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**MÉLANGES OFFERTS À HOSSAM ELKHADEM
PAR SES AMIS ET SES ÉLÈVES**

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SIMILAR MATHEMATICS IN DIFFERENT CULTURES:
JAMSHĪD AL-KĀSHĪ AND LUDOLPH VAN CEULEN ON THE
DETERMINATION OF π

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1 Introduction

One of the highlights of the medieval Islamic mathematical tradition is the determination of π to 16 decimals by Jamshīd Ibn Mas'ūd al-Kāshī or al-Kāshānī (died 1429).² Al-Kāshī held the world record until 1596, when Ludolph Van Ceulen determined π to 20 decimals in his work *Van den Circkel*.³ In the same work, Van Ceulen included a π -determination in 18 decimals by exactly the same method as al-Kāshī. Van Ceulen did not know al-Kāshī's work, which remained unknown in the West until 1925.⁴ The curious resemblance can be explained by the fact that both mathematicians were inspired by Archimedes' work. The 65-th birthday of Hossam Elkhadem provides a fitting occasion to compare the two π -determinations to 16 and 18 decimals by the two brilliant mathematicians from the Islamic world and Europe.

2 Al-Kāshī and his work on π

Al-Kāshī was born in Kāshān in Iran. In 1421 he moved to Samarkand to work as a mathematician and astronomer at the court of Ulugh Beg.⁵ His π -determination is the subject of his *Treatise on the Circumference (al-Risāla al-muhitīyya)*. This treatise probably dates back to the period when al-Kāshī was still

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² On the life and works of al-Kāshī see, e.g., [19], [17], pp. 269-271).

³ In 1593, the Belgian mathematician Adriaan Van Roomen, who did not know al-Kāshī's work either published *Ideae mathematicae pars prima*, in which he determined π to 15 decimals using a regular 15·2²⁴-gon, see [7, vol. 1, p. 174]. Van Roomen and Van Ceulen became close friends.

⁴ Al-Kāshī's work on π was first mentioned by David Eugene Smith [18 vol. 2, pp. 238-240], who had been informed by Salih Mourad from Istanbul. The similarities between *Van den Circkel* and al-Kāshī's treatise were briefly noticed by Luckey [10, p. 54].

⁵ The date of his move has been determined by Qorbānī in [13, pp. 7-9].

in Kashān, because it does not contain a dedication to an emperor. In 1953 the treatise appeared in a posthumous German translation by Paul Luckey (1884-1949) [10], with an excellent Arabic edition based on a manuscript in Istanbul. In the appendix to this paper I will discuss a manuscript of al-Kāshī's treatise which has recently been discovered in Meshed in Iran. This manuscript is a direct copy of al-Kāshī's original, made by the mathematician Bahā' al-Dīn al-'Āmuli (1547-1622 CE),⁶

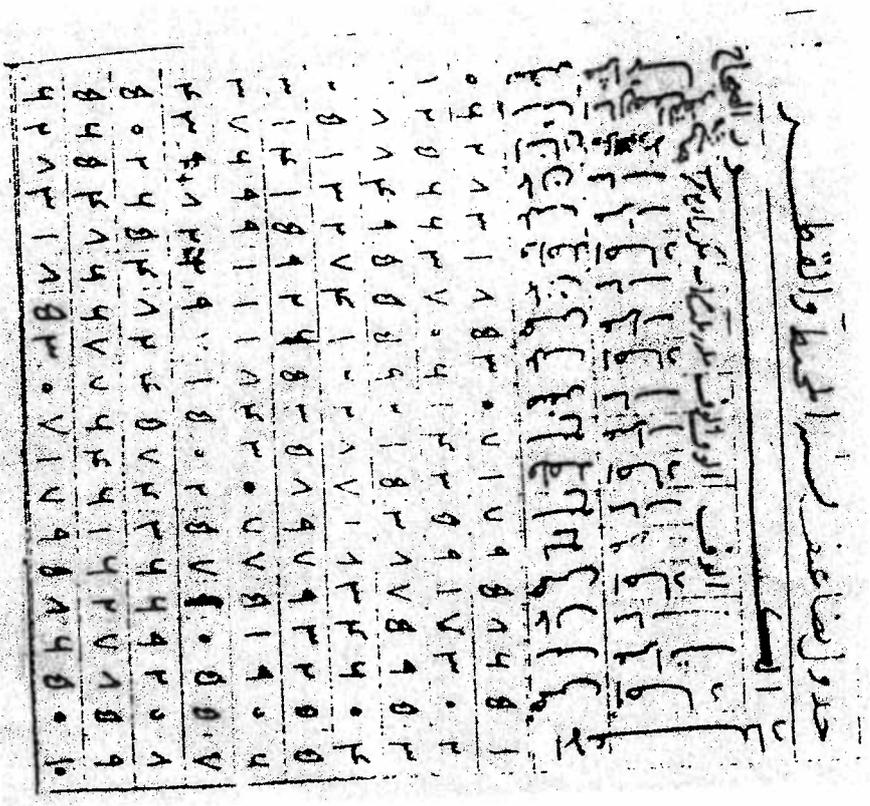


Figure 1

⁶ On al-'Āmuli see [13, p. 160], [17, pp. 348-350].

Because al-Kāshī's treatise on π is not yet available in an English translation, incorrect statements about it have been made in the Western literature. For example, a recent survey of π -determinations [3] states that al-Kāshī determined π to 14 decimals. This statement is easily refuted by Figure 1, which displays al-Kāshī's "table of the multiples of the ratio of the circumference and the diameter"⁷ in the Meshed manuscript. In this table, al-Kāshī lists in modern terms the multiples of 2π in decimal notation; the symbol π was not yet used in al-Kāshī's time. On the fifth line,⁸ al-Kāshī writes 10π as 31 4159 26535 89793 25. Here we see the first 16 correct decimals in the decimal expansion of $\pi = 3.14159 26535 89793 238 \dots$

The decimal system for fractions was discovered by al-Kāshī and several other authors in the medieval Islamic tradition, but it never became popular. Al-Kāshī made his computations in the sexagesimal system, which had been developed in ancient Babylon, and which was widely used in medieval Islamic astronomy. In his *Treatise on the Circumference*, al-Kāshī wanted to compute π with such an accuracy that in a circle with radius equal to the radius of the universe, the maximal error in the circumference is less than the breadth of one hair.

Al-Kāshī and his contemporaries accepted the cosmology of Ptolemy (ca. 150 CE, see [15]), who believed that the earth is surrounded by the concentric spheres of the moon, Mercury, Venus, the sun, Mars, Jupiter, Saturn, the fixed stars, and by an outermost sphere. Ptolemy and his Islamic successors determined the distance from the earth to the moon and the sun on the basis of measurements of lunar parallax and the apparent sizes of the moon, the sun, and the earth shadow during solar and lunar eclipses. These methods are correct but not very accurate. Thus, a relatively small error in the measurement of the earth shadow produced a distance between the earth and the sun which is only 1/20-th of the correct value.

Ptolemy assumed that the maximal distance between the earth and the sun is equal to the minimal distance between the earth and Mars. The Ptolemaic model produces essentially correct ratios between the maximal and minimal distances from the earth to each planet (but not the values of the distances themselves). From the supposed minimal distance of Mars, and the ratio between its minimal and maximal distance, Ptolemy could now find the maximal distance of Mars, which he supposed to be equal to the minimal distance of Jupiter, and so on. Finally, he assumed that the maximal distance of Saturn is equal to the minimal distance of the fixed stars. Ptolemy believed that all fixed stars were attached to a thin sphere with

⁷ Al-Kāshī actually means: half the diameter.

⁸ In his table, al-Kāshī uses Hindu-Arabic numbers but no decimal point. He writes the decimals in columns for "tens," "times the diameter," and multiples of 10^4 times the diameter for $1 \leq n \leq 16$. The fifth line actually ends with 255, but the last 5 refers to the multiple $5 \cdot 2\pi$.

a very slow (precessional) motion with respect to the outermost sphere containing the celestial equator. This outermost sphere rotated once every day around the center of the universe, which coincided with the center of the earth. Ptolemy concluded that the radius of the universe was approximately 20,000 earth-radii.

On the basis of new astronomical observations, the medieval Islamic astronomers changed some of the parameters in Ptolemy's models, but they computed the radius of the universe along the same lines with similar results. In his work *Starway to Heavens (sullam al-samā')*, al-Kāshī stated that the radius of the universe is 26,328 earth-radii [2, p. 251]. In his *Treatise on the Circumference*, al-Kāshī required that in a circle with radius R equal to 600,000 earth-radii, the inaccuracy in the circumference $2\pi R$ should be less than the breadth of a hair. Then the inaccuracy is much less than the breadth of a hair for all circles which can exist in the physical universes.

For the approximation of π , al-Kāshī used a method of Archimedes, which is as follows in modern notation. Consider a circle with diameter 1. Then the circumferences of the inscribed hexagon, the circle, and the circumscribed hexagon are 3 , π , and $2\sqrt{3}$ respectively. The circumference of the circle is greater than the circumference of the inscribed hexagon and less than the circumference of the circumscribed hexagon. Thus $3 < \pi < 2\sqrt{3} = 3.46\dots$ Using lower and upper bounds of the sides of an inscribed and circumscribed regular n -gon, Archimedes computed lower and upper bounds of the sides of the inscribed and circumscribed regular π -gon. Using the hexagon, he approximated the sides of the inscribed and circumscribed regular 12-, 24-, 48-, and 96-gon. From the 96-gon he obtained $3\frac{10}{19} < \pi < 3\frac{1}{7}$.

The algebraic expressions of the sides of these polygons involve irrational numbers, which Archimedes avoided by means of complicated estimates such as (in modern terms) $265/153 < \sqrt{3} < 1351/780$. Archimedes did not use a decimal or sexagesimal system for fractions.

Al-Kāshī mentioned Archimedes' approximation but remarked that the result $3\frac{10}{19} < \pi < 3\frac{1}{7}$ is much too inaccurate for his purpose. After the 96-gon ($96 = 3 \cdot 2^4$), al-Kāshī considered 24 more polygons, namely the 192-gon, the 384-gon, and so on, until the $3 \cdot 2^{28}$ -gon. He showed that the circumferences of the inscribed and circumscribed $3 \cdot 2^{28}$ -gon of a circle with radius 600,000 earth-radii differ by less than a breadth of a hair, so his approximation of the circle by one of these circumferences produces a sufficiently accurate value of π .

Al-Kāshī worked in a circle with radius 60 units, as was usual in the trigonometry of his time. He simplified Archimedes' method of computation in the

⁹ For Archimedes' method see [1, pp. 93-94], reprinted in [4, pp. 9-14].

following way in modern terms:

The sides of the inscribed and circumscribed n -gons in a circle with radius 60 are given by the formulas

$$120 \sin \frac{180^\circ}{n}, 120 \tan \frac{180^\circ}{n}.$$

Al-Kāshī first computed, for $n = 3, 6, 12, \dots, 3 \cdot 2^8$,

$$k_n = 120 \cos \frac{180^\circ}{n}.$$

The magnitudes k_n satisfy the simple relation

$$k_{2n} = \sqrt{60(120 + k_n)},$$

a consequence of the modern formula

$$2 \cos \frac{\alpha}{2} = \sqrt{2 + 2 \cos \alpha}.$$

Note that $k_3 = 60$. Al-Kāshī computed, in modern notation:

$$\text{up to } k_{3 \cdot 2^{28}} \quad k_6 = 60\sqrt{3}, \quad k_{12} = 60\sqrt{2 + \sqrt{3}}, \quad k_{24} = 60 \cdot 2 + \sqrt{2 + \sqrt{3}}, \dots$$

Al-Kāshī also showed that the computations are sufficiently accurate if the root extractions are carried out in 20 sexagesimals (two integer and 18 fractional). Figure 2 is Lueky's transcription [10, p. 12] of al-Kāshī's computation of $\sqrt{3 \cdot 60^2}$. For the transcription of sexagesimal numbers, I use the system of O. Neugebauer, in which a semicolon is the separation sign between integer and fractional part, and sexagesimals are separated by a comma. In this system, the result of al-Kāshī's computation can be expressed as

$$\sqrt{3 \cdot 0,0} = 1,43;55,22,58,27,57,56,0,44,25,31,42,1,56,22,42,48,58,57\dots$$

(meaning: $= 1 \cdot 60 + 43 + \frac{55}{60} + \frac{22}{60^2} \dots$). In this computation, al-Kāshī did not use Hindu-Arabic numbers but the abjad-system, which is a transcription of the following system that had been used by Ptolemy and his Greek successors. The letters of the Greek alphabet have numerical values $\alpha = 1, \beta = 2, \gamma = 3, \dots, \theta = 9, \iota = 10, \kappa = 20, \lambda = 30, \mu = 40, \nu = 50, \dots$. Ptolemy wrote the number 53 as $\nu\gamma$, that is the

Luckey used (see the facsimile in [16]), and L for Luckey's edition,¹⁹ translation and commentary [10]. A notation such as A 55:3 or L 55:3 refers to line 3 of A or L, L 85n37 refers to footnote 37 of page 85 of L.

Because the Meshed manuscript was copied from al-Kāshī's original by a mathematician, it is unlikely that mathematical errors were introduced during the copying process. Thus, the new manuscript A suggests that the following inessential mathematical errors were already found in al-Kāshī's lost original:

(1) L 78:1 = I 2b:14-15 = A 7:1-2. The text contains a number "366 and a fraction," which Luckey interpreted as 120π with π approximated by $3\frac{1}{7}$. Luckey then emended the text to "377 and a fraction," see L 7n1, L 55:8. Actually $120\pi = 376.99112\dots$ so the "366 and a fraction" may be a slip of al-Kāshī's mind for "376 and a fraction."

(2) In L 80:3 = I 3a:10-11 = A 9:4-6 and L 83:2 = I 18a:2 = A 40:3, the manuscript gives the decimal expression of $3\cdot 2^{28}$ in words as 800, 335, 168. Actually $3\cdot 2^{28} = 805, 306, 368$, as noted by Qorbānī [13, p. 148]. The error shows that al-Kāshī was relatively unfamiliar with computations in the decimal system. Al-Kāshī's sexagesimal expression $3\cdot 2^{28} = 1, 2, 8, 16, 12, 48 (= 1\cdot 60^5 + 2\cdot 60^4 + 8\cdot 60^3\dots)$ is of course correct.

The manuscript A suggests that some small changes be made to Luckey's edition. The most important changes will be listed here. The futility of these changes is a witness of the excellent quality of Luckey's work.

(1) L 85:6n37, and L 85:30n46 change Luckey's emendation *wa'l-nāqisa to aw al-bāqiya* as in A 44:2, 45:17 and I 20a:1, 20a:20. In L 20:9 (title of Chapter 7) and L 21:15 change *überschießenden und mangelhaften Brüchen* to *überschießenden oder übrig bleibenden Brüchen*.

(2) See L 87:6-7, L 22. A 47:2 has a somewhat different vocalization of the Arabic mnemonic verse which al-Kāshī composed for the decimal digits of $2\pi = 6.28318\ 53071\ 79586\ 5$, see [13, p. 152]. Thus an alternative reading of L 22:1 is: *wahḡā haḡā saz azāhī huwāhīu*. In al-Kāshī's Persian verse for the decimal digits of 2π , Luckey read the two words *yek rā* as 1 and he remarked that the following digit 7 is missing in the verse. A 47:4 has *zā* instead of *rā*, and Qorbānī points out [13, p. 152] that the abjad number *zā* represents 7.

(3) Change L 89:5 = I 19b *wagt* to A 49:5 *diqqa* in the text in the bottom part of Table 19a, L 24, change the last word *Rechenzeit* to *Rechengenauigkeit*.

¹⁹ The German translation and commentary of al-Kāshī's *Treatise on the Circumference* [10] by Paul Luckey (1884-1949) appeared posthumously in 1953. Luckey had made a handwritten edition of the Arabic text, which was used by his editors, Siegel and Giesecke, to prepare the Arabic edition in [10]. I call this Arabic edition "Luckey's edition," even though Luckey did not intend to publish a critical edition because he only had access to one manuscript of al-Kāshī's treatise.

(4) Al-Kāshī's *Treatise on the Circumference* contains a large number of numerical tables. In the first row of table 22a, right side, al-Kāshī presented his approximation of $120 \sin 0.25^\circ$ without giving details about its computation.²⁰ I gives the value 0; 31, 24, 56, 58, 35, 58, 41, 17. Because the mathematically correct value of $120 \sin 0.25^\circ$ is 0; 31,24,56,58,35,58,41,46..., Luckey emended the last sexagesimal to 47 [10, p. 30]. Manuscript A also has 17, we have to accept 17 as the eighth sexagesimal of al-Kāshī's approximation. This value is consistent with the second line of the table because $(0; 31, 24, 56, 58, 35, 58, 41, 17)^2 = 0, 16, 26, 57, 15, 2, 9, 46, 0$ as in the second line of the table, whereas $(0; 31, 24, 56, 58, 35, 58, 41, 47)^2 = 0; 16, 26, 57, 15, 2, 9, 46, 31, 28\dots$

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²⁰ This sine cannot be computed by square-root extractions only, but it can be expressed as a root of a cubic equations with coefficients that can be constructed by means of square-root extractions.

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