ARCHIVES ET BIBLIOTHÈQUES DE BELGIQUE
ARCHIEF- EN BIBLIOTHEEKWEZEN IN BELGIË
NUMÉRO SPÉCIAL 83 EXTRANUMMER

MÉLANGES OFFERTS À HOSSAM ELKHADEM PAR SES AMIS ET SES ÉLÈVES

ÉDITEURS
FRANK DAELEMANS
JEAN-MARIE DUVOSQUEL
ROBERT HALLEUX
DAVID JUSTE

BRUXELLES 2007

SIMILAR MATHEMATICS IN DIFFERENT CULTURES: JAMSHĪD Al-KĀSHĪ AND LUDOLPH VAN CEULEN ON THE DETERMINATION OF π

JAN P. HOGENDIJK[1]

1 Introduction

One of the highlights of the medieval Islamic mathematical tradition is the determination of π to 16 decimals by Jamshīd ibn Masʿūd al-Kāshī or al-Kāshānī (died 1429).[2] Al-Kāshī held the world record until 1596, when Ludolph Van Ceulen determined π to 20 decimals in his work *Van den Cirkel.*[3] In the same work, Van Ceulen included a π-determination in 18 decimals by exactly the same method as al-Kāshī. Van Ceulen did not know al-Kāshī's work, which remained unknown in the West until 1925.[4] The curious resemblance can be explained by the fact that both mathematicians were inspired by Archimedes' work. The 65-th birthday of Hossam Elkhadem provides a fitting occasion to compare the two π-determinations to 16 and 18 decimals by the two brilliant mathematicians from the Islamic world and Europe.

2 Al-Kāshī and his work on π

Al-Kāshī was born in Kāshān in Iran. In 1421 he moved to Samarkand to work as a mathematician and astronomer at the court of Ulugh Beg.[5] His π-determination is the subject of his *Treatise on the Circumference (al-Risāla al-muḥīṭiyya).* This treatise probably dates back to the period when al-Kāshī was still

[1] Mathematics Department, Utrecht University, P.O. Box 80.010, 3508 TA Utrecht, Netherlands.
[2] On the life and works of al-Kāshī see, e.g., [19], [17, pp. 269-271].
[3] In 1593, the Belgian mathematician Adriaan Van Roomen, who did not know al-Kāshī's work either, published *Ideae mathematicae pars prima,* in which he determined π to 15 decimals using a regular $15\cdot2^{24}$-gon, see [7, vol. 1, p. 174]. Van Roomen and Van Ceulen became close friends.
[4] Al-Kāshī's work on π was first mentioned by David Eugene Smith [18 vol. 2, pp. 238-240], who had been informed by Salih Mourad from Istanbul. The similarities between *Van den Cirkel* and al-Kāshī's treatise were briefly noticed by Luckey [10, p. 54].
[5] The date of his move has been determined by Qorbānī in [13, pp. 7-9].

in Kāshān, because it does not contain a dedication to an emperor. In 1953 the treatise appeared in a posthumous German translation by Paul Luckey (1884-1949) [10], with an excellent Arabic edition based on a manuscript in Istanbul. In the appendix to this paper I will discuss a manuscript of al-Kāshī's treatise which has recently been discovered in Meshed in Iran. This manuscript is a direct copy of al-Kāshī's original, made by the mathematician Bahā' al-Dīn al-'Āmulī (1547-1622 CE).[6]

Figure 1

[6] On al-'Āmulī see [13, p. 160], [17, pp. 348-350].

Because al-Kāshī's treatise on π is not yet available in an English translation, incorrect statements about it have been made in the Western literature. For example, a recent survey of π-determinations [3] states that al-Kāshī determined π to 14 decimals. This statement is easily refuted by Figure 1, which displays al-Kāshī's "table of the multiples of the ratio of the circumference and the diameter"[7] in the Meshed manuscript. In this table, al-Kāshī lists in modern terms the multiples of 2π in decimal notation; the symbol π was not yet used in al-Kāshī's time. On the fifth line,[8] al-Kāshī writes 10π as 31 4159 26535 89793 25. Here we see the first 16 correct decimals in the decimal expansion of π = 3.14159 26535 89793 238 …

The decimal system for fractions was discovered by al-Kāshī and several other authors in the medieval Islamic tradition, but it never became popular. Al-Kāshī made his computations in the sexagesimal system, which had been developed in ancient Babylon, and which was widely used in medieval Islamic astronomy. In his *Treatise on the Circumference*, al-Kāshī wanted to compute π with such an accuracy that in a circle with radius equal to the radius of the universe, the maximal error in the circumference is less than the breadth of one hair.

Al-Kāshī and his contemporaries accepted the cosmology of Ptolemy (ca. 150 CE, see [15]), who believed that the earth is surrounded by the concentric spheres of the moon, Mercury, Venus, the sun, Mars, Jupiter, Saturn, the fixed stars, and by an outermost sphere. Ptolemy and his Islamic successors determined the distance from the earth to the moon and the sun on the basis of measurements of lunar parallax and the apparent sizes of the moon, the sun, and the earth shadow during solar and lunar eclipses. These methods are correct but not very accurate. Thus, a relatively small error in the measurement of the earth shadow produced a distance between the earth and the sun which is only 1/20-th of the correct value.

Ptolemy assumed that the maximal distance between the earth and the sun is equal to the minimal distance between the earth and Mars. The Ptolemaic model produces essentially correct ratios between the maximal and minimal distances from the earth to each planet (but not the values of the distances themselves). From the supposed minimal distance of Mars, and the ratio between its minimal and maximal distance, Ptolemy could now find the maximal distance of Mars, which he supposed to be equal to the minimal distance of Jupiter, and so on. Finally, he assumed that the maximal distance of Saturn is equal to the minimal distance of the fixed stars. Ptolemy believed that all fixed stars were attached to a thin sphere with

[7] Al-Kāshī actually means: half the diameter.
[8] In his table, al-Kāshī uses Hindu-Arabic numbers but no decimal point. He writes the decimals in columns for "tens," "times the diameter," and multiples of 10^{-n} times the diameter for $1 \leq n \leq 16$. The fifth line actually ends with 255, but the last 5 refers to the multiple $5 \cdot 2\pi$.

a very slow (precessional) motion with respect to the outermost sphere containing the celestial equator. This outermost sphere rotated once every day around the center of the universe, which coincided with the center of the earth. Ptolemy concluded that the radius of the universe was approximately 20,000 earth-radii.

On the basis of new astronomical observations, the medieval Islamic astronomers changed some of the parameters in Ptolemy's models, but they computed the radius of the universe along the same lines with similar results. In his work *Stairway to Heavens (sullam al-samā')*, al-Kāshī stated that the radius of the universe is 26,328 earth-radii [2, p. 251]. In his *Treatise on the Circumference*, al-Kāshī required that in a circle with radius R equal to 600,000 earth-radii, the inaccuracy in the circumference $2\pi R$ should be less than the breadth of a hair. Then the inaccuracy is much less than the breadth of a hair for all circles which can exist in the physical universes.

For the approximation of π, al-Kāshī used a method of Archimedes, which is as follows in modern notation. Consider a circle with diameter 1. Then the circumferences of the inscribed hexagon, the circle, and the circumscribed hexagon are 3, π, and $2\sqrt{3}$ respectively. The circumference of the circle is greater than the circumference of the inscribed hexagon and less than the circumference of the circumscribed hexagon. Thus $3 < \pi < 2\sqrt{3} = 3.46...$ Using lower and upper bounds of the sides of an inscribed and circumscribed regular n-gon, Archimedes computed lower and upper bounds of the sides of the inscribed and circumscribed regular π-gon. Using the hexagon, he approximated the sides of the inscribed and circumscribed regular 12-, 24-, 48-, and 96-gon. From the 96-gon he obtained $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

The algebraic expressions of the sides of these polygons involve irrational numbers, which Archimedes avoided by means of complicated estimates such as (in modern terms) $265/153 < \sqrt{3} < 1351/780$. Archimedes did not use a decimal or sexagesimal system for fractions.[9]

Al-Kāshī mentioned Archimedes' approximation but remarked that the result $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is much too inaccurate for his purpose. After the 96-gon ($96 = 3.2^4$), al-Kāshī considered 24 more polygons, namely the 192-gon, the 384-gon, and so on, until the 3.2^{28}-gon. He showed that the circumferences of the inscribed and circumscribed 3.2^{28}-gon of a circle with radius 600,000 earth-radii differ by less than a breadth of a hair, so his approximation of the circle by one of these circumferences produces a sufficiently accurate value of π.

Al-Kāshī worked in a circle with radius 60 units, as was usual in the trigonometry of his time. He simplified Archimedes' method of computation in the

[9] For Archimedes' method see [1, pp. 93-94], reprinted in [4, pp. 9-14].

following way in modem terms:

The sides of the inscribed and circumscribed n-gons in a circle with radius 60 are given by the formulas

$$120 \sin \frac{180^\circ}{n}, 120 \tan \frac{180^\circ}{n}.$$

Al-Kāshī first computed, for $n = 3, 6, 12, \ldots, 3\,2^{28}$,

$$k_n = 120 \cos \frac{180^\circ}{n}.$$

The magnitudes k_n satisfy the simple relation

$$k_{2n} = \sqrt{60(120 + k_n)},$$

a consequence of the modern formula

$$2 \cos \frac{\alpha}{2} = \sqrt{2 + 2\cos\alpha}.$$

Note that $k_3 = 60$. Al-Kāshī computed, in modern notation:

$$k_6 = 60\sqrt{3}, \; k_{12} = 60\sqrt{2+\sqrt{3}}, \; k_{24} = 60\sqrt{2+\sqrt{2+\sqrt{3}}}, \ldots$$

up to $k_{3.2^{28}}$

Al-Kāshī also showed that the computations are sufficiently accurate if the root extractions are carried out in 20 sexagesimals (two integer and 18 fractional). Figure 2 is Luckey's transcription [10, p. 12] of al-Kāshī's computation of $\sqrt{3.60^2}$. For the transcription of sexagesimal numbers, I use the system of O. Neugebauer, in which a semicolon is the separation sign between integer and fractional part, and sexagesimals are separated by a comma. In this system, the result of al-Kāshī's computation can be expressed as

$$\sqrt{3,0,0} = 1,43;55,22,58,27,57,56,0,44,25,31,42,1,56,22,42,48,58,57\ldots$$

(meaning: $= 1\,60 + 43 + \frac{55}{60} + \frac{22}{60^2} \ldots$). In this computation, al-Kāshī did not use Hindu-Arabic numbers but the abjad-system, which is a transcription of the following system that had been used by Ptolemy and his Greek successors. The letters of the Greek alphabet have numerical values $\alpha = 1, \beta = 2, \gamma = 3 \ldots \theta = 9, \iota = 10, \kappa = 20, \lambda = 30, \mu = 40, \nu = 50. \ldots$ Ptolemy wrote the number 53 as $\nu\gamma$, that is the letter for 50 followed by the letter for 3. For zero, Ptolemy used the symbol ō (the first letter o of the Greek word *ouden*, meaning: "nothing," with an overbar). The Arabic translators substituted for the different letters of the Greek alphabet the letters of the Arabic alphabet with the same numerical values. Thus α became *alif*, β became *bā'*, and so on. The translators maintained the symbol ō for zero.

By means of 27 further square-root extractions as in Figure 2, al-Kāshī found $k_{3.2^{28}}$, and hence the circumference I of the inscribed 3.2^{28}-gon as

$$I = 3 \cdot 2^{28} \times \sqrt{120^2 - \kappa^2_{3.2^{28}}}\,.$$

Tafel 3b

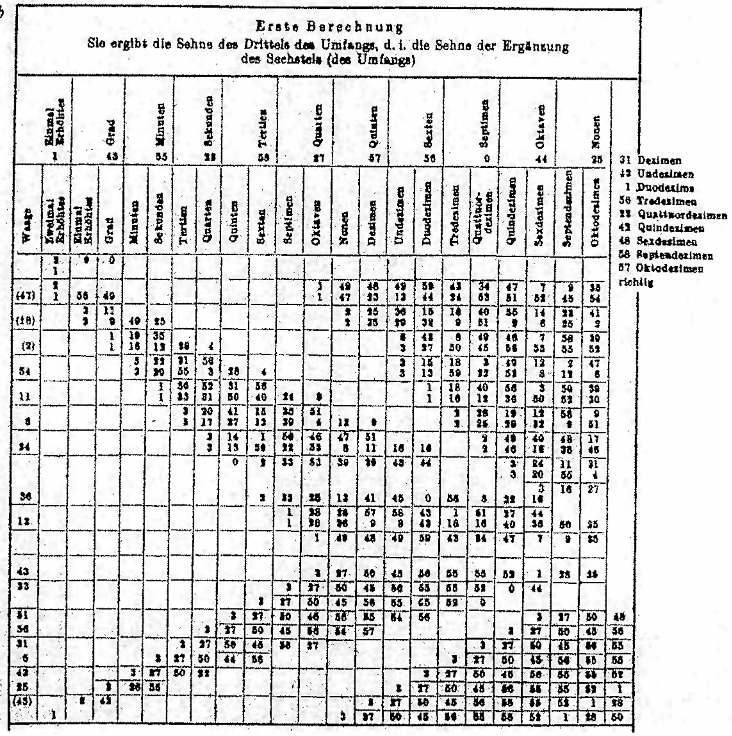

Figure 2: Luckey's transcription of al-Kāshī's computation of $\sqrt{3.60^2}$. (see [10, p. 12], al-Kāshī's verification by computation of the square has been omitted)

Al-Kāshī then determined the circumference C of the circumscribed polygon by a method involving similar triangles, equivalent to the formula $C/I = 120/k_n$ for $n = 3.2^{28}$. In this way he found upper and lower bounds for the circumference of the circle with radius 60, corresponding to the following upper and lower bounds for 2π:

$$2\pi < 6;\ 16, 59, 28, 1, 34, 51, 46, 14, 50, 15$$
$$2\pi > 6;\ 16, 59, 28, 1, 34, 51, 46, 14, 49, 45$$

Initially, al-Kāshī found 46 as the last sexagesimal of the lower bound, but he then estimated the next sexagesimal place in all his computations, and corrected 46 to 45.

Al-Kāshī chose for 2π the average 6;16,59,28,1,34,51,46,14,50, and he converted this number into the decimal system of fractions as 6.28318 53071 79586 5 in modern notation.[10]

3 Ludolph Van Ceulen and bis work *Van den Cirkel*

Ludolph Van Ceulen was born in Hildesheim in Germany in 1540. Before 1580 he settled in Holland, where he made his living as a teacher of arithmetic, surveying and fencing. In 1600 he obtained a position at an engineering school in Leiden, to teach mathematics in Dutch. He died in 1610 in Leiden.[11] In 1584 and 1586, Symon van der Eycke published two quadratures of the circle, which he had obtained by wrong applications of the method in Archimedes' *Measurement of the Circle.* These two quadratures are equivalent to $\pi = (39/22)^2$ (= 3.142562...) and $\pi = \sqrt{320-8}$ (= 3.144605...).[5, pp. 134-138]. Van Ceulen did not know classical languages and Archimedes' *Measurement of the Circle* was only available in Latin and Greek, but his friend Jan de Groot translated the text in Dutch for him, see the preface to [8]. Van Ceulen then published proofs that the quadratures of Simon van der Eycke were incorrect. Van Ceulen continued his computations by increasing the number of sides of the inscribed and circumscribed polygons. In 1596 he published his great treatise in Dutch entitled *Van den Cirkel* (On the Circle) [8]. In this work he discusses not only π-determinations but also the numerical determination of sides of arbitrary polygons, and sine, tangent and interest tables. Van Ceulen eventually found 35 decimals of π, which were first published in the form of an inscription on his tombstone in a church (the Pieterskerk) in Leiden. The tomb was destroyed in the mid-nineteenth century, but on the occasion of the world mathematics year 2000, a reconstruction of the inscription was placed in the

[10] Al-Kāshī upper and lower bounds of 2π are equivalent to 3.14159 26535 89793 230 < π < 3.14159 26535 89793 254. The average π = 3.14159 26535 89793 242 is correct to 17 decimals [10, p. 67].
[11] On Van Ceulen and his work *Van den Cirkel* see [5], [6], [9].

church [12].

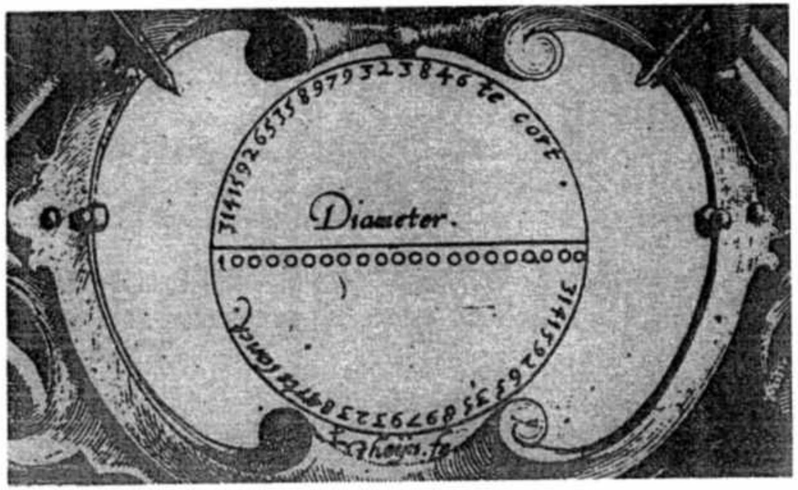

Figure 3: Part of Van Ceulen's computation of the side of a $3 \cdot 2^{31}$-gon in [8, f. 13]. The figure displays the quantities $C_{3 \cdot 2^n}$ for $n = 20 \ldots 31$. At the bottom Van Ceulen printed the resulting upper and lower bounds of π.

In *Van den Cirkel*, Van Ceulen presents four different π-determinations to 12, 16, 18 and 20 decimals, using inscribed and circumscribed polygons with $5 \cdot 2^{25}$, 2^{30}, $3 \cdot 2^{31}$ and $15 \cdot 2^{32}$ sides respectively. We will only pay attention to the third π-determination, in which Van Ceulen uses the same regular polygons as al-Kāshī. There, van Ceulen computed for n = 48,96,..., $3 \cdot 2^{31}$ the quantity $c_n = 2 \cos(180°/n)$ (my notation), using the relationship $c_{2n} = \sqrt{2 + c_n}$. Some of his results are displayed in Figure 3. As we have seen above, al-Kāshī computed $k_n = 60c_n$. Van Ceulen's relationship between c_{2n} and c_n corresponds to al-Kāshī's relationship between k_{2n} and k_n. Thus the two mathematicians improved the Archimedean method in essentially the same way.

Unlike al-Kāshī, Van Ceulen used algebraic notation, so he could express the quantities c_6, c_{12}... algebraically as $\sqrt{3}$, $\sqrt{2 + \sqrt{3}}$ etc. (compare [8, f. 3b]). Van Ceulen performed his square-root extractions in the decimal system, which had been developed by the Dutch mathematician Simon Stevin (1548-1620), independently of al-Kāshī and other medieval Islamic mathematicians. If we transform Van Ceulen's numbers c_n into the sexagesimal system, we obtain for $n \leq 3 \cdot 2^{28}$ ail sexagesimals of al-Kāshiī's numbers k_n The fact that $k_n = 60c_n$ causes a shift of one sexagesimal position. For example, on une 28 in Figure 3, Van Ceulen's number[12] $c_{3 \cdot 2^{28}}$ = 1 99999 99999 99999 98478 13027 08290 02173 7702 is in sexagesimal notation 1; 59, 59, 59, 59, 59. 59, 59, 59, 59, 50, 47, 52, 12, 30, 48, 37, 49, 54, 40, 26, 21 .... The first twenty sexagesimals occur in the result of al-Kāshī's 28-th square-root extraction: 1, 59; 59, 59...40, see [10, p. 11]. Because Van Ceulen carried the computation three steps further (to a 3.2^{31}-gon), and because he used 39 decimals in the end of the computation, he succeeded in determining π to 18 decimals.[13]

Al-Kāshī had explicitly connected his computation of π to the size of the universe. Van Ceulen says, somewhat more vaguely, that by his π-determination in 20 decimals "all circles can be measured so perfectly that the result is neither too much nor too little (so to say) by the breadth of a hair, even if the circle were greater than the whole earth".[14] Note that Van Ceulen invokes the breath of a hair, just like al-Kāshī. As a further motivation an Ceulen cites in his introduction to [8] the following problem from an unspecified printed book: find the area of a circular piece of land with radius 7,000 rods (1 rod = ca. 3.7 meter). According to Archimedes' value $\pi = 3\frac{1}{7}$ the surface area would be 38500000 square rods. Van Ceulen says that by his better π-values, one would find a surface area between $3848510\frac{6475}{1000000}$ and $3848510\frac{6475}{1000000}$, "not a hair-breath more or less." Van Ceulen says that application of the correct 7π-value would make a difference of thousands of guilders if the piece of land were sold. At the end of his first π-determination in 12 decimals, Van Ceulen says that this is enough to find "the magnitude (i.e., surface area) of a circle of a circumference of many hundreds of miles, so near that it is not too much or too little by the breadth of a hair." His own "inclination" was the reason why lie continued with the second, third and fourth π-determinations [8, f. 12a].

Van Ceulen's approximation of π to 20 decimals apparently became known in Iran one century after his death. The Iranian author Muḥammad Bāqir Yazdī, grandson of the famous mathematician Muḥammad Bāqir Yazdī, says around 1700 in a passage cited by Qorbānī [14, p. 5] that some mathematician from Europe had shown the following: if the diameter of a circle is 1 followed by 11 zeros, the circumference is 314 159 265 481. I have been unable to identify the author of this approximation. Muḥammad Bāqir Yazdī then says that someone else found by a more accurate computation: if the diameter is 1 followed by 20 zeros, the circumference is between 314 159 265 358 979 323 847 and ... 846. The "someone

---

[12] The last three decimals of Van Ceulen's number are incorrect and should be 692.

[13] Van Ceulen obtained an estimate of π between ...238 and ...239, see the bottom of Figure 3. The inaccuracy is 10^{-18}, but the computation did not show whether the 18th decimal is 8 or 9.

[14] Door dese reden kunnen alle Circkels ghemeten werden volcomen / dat sulcx gheen hayr-breedt (bij maniere van spreken) te veel ofte te weynigh comt / al waer de Circkel grooter dan't geheele Aertrijc." See p. 2 of the preface in [8].

else" is probably Van Ceulen, because the approximation is expressed in the same way on the front page of his work (Figure 4). The late professor Qorbānī considers this transmission from Europe to the Islamic world as the event which defines the end of the medieval Islamic period in mathematics [14, p. 4].

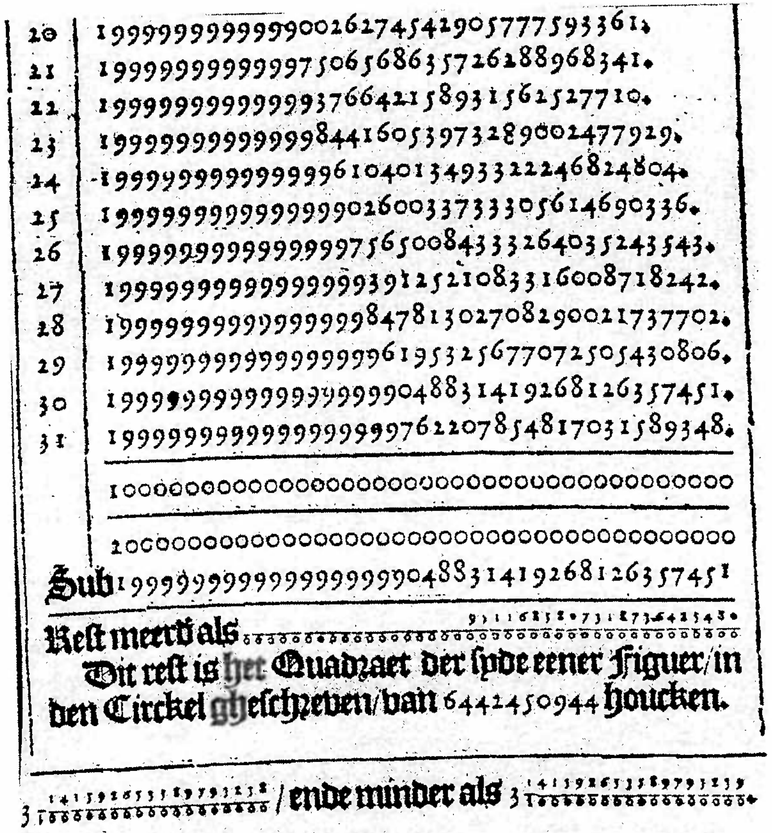

Figure 4. Part of the front page of Van Ceulen's work "Van den Cirkel" [8].

Al-Kāshī and Van Ceulen worked in very different mathematical communities. Van Ceulen was working in a period in which there was much interest in π in Western Europe. He and some of his contemporaries spent much time in refuting the wrong quadratures that were published. As far as we know, none of al-Kāshī's the wrong quadratures that were published. As far as we know, none of al-Kāshī's contemporaries were working on the determination of π. Al-Kāshī's immediate predecessors were al-Būzjānī (died 998) and al-Bīrūnī (died 1043), whose values of π were by-products of their computations of the sine of one half or one-quarter of a degree, using a regular polygon of at most 720 sides. In the determination of π, and in computational mathematics as a whole, al-Kāshī was a pioneer.

Appendix. New information from al-ʿĀmulī's manuscript of al-Kāshī's treatise

In 1989, the eminent Iranian historian of mathematics A. Qorbānī drew the attention to the manuscript no. 5389 in the Holy Shrine Library in Meshed.[15] This manuscript contains al-Kāshī's *Treatise on the Circumference,* with the following colophon: "This has been written by its author, the most insignificant servant of God Most High, Jamshīd ibn Masʿūd ibn Maḥmūd ibn Muḥammad, the Physician, al-Kāshānī, called Ghiyāth, may God treat him well, in the middle of the great month Shaʿbān of the year 827 of the Hijra" (corresponding to the end of July 1424 CE).[16] Qorbānī concluded from this colophon that the manuscript was an autograph [13, pp.131-153]. Professors E. Neubauer and F. Sezgin (Frankfurt) have pointed out to me that the handwriting does not resemble the usual fifteenth-century Persian style. They also referred to the following passage on the second page of the manuscript (reproduced in [13, p. 125]): "The Treatise on the Circumference; it is a copy of the original in the handwriting of the author, the most glorious and most excellent master, the Ptolemy of his time, our master Ghiyāth al-Dīn Jamshīd al-Kāshī; it was written by the poor Muḥammad Bahā' al-Dīn al-ʿĀmulī.[17] The Meshed manuscript is therefore a copy of the original text of al-Kāshī by Bahā' al-Dīn al-ʿĀmulī, who was a well-known mathematician himself, and who was born in Baalbek, now in Lebanon. This manuscript is of better quality than the Istanbul manuscript used by Luckey, and it confirms many emendations which Luckey made to restore the original text.[18]

In what follows I will use the notations **A** for the Meshed manuscript (Holy Shrine Library, 5389), **I** for the manuscript Istanbul, Askeri Müze 756, which

[15] See [11, vol. 3, p. 52, no. 162 ; vol. 8, p. 42], and see [13, pp 124-130] for facsimiles of six pages.
[16] Katabahu al-mu'allif aṣghar ʿibād Allāh taʿālā. Jamshīd ibn Masʿūd ibn Maḥmūd ibn Muḥammad al-ṭabīb al-Kāshānī al-mulaqqab bi-Ghiyāth aḥsana Allāh aḥwālahu fī awāsiṭ shaʿbān al-muʿaẓẓam sana 827 al-hijrīyya.
[17] al-risāla al-muḥīṭiyya wa-hiya nuskhat al-aṣl bi-khaṭṭ mu'allifihā al-mawlā al-ajall al-afḍal baṭlamyūs zamānihi mawlānā Ghiyāth al-dīn Jamshīd al-Kāshī ḥarrarahu al-faqīr Muḥammad Bahā' al-Dīn al-ʿĀmulī.
[18] These are the emendations to the Arabic text in **L** 75-95 indicated by notes 2, 15, 17, 22, 25, 27, 28, 29, 31, 33, 34, 35, 42, 43, 47, 50, 56, 58, 60, 61, 62, 67, as well as the following corrections to the tables: **L** 80 note 4 to p. 81 ; **L** 80 notes 1,2,4,5 to p. 82; **L** 83 note 1 ; **L** 90 notes 1,2,3,4 ; **L** 93 notes 1,2,3 ; **L** 94 note 1.

Luckey used (see the facsimile in [16]), and **L** for Luckey's edition,[19] translation and commentary [10]. A notation such as **A** 55:3 or **L** 55:3 refers to line 3 of **A** or **L**, **L** 85n37 refers to footnote 37 of page 85 of **L**.

Because the Meshed manuscript was copied from al-Kāshī's original by a mathematician, it is unlikely that mathematical errors were introduced during the copying process. Thus, the new manuscript **A** suggests that the following inessential mathematical errors were already found in al-Kāshī's lost original:

(1) **L** 78:1 = **I** 2b:14-15 = **A** 7:1-2. The text contains a number "366 and a fraction," which Luckey interpreted as 120π with π approximated by $3\frac{1}{7}$. Luckey then emended the text to "377 and a fraction," see **L** 7n1, **L** 55:8. Actually 120π = 376.99112... so the "366 and a fraction" may be a slip of al-Kāshī's mind for "376 and a fraction."

(2) In **L** 80:3 = **I** 3a:10-11 = **A** 9:4-6 and **L** 83:2 = **I** 18a:2 = A 40:3, the manuscript gives the decimal expression of $3 \cdot 2^{28}$ in words as 800, 335, 168. Actually $3 \cdot 2^{28}$ = 805, 306, 368, as noted by Qorbānī [13, p. 148]. The error shows that al-Kāshī was relatively unfamiliar with computations in the decimal system. Al-Kāshī's sexagesimal expression $3 \cdot 2^{28}$ = 1, 2, 8, 16, 12, 48 (= $1 \cdot 60^5 + 2 \cdot 60^4 + 8 \cdot 60^3$...) is of course correct.

The manuscript **A** suggests that some small changes be made to Luckey's edition. The most important changes will be listed here. The futility of these changes is a witness of the excellent quality of Luckey's work.

(1) **L** 85:6n37, and **L** 85:30n46 change Luckey's emendation *wa'l-nāqisa* to *aw al-bāqiya* as in **A** 44:2, 45:17 and **I** 20a:1, 20a:20. In **L** 20:9 (title of Chapter 7) and **L** 21:15 change *überschießenden und mangelhaften Brüchen* to *überschießenden oder übrig bleibenden Brüchen.*

(2) See **L** 87:6-7, **L** 22. **A** 47:2 has a somewhat different vocalization of the Arabic mnemonic verse which al-Kāshī composed for the decimal digits of 2π = 6.28318 53071 79586 5, see [13, p. 152]. Thus an alternative reading of **L** 22:1 is: *waḥğā ḥahağ ṣaz azaṭh^in ḥuwahhu.* In al-Kāshī's Persian verse for the decimal digits of 2π, Luckey read the two words *yek rā* as 1 and he remarked that the following digit 7 is missing in the verse. **A** 47:4 has *zā'* instead of *rā*, and Qorbānī points out [13, p. 152 that the abjad number *zā'* represents 7.

(3) Change **L** 89:5 = **I** 19b *waqt* to **A** 49:5 *diqqa;* in the text in the bottom part of Table 19a, L 24, change the last word *Rechenzeit* to *Rechengenauigkeit.*

[19] The German translation and commentary of al-Kāshī's *Treatise on the Circumference* [10] by Paul Luckey (1884-1949) appeared posthumously in 1953. Luckey had made a handwritten edition of the Arabic text, which was used by his editors, Siggel and Gieseke, to prepare the Arabic edition in [10]. I call this Arabic edition "Luckey's edition," even though Luckey did not intend to publish a critical edition because he only had access to one manuscript of al-Kāshī's treatise.

(4) Al-Kāshī's *Treatise on the Circumference* contains a large number of numerical tables. In the first row of table 22a, right side, al-Kāshī presented his approximation of 120 sin 0.25° without giving details about its computation.[20] I gives the value 0; 31, 24, 56, 58, 35, 58, 41, 17. Because the mathematically correct value of 120 sin 0.25° is 0; 31,24,56,58,35,58,41,46..., Luckey emended the last sexagesimal to 47 [10, p. 30]. Manuscript **A** also has 17, we have to accept 17 as the eighth sexagesimal of al-Kāshī's approximation. This value is consistent with the second line of the table because $(0; 31, 24, 56, 58, 35, 58, 41, 17)^2$ = 0; 16, 26, 57, 15, 2, 9, 46, 0 as in the second line of the table, whereas $(0; 31, 24, 56, 58, 35, 58, 41, 47)^2$ = 0; 16, 26, 57, 15, 2, 9, 46, 31, 28...

Acknowledgement. Part of the research for this paper was done during my sabbathical leave at the Mathematics Department of the University of Virginia, Charlottesville (USA). I thank Professors Karen Parshall and Jim Howland for their hospitality.

References

[1] T.L. Heath, *The Works of Archimedes.* Cambridge: University Press, 1897, reprint ed. New York: Dover, no date.

[2] M. Bagheri, A Newly Found Letter of al-Kāshī on Scientific Life in Samarkand. *Historia Mathematica* **24** (1997), 241-256.

[3] David H. Bailey, Jonathan M. Borwein, Peter B. Borwein, Simon Plouffe, The Quest for Pi. *Mathematical Intelligencer* **19** (1997), no. 1, 50-57.

[4] J. Lennart Berggren, J. Borwein, P. Borwein, *Pi: A Source Book.* New York: Springer Verlag, 1997, 2nd edition 2000.

[5] D. Bierens de Haan, Bouwstoffen voor de Geschiedenis der wis- en natuurkundige wetenschappen in de Nederlanden, VIII, Ludolph van Ceulen. *Verslagen en Mededelingen der Koninklijke Akademie van Wetenschappen, afdeling Natuurkunde, tweede reeks, negende deel,* Amsterdam 1876, pp.

[20] This sine cannot be computed by square-root extractions only, but it can be expressed as a root of a cubic equations with coefficients that can be constructed by means of square-root extractions.

322-369.

[6] H. Bosmans S.J., Un émule de Viète: Ludolphe van Ceulen, Analyse de son "Traité du Cercle." *Annales de la Société Scientifique de Bruxelles* **34** (1910), pp. 88-139.

[7] A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie* [Lectures on the History of Trigonometry], 2 vols. Leipzig: Teubner, 1900-1903, reprint ed. in 1 vol.: Niederwalluf: Sändig, 1971.

[8] Ludolph Van Ceulen, *Van den Cirkel* [On the Circle]. Delft: Jan Andriesz, 1596.

[9] Friedrich Katscher, *Einige Entdeckungen über die Geschichte der Zahl Pi, sowie Leben und werk von Christoffer Dyvbad und Ludolph van Ceulen. Wien 1979, Österreichische Akademie der Wissenschaften*, Mathematisch-Naturwissenschaftliche Klasse, Denkschriften, 116. Band, 7. Abhandlung.

[10] Paul Luckey, Die Lehrbrief über den Kreisumfang (*ar-risāla al-muḥīṭīya*) *von Ǧamšīd b. Mas'ūd al-Kāšī, übersetzt und erläutert von P. Luckey, herausgegeben von A. Siggel.* Berlin 1953: Abhandlungen der deutschen Akademie der Wissenschaften zu Berlin, Klasse für Mathematik und allgemeine Naturwissenschaften, Jahrgang 1950 nr. 6. Reprinted in F. Sezgin, ed., *Islamic Mathematics and Astronomy*, Frankfurt: Institut für Geschichte der arabisch-islamischen Wissenschaften, 1998, vol. 56, pp. 227-329.

[11] *Fehrest-e Ketābkhāne-ye Āstān Qods Reżawī, ta'līf Aqā-ye'Abd al-'Alī Ektābī.* Meshed: A.H. (solar) 1305-1350/1926-1971 CE. Vols. 7-8 are by A. Gulchī Ma'ānī.

[12] R.M. Th. Oomes, J.J.T.M. Tersteeg, J. Top, Het Grafschrift van Ludolph van Ceulen [The inscription on the tomb of Ludolph van Ceulen]. *Nieuw Archief voor Wiskunde* 5th series, **1** (2000), 156-161.

[13] *Abo'l-Qāsem Qorbānī, Kāshānī-Nāmeh, Aḥwāl wa Āthār-e Ghiyāth al-Dīn Jamshīd Kāshānī.* Tehran, Markaz- Nashr-e Dāneshgāhī, 2nd edition, 1368 A.H. (solar) /1989 CE.

[14] *Abo'l-Qāsem Qorbanī, Zendegīnāme-ye riyażīdānān-e dowre-ye Eslāmī.*

Tehran, no date (ca. 1995).

[15] Olaf Pedersen, *A Survey of the Almagest.* Odense: Odense University Press, 1974.

[16] Dzhemshid Giyaseddin al-Kashi, *Klyuch Arifmetiki, Traktat ob Okruzhnosti* [Key to Arithmetics, Treatise on the Circumference], Per. B.A. Rosenfeld, comm. A.P. Yuschkevitch, B.A. Rosenfeld. Moskva: Gosudarstvennoe Izdatelstvo, 1956.

[17] B.A. Rosenfeld, E. Ihsanoglu, *Mathematicians, Astronomers* and *Other Scholars of Islamic Civilization and their works (7th-19th c.).* Istanbul: IRCICA, 2003.

[18] D.E. Smith, *History of Mathematics.* Boston: Ginn & Co., 1923-25, 2 vols.

[19] A.P. Youschkewitch, B.A. Rosenfeld, Article: al-Kāshī, in C.G. Gillispie, ed., *Dictionary of Scientific Biography,* vol. 7. New York: Scribner's Sons, 1973, pp. 255-262.