertex, the basis, the transversal and the cutting plane are given in position, all the rest is similarly given in its derivation, kind and position.

These conclusions call for some comments.²⁶ In [1], Desargues refers to the cone with base the conic section Γ and apex point θ as in Figure 19. As we have seen, this cone intersects a plane parallel to the plane θAD in a conic Γ' with foci X' and Q' . We note that Desargues' proof is only correct in the case where Γ' is an ellipse. If Γ' is a parabola, point F must lie on Γ and it must coincide with one of the points X and Q , say X ; the other point Q must be on line HF (for the projection of HF is the axis of the parabola). Clearly, Desargues had not yet studied this case in detail. If line l meets Γ at two points, Γ' is a hyperbola. If X and Q are to be projected on the foci of Γ' , they must lie "on the concave side" of Γ , i.e. inside the cone with apex θ and base Γ . Now point F is outside the cone, so A must be inside the cone, and no tangent AH to Γ can be drawn. In order to include the case of the hyperbola Γ' , Desargues would have had to generalize his proposition. This is not difficult, but it is clear that Desargues had only the ellipse in mind when he wrote his proposition.

In part [2] of the conclusion, Desargues assumes that the conic Γ , line l and the angle between the base plane V and the intersecting plane W are given. Note that because Γ and l are given, the involution π is also determined. Desargues then makes one of the five additional assumptions [2a], ..., [2e]. His assumptions are not completely accurate,²⁷ but they amount to saying that in [2a] and [2b] the involution σ is assumed, and in [2c], [2d] and [2e] the involution φ is assumed (so by the proposition that he has just proved, the existence of σ is also proved). He then correctly concludes that the apex θ of the cone is determined, and hence also the shape of the intersection Γ' and (once the intersecting plane has been fixed) the centre F' . Desargues says that the foci of Γ' are also given, and this is correct in the cases [2c], [2d] and [2e], where X and Q are determined. However, Desargues assumes in [2a] and [2b] a converse of the proposition that he has just proved, namely

that he can prove the existence of φ , and construct its fixed points X and Q , from σ and π .

In case [3] σ and π are again given (σ is determined by the distance between the vertex of the cone and line l). Desargues says that "all the rest" is similarly given. If this is taken to include the foci, Desargues again assumes the converse of the proposition that he has proved, as in cases [2a] and [2b].

I now return to the question of why Desargues proved his very complicated proposition. We can try to answer this question by looking at his own conclusion. The immediate consequence [1], to the effect that certain points X, Q in the base plane can be projected onto the foci of the conic section Γ' , does not seem to be of great intrinsic interest, and therefore does not explain the great amount of work that Desargues put into his proposition. The same is true for the technical consequences [2a] ... [2e] (although these may be interesting as preliminaries to something else). In connection with the foci, conclusion [3] is of a more fundamental nature, and it is interesting to compare [3] with the following passage in a letter by Desargues to Mersenne on April 4, 1638. Desargues mentions his discoveries concerning diameters and ordinates, which we discussed in Section 3 of this paper, and he then says:

And similarly by another simple and same reasoning, one sees a similar generation in each kind of conic sections of the points that one calls foci, and following this, (one sees) their positions and some properties common to them in every kind of conic section. All this without a special treatment of the parabola, and without making an exception for the circle; neither for the foci nor for the different kinds of lines that receive ordinates, nor for the different ordinates. [T 84].

This passage shows that in 1638, Desargues was working on an alternative theory of the foci, essentially different from the theory in the *Conics*, and using projective methods. In my view, the last proposition in the *BrP* is a product of this research. However, what we have in the *BrP* is not a complete new theory of foci based on projective methods. Desargues' conclusion [3] is based on the converse of the proposition he proved. As I explained above, there is a serious difficulty here, for the introduction of foci (and the proof that they are "given in derivation, kind and position") involves not only the definition of φ from σ and π and the proof that φ is an

involution, but also the discussion of the existence of the fixed points of φ .

One may well ask whether Desargues was really bothered by these gaps. I think he was. The fact that he introduced the foci in the manner of the *Conics*, as discussed in Section 4, shows that he did not yet consider his new theory to be an adequate replacement for that of Apollonius. His last proposition is therefore an interesting example of a great mathematician at work.

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NOTES

1. This and other English translations of French passages are taken from Field and Gray, unless indicated otherwise.
2. Printed in 1642, reprinted in Poudra, vol. 2, pp. 355–379.
3. The propositions "from the end of the third book" of the *Conics* are probably III:45–52 (see Section 4), not the very last propositions III:53–56, which are only loosely related to the *BrP*.
4. Poudra, vol. 2, p. 378.
5. This of course does not mean that modern historians have not seen the simplification. K. Fladt says in *Geschichte und Theorie der Kegelschnitte*, p. 42:
Desargues zieht durchaus und grundsätzlich unendlich ferne Elemente mit in Betracht. So ergeben sich die Eigenschaften konjugierter Durchmesser und der Hyper-

belasymptoten unmittelbar aus der allgemeinen Polartheorie.

Fladt does not give details and he does not refer to Apollonius.

6. See Chasles, *Aperçu*, p. 18 and Section 2 of this paper.
7. Chasles says in *Aperçu*, p. 79, in connection with the theorem of Desargues on a conic and an inscribed quadrilateral:

C'est que ce théorème, par sa nature, permettait à Desargues de considérer, sur un cône à base circulaire, des sections tout à fait arbitraires, sans faire usage du triangle par l'axe, comme le dit Pascal; tandis que les Anciens, et tous les écrivains après eux, n'avaient coupé le cône que par des plans perpendiculaires à ce triangle par l'axe. Cette grande innovation nous paraît être le principal mérite du traité des coniques de Desargues.

This passage was written by Chasles in 1839, before the *BrP* was rediscovered, but the passage was not modified in the second edition of the *Aperçu* (1875) (which is not surprising, because it is based on a misinterpretation of the *Conics*, not of the *BrP*). Pascal said correctly that Apollonius used the axial triangle in his proofs and that Desargues did not (see Section 3). See section 2 of this paper for the correct meaning of the term "axial triangle".
8. For example Zeuthen (1866), p. 65.
9. Thus Taton (p. 166) and Field and Gray (p. 57) confuse the "principal diameter" with the axis, although these concepts are not equivalent, compare Section 2. If the intersecting plane is perpendicular to the axial triangle (compare note 7), the principal diameter is an axis.
10. See Taton, p. 140 footnote 51 and p. 155 footnote 66.
11. For example: fix a conic section and a line m , and for any point P on m let p be the intersection of m with the polar of P . Then the couples p, P are in involution [T 157–158, FG 120]. Other examples are Desargues' theory of six points in involution, and his theorem on the inscribed quadrilateral in a conic (Section 3).
12. Desargues also considers the asymptotes as diameters, and he did not worry about whether or not (and in what sense) the asymptotes satisfy a definition of diameter. He probably believed that the geometer does not construct objects or relations in his mind, but that the objects and relations exist already and are only described by the geometer.
13. Note that if $PX: XQ = PY: YQ$, also $XP: PY = XQ: QY$, but in general $PQ: QX \neq PY: YX$. Thus if P, Q and X, Y are four points in involution, X, Y and P, Q are also four points in involution, but P, X and Q, Y are in general not four points in involution. Therefore Desargues' terminology is somewhat unfortunate, and it would have been better to use the term two pairs of points in involution. Nevertheless, I will continue to use the terminology of Desargues.
14. The term "ordinate" in the (Latin translation of) the *Conics* may well have inspired the following terminology of Desargues: *ordinal* (lines through P which do not meet the conic) and *ordnance* (a collection of lines through a fixed point P , without reference to a conic).
15. The few theorems before *Conics* III:37,38 deal with the following special cases for the hyperbola: Y at infinity (*Conics* III:30–34) and A at infinity (*Conics* III:35–36). Of course Apollonius deals not with points at infinity but with parallel lines.
16. Special cases of this theorem were used by Apollonius: in Figure 7, P is on the polar of C , and C is on the polar of P .

17. The mathematical equivalent had also been proved (for ordinary points) in prop. 145 of Book VII of the *Mathematical Collection* (ed. Jones, p. 278). Pappus does not discuss the cases where (in the terminology of Desargues) one of the points is at infinity. If P and P' are outside the cone (as in Figure 10), the proof can be simplified. In this case, the polar of P intersects the circle in the base plane at points A and B in such a way that PA and PB are tangents. Therefore planes TPA and TPB are tangent to the cone, so $P'A'$ and $P'B'$ are also tangents. Hence p' is the polar of P' . Compare Figure 6.
18. Desargues uses "costé droit", which is equivalent to the Latin *latus rectum*. Field and Gray translate "normal side".
19. Zeuthen, 1903, p. 184. Taton, p. 168 and Field and Gray p. 58 state that the connection was first noted by Zacharias in 1922, p. 84 note 79.
20. *Conics* I:53,55,58.
21. Figure 6.18 in *FG* (p. 111) contains printer's errors.
22. Poudra, vol. 1, p. 288.
23. The explanation of Zacharias (p. 84, note 79) is incorrect for the hyperbola.
24. Interesting mathematical discussions of the theorem are found in Zeuthen, 1903, p. 184, Fladt, 1965, p. 42, Cremona, pp. 90–91.
25. Desargues says that a large part of the proof is due to a certain Jean Pujos.
26. The commentary on [1] in Poudra, vol. 1, pp. 293–295 (followed by Zacharias, p. 85 and Taton, p. 172) is unclear, because Poudra states that H is an arbitrary point on the conic section, and because his explanation only involves H , A , F , X and Q . In fact, H is determined by X , Q and F . Point G is arbitrary, but Poudra (followed by Zacharias and Taton) does not mention G , Z and R at all. Poudra added extra lines with Greek letters to the figure, which have been taken over by Taton, p. 171 (reproduced in Field and Gray, p. 134). Desargues does not use Greek letters in the BrP . My notation θ is analogous to that of Poudra.
27. Condition [2b] is accurate, but [2a] is not sufficient, for to determine the involution σ it is necessary to know one more pair of knots of the tree on p (line AV). Since F is given as the pole of l with respect to Γ , any of the points P [2c], X or Q [2d] determine point A . Now the involution φ is determined, because we know $\varphi(F) = A$ and the image of one other point (P , X or Q). In condition [2e], Desargues gives too much information; it would have been more correct if he had said: *one* of the pairs of knots of the tree, such as RZ .