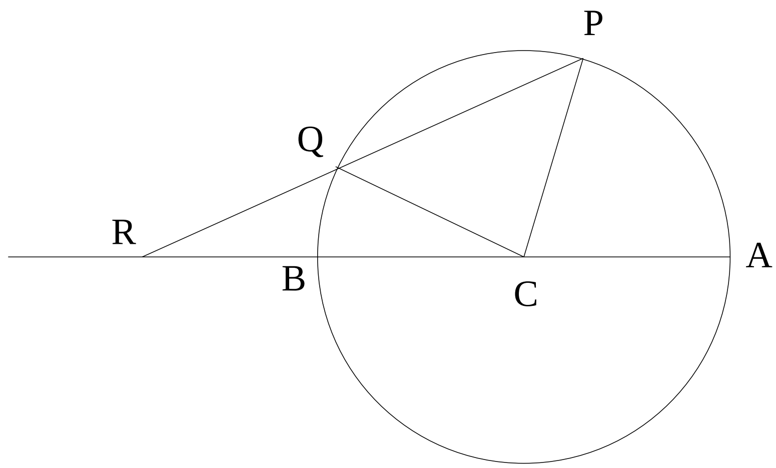
The trisection of the angle.

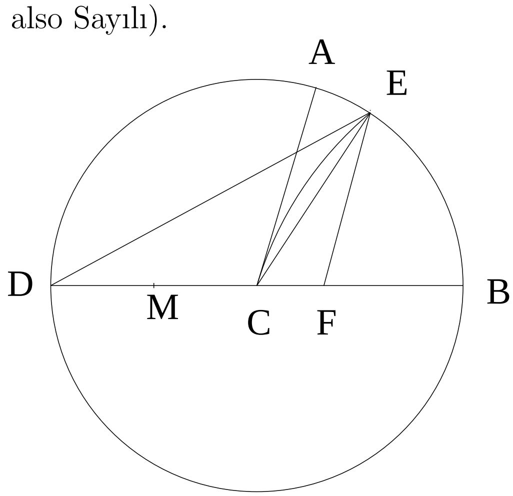
The trisection of the angle was one of the three famous problems of classical Greek geometry (the other two problems being the suplication of the cube and the quadrature of the circle). Around 300 B.C., Euclid showed in *Elements* I:10 how to divide a given angle into two equal parts by means of ruler and compass. The trisection of the angle is the problem to divide a given angle into three equal parts. The European mathematician Pierre L. Wantzel (1814-1848) proved in 1837, using the theory of algebraic equations, that the trisection of the angle is not in general possible by means of ruler and compasses (Kline p. 764). There are some special angles which can be trisected by ruler and compass, for example an angle of 90 degrees.

In the second or third century B.C. Greek geometers found trisections of the angle by other means (see Heath, vol. 1, pp. 235-244).

One Greek trisection can be reconstructed from the *Book of Lemmas* (kitāb al-ma'khūdhāt) which has been attributed to Archimedes and which has come down to us only in a medieval Arabic translation. (Figure 1) Draw a circle with centre C and let it be required to trisect the angle PCA with points P and A on the circle. Draw the diameter AB through A and extend it on the side of B. Now insert a segment QR equal to the radius of the circle between the outer side of the circle and the extended diameter, in such a way that point Q is on the circle between P and B and point P is on the rectilinear extension of RQ. Then $\angle QCR$ is one-third of angle PCA. This construction is an example of a type of constructions called *neusis* in Greek geometry. A neusis is the insertion of a straight segment of given length between two given lines, in such a way that the segment or its rectilineal extension passes through a given point. Some Greek geometers (such as Archimedes) accepted this type of constructions without further justification.



Other Greek geometers preferred to work with conic sections. Two trisections of the angle by means of a circle and a hyperbola survive in the *Collections* of Pappus of Alexandria (third century A.D.). One of these trisections was transmitted into Arabic, and it occurs in the works of Aḥmad ibn Mūsā and Thābit ibn Qurra (third century H.). A very simple trisection of the angle was invented by Abū Sahl al-Kūhī in the fourth century of the Hijra. This trisection was plagiarized by Aḥmad ibn Muḥammad ibn ᶜAbdaljalīl al-Sijzī (for Arabic texts on the trisection see Knorr pp. 267-309 and for facsimiles of the manuscripts see Knorr pp. 358-363, 370, for al-Kūhī see also Sayılı).



Al-Kūhī supposes that $\angle ACB$ is the angle to be trisected (Figure 2). Draw a circle with C as centre and arbitrary radius and assume that points A and B are on this circle. Let BC extended meet the circle at D and let M be the midpoint of CD. He then draws the orthogonal hyperbola which has centre M, passes through C and has the property that line AC is tangent to it. This hyperbola will intersect the circle at a point E between A and B. Then $\angle EDB$ is one-third of angle ACB. Proof: Draw EF parallel to AC to meet BC at F, and draw EC.

Since point E is on the hyperbola we have (by Apollonius' *Conics* I:12) $EF^2 = FC \cdot FD$, so $FC : FE = FE : FD$. Because $\angle CFE = \angle EFD$, the triangles CFE and EFD are similar, so $\angle CEF = \angle EDF = \angle EDC$. Since point E is on the circle, we have $EC = CD$ so $\angle CED = \angle EDC$. Therefore $\angle ACB = \angle EFB = \angle FED + \angle EDF = 3\angle EDC$.

Al-Kūhī's assumption that the hyperbola can be drawn is based on the theory in the end of Book 1 of the *Conics* of Apollonius of Perga (200 B.C.) which had been translated into Arabic in the third century of the Hegira. Apollonius explains the three-dimensional construction of a cone which intersects the plane in the desired conic section. Al-Kūhī wrote a treatise on a special type of compasses with which this type of construction could be carried out. There is no evidence that this so-called "perfect compass" (al-birkār al-tāmm) was reslly used in practice. In the time of al-Kūhī, the trisection of the angle was of theoretical importance only. This situation changed when it was realized that the trisection of the angle is related to the problem to compute the sine of one degree, a fundamental quantity in trigonometrical tables. If an angle of α degrees can be constructed by means of ruler and compass, one can always compute the sine of α with arbitrary accuracy by means of square root extractions. In this way one can compute e.g. the sine of 3 degrees.

Jamshīd Ghiyāth al-Dīn al-Kāshī showed in his treatise *On the Chord and the Sine* (fi'l-watar wa'l-jayb), which is now lost, but survives in later commentaries, that the trisection of a given angle can be reduced to the problem to solve a cubic equation with of the form $x^3 + q = px$ with $p, q > 0$ known quantities. He developed an iterative process to compute the root x of such an equation. In the case of the trisection of an angle of 3 degrees, this process produces a rapidly converging series of approximations to the sine of 1 degree. He then made the accurate value of the sine of 1 degree was the basis of a new sine table (Juschkewitsch pp. 319-323).

In Chapter 3 of Book 3 of al-Qānūn al-Mas^{c}ūdī, al-Bīrūnī studied the side of the regular nonagon in a circle with known radius, in order to compute the chord of 40 degrees, that is to say, two times the sine of 20 degrees (Al-Bīrūnī vol. 1, pp. 286-292). He showed that this quantity can also be found if either of the cubic equations $x^3 = 1 + 3x$ or $x^3 + 1 = 3x$ can be solved, and he gave an approximation of the root by means of which the sine of 20 degrees can be computed with an accuracy of 7 decimals (Schoy, pp. 78-82). This problem is equivalent to the trisection of an angle of 60 degrees (Juschkewitsch, p. 311)

In European mathematics, the trisection occurs again in the work of Francois Viète. He used it in order to solve the cubic equation of the form $x^3 + q = px$. The algebraic solution leads to complex numbers which Viète wanted to avoid (Kline, pp. 266-272).

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Duplication of the Cube

The duplication of the cube is one of the three famous problems of classical Greek geometry (the other two problems being the trisection of the angle and the quadrature of the circle). The duplication of the cube is to construct a cube whose volume is two times the volume of a given cube. This geometrical problem which made its first appearance in early Greek geometry before 450 B.C. or earlier. There are legends which suggest a religious origin of the problem: a certain altar had to be doubled in size in such a way that the shape is kept the same. Algebraically, the problem is equivalent to the construction of $\sqrt[3]{2}$, and it was first proved by Wantzel in 1837 that the problem cannot be solved by means of ruler and compass (Kline p. 764).

Hippocrates of Chios (ca. 450 B.C.) reduced the problem to the construction of two mean proportionals x and y between two given lines a and b, i.e. two segments such that $a : x = x : y = y : b$, and this is the problem which was solved by later Greek mathematicians. If $b = 2a$ we have $x^3 = 2a^3$ so x is the side of the cube whose volume is two times the volume of the cube with given side a.

In the fourth century BC and later, many constructions of two mean proportionals between two given lines were found by Greek geometers. Here is a list of the geometers and the means which they used in their construction (for details see Heath vol. 1, pp. 244-270):

- Archytas of Taras (first half of fourth century BC) gave a three-dimensional construction using the intersection of a cylinder, a right cone and a torus

- Menaechmus (ca. 350 BC) used conic sections, a hyperbola and a parabola. The idea of this solution is easy to describe algebraically. In modern terms. Menaechmus uses a point of intersection of the hyperbola with equation $xy = ab$ so $a : x = y : b$ and the parabola with equation $y^2 = bx$, to construct x and y such that $a : x = x : y = y : b$.

- A solution using an instrument which is basicly a combination of rulers has been attributed to the philosopher Plato (427 - 347 B.C.). The attribution must be erroneous because Plato despised the use of such instruments.

- Another mechanical solution has been attributed to Eratosthenes (third century B.C.)

- A solution by means of two parabolas is to be found in a work by Diocles (first century B.C.) entitled *On Burning Mirrors*. The equations of the two parabolas are in modern coordinates $x^2 = ay$ and $y^2 = bx$.
- Diocles also provided another solution by means of a curve called the cissoid.
- Nicomedes (second century B.C.) gave a solution by means of a neusis (see for a discussion of the concept of neusis the article on the trisection of the angle)
- Another solution by means of a circle and a hyperbola is found in works of various geometers, including Apollonius of Perga (ca. 200 B.C.). This solution was well known to the medieval Islamic mathematicians.
- Another solution using a moving ruler is given by Sporus and Pappus (both third century A.D.)

Most of the Greek solutions were transmitted into Arabic through the commentary of Eutocius on Book 2 of Archimedes' *On the Sphere and Cylinder*, or otherwise, and some solutions were translated into Latin. Most medieval Islamic and Latin geometers contented themselves by studying the many existing solutions of the problem and they did not develop new solutions (except the Andalusian king al-Mu'taman ibn Hūd, who combined a solution by Menaechmus and a solution attributed to Apollonius, and obtained a construction of two mean proportionals by means of a parabola and a circle.). Later we find geometric solutions of the problem repeated in algebraic works, for example in the *Algebra* of Umar Khayyām in the discussion of the geometric solution of $x^3 = c$ (Daneshnameh-ye Khayyam pp. 209-210). For Umar Khayyām and his contemporaries and successors, the numerical approximation of $\sqrt[3]{c}$ was a more interesting problem than the geometric construction of the root x of $x^3 = c$, which was of theoretical importance only. Khayyām says that Ibn al-Haytham solved the problem to construct four mean proportionals $x_1 \dots x_4$ between two given lines a and b (Daneshnameh-ye Khayyam p. 236). Algebraically, this problem is equivalent to the solution of the equation $x_1^5 = a^4 b$. In his *Algebra*, Khayyām also says that he described the numerical (approximative) solution of the equation $x^n = c$ in another work, which is now lost.

The 17th century European mathematicians were interested in generalizations of the problem. In his *Geometry* (1637), René Descartes studied the problem to construct n mean proportionals between two given lines a and b, that is to say n segments $x_1 \ldots x_n$ such that $a : x_1 = x_1 : x_2 = \ldots = x_{n-1} : x_n = x_n : b$ (see Bos 1981 p. 309). Descartes studied solutions of this problem by means of algebraic curves and constructions of such curves. This tied in with the general subject to construct the roots of algebraic equations by means of curves. By 1750 A.D. this theory had died out (Bos 1984).

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The quadrature of the circle

The quadrature of the circle is one of the three classical problems of Greek geometry (the others being the duplication of the cube and the trisection of the angle). The quadrature of the circle is the problem to construct a square equal in area to a given circle. The problem can be solved if one can find a figure bounded by straight lines which is equal in area to a circle.

Generations of ancient Greek geometers struggled with this problem and variations on it. Around 450 BC, Hippocrates of Chios showed that it is possible to construct by means of ruler and compass, a square equal to a lunule of a certain type; a lune is a figure bounded by two arcs of a circle. He also showed that it is possible to find a square equal in area to a lune of another type plus a circle. Of course this does not lead to a solution of the quadrature of the circle (Heath vol. 1 pp. 183-200).

Antiphon of Athens (ca. 400 B.C.) observed that the quadrature of the circle may be solved approximately, because it is possible to construct squares equal in area to a regular polygons with 4,8,16 ... sides inscribed in a given circle. Deinostratos (middle of fourth century BC) used the quadratrix, a transcendental curve, for the quadrature of the circle. Archimedes (3d century B.C.) proved that the circle is equal in area to a right-angled triangle with basis equal to the circumference of the circle and height equal to the radius. Therefore the quadrature of the circle can be found if one can rectify the circle, i.e. to construct a straight line segment equal to the circumference of the circle. Archimedes constructed such a segment by means of a tangent to a spiral. None of the solutions thus far use the standard Euclidean means (ruler and compass); as a matter of fact, a solution by means of ruler and compass is impossible.

Archimedes also opened an approach to the problem which eventually turned out to be more fruitful. By considering inscribed and circumscribed regular 96-gons, he showed that the ratio between the circumference and the diameter of the circle (which will be indicated by the modern symbol π, introduced by W. Jones in 1706) can be estimated by $3\frac{10}{71} < \pi < 3\frac{1}{7}$. His approach was further developed by the mathematicians of medieval Islam. Jamshīd Ghiyāth al-Dīn al-Kāshī approximated π to 16 decimals by considering inscribed and circumscribed $3 \cdot 2^{28}$-gons (Juschkewitch pp. 313-319). In practice, most people used the approximation $\pi \approx 3\frac{1}{7}$; Chinese mathematicians found $\pi \approx \frac{355}{113}$ which is accurate to 6 decimals. Medieval Islamic mathematicians began to wonder about the solubility of the problem. In

Book III of the Qānūn al-Mas^{c}ūdī, al-Bīrūnī conjectured that π is an irrational quantity, and Ibn al-Haytham suggested that although a square may exist equal to a given circle, it may not be possible for humans to find it.

Attempting to solve the age-old problem of the quadrature of the circle, European mathematicians in the 17th century found ways to represent π by series. Examples are the infinite product

$$\frac{4}{\pi} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \ldots$$

found by John Wallis in his *Arithmetica Infinitorum* by interpolating in a triangle with binomial coefficients (in modern terms he studied values of the *Beta*-function at $(\frac{1}{2}, \frac{1}{2})$. Another example is the "arithmetical quadrature of the circle" by Gottfried Wilhelm Leibniz, with the result $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$ (Katz pp. 525-527). This development is related to the new differential and integral calculus in the 17th century. Using the new calculus, similar but more complicated series were found and used to approximate π in a more efficient way than the method of Archimedes. For example, John Machin (1680-1752) found $\frac{\pi}{4} = \arctan \frac{1}{5} - \arctan \frac{1}{239}$ in 1706 (Beckman p. 145). This identity provides a rapidly converging series. The same developments were found (on a smaller scale and without the theoretical foundation of calculus) in Kerala (India) in the school of Mādhava from 1450 on (Katz pp. 494-496). In modern times, π has been approximated to several billions of decimals by means of computers (see for an introduction Berggren, Borwein and Borwein).

The problem of the "quadrature of the circle" was finally put to rest by Ferdinand Lindemann 91852-1939), who proved in 1882 that the number π is transcendental (Kline pp. 981-982). This means that it cannot be the root of an algebraic equation with integer coefficients. It follows that the geometric problem to find a square equal in area to a given circle can be solved neither by ruler and compass, nor by other algebraic curves (such as conic sections, which can be used to solve the trisection of the angle and the duplication of the cube.) Lindemann's proof is quite complicated but later simper proofs have been found by Niven, which can be understood by any graduate student in mathematics. (Niven's proof that π is an irrational number uses simple calculus and can be understood by any beginning student in mathematics and physics; see Berggren, Borwein and Borwein p. 276.)

Janos Bolyai (1802-1860) showed in paragraph 41 of his *Appendix* (that is his treatise on non-Euclidean geometry which appeared in 1832) that the "quadrature of the circle" is possible for some circles in Non-Euclidean geometry, because the area of such circles is equal to $c \cdot r^2$ with c a variable number depending on the radius (Stäckel pp. 214-216).

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Precession of the Equinoxes

We begin with a modern and simplified explanation of the concept of equinoxes and their precession. The earth makes one rotation per day arond the axis through the north and south pole. This axis is not perpendicular to the plane of the earth's revolution around the sun, but it is tilted to the line perpendicular to this plane by a fixed angle of $\epsilon \approx 23\frac{1}{2}$ degrees. For an observer on earth, the plane of the earth's equator makes the same angle of ϵ degree with the plane of the apparent revolution of the sun around the earth. This can be easily visualized using the concept of the celestial sphere. We define the celestial sphere as a sphere with a very large radius, so large that the radius of the earth can be neglected, and the earth can be considered as a point at its centre. (Note that the celestial sphere is a mathematical tool whihch has no physical substance.) Now we can project all stars on the celestial sphere by means of the straight segment through the (center of the) star and the (center of the) earth. Since the stars have a very small proper motion, they can be considered as "fixed" on the celestial sphere. We also project the North Pole and South Pole of the earth on this sphere and call their projections the celestial north and south pole. For an observer on earth, the celestial sphere seems to rotate around the earth once a day, about an axis through the celestial north and south pole. The planes of the earth's equator and of the apparent revolution of the sun around the earth intersect the celestial sphere in two great circles, which are called the (celestial) equator and the ecliptic respectively. The celestial equator divides the celestial sphere in the northern hemisphere, containing the celestial north pole, and the southern hemisphere, containing the celestial south pole.

For an observer on earth, the sun seems to move around the ecliptic, against the background of the fixed stars, and it makes one revolution per year. The sun is at an intersection of the ecliptic and the celestial equator at two moments during this revolution. Since the length of daylight is the same as the length of night at these moments, the two points of intersection of the ecliptic and the celestial equator are called the two equinoxes. The vernal equinox defines the beginning of spring and the beginning of the Iranian New Year; it is the moment when the (centre of the) sun leaves the southern hemishere of the celestial sphere and enters the northern hemisphere. The autumnal equinox is the moment when the (centre of the) sun leaves the northern hemisphere and enters the sourthern hemisphere. The angle between the ecliptic and celestial equator is ϵ degrees.

Unfortunately, this is not all there is to say. Because the earth rotates, it is not a perfect sphere but it is flattened out at the equator. Because of the gravitational attraction of the sun and the moon on these part of the earth which 'bulge' out at the equator, the axis of the earth makes a very slow rotation with a period of about 25770 years with respect to the fixed stars. During this rotation, the angle between the equator and the ecliptic remains approximately the same (there is a small variation called nutation with a period of 18 years, which is also caused by the motion of the sun and moon and which we will not discuss further). This rotation has two effects:

- The celestial north pole seems to move against the background of the fixed stars in a circle. Now the pole is close to the pole star, but this was not the case 2000 years ago and will no longer be the case in the year 4000 A.D.

- The two intersections between the celestial equator and ecliptic, i.e. the vernal and autumnal equinox, also seems to move against the background of the fixed stars. This effect is called the precession of the equinoxes.

Since the period of this motion is 25770 years, the intersection with the equator moves in the ecliptic with a velocity of one degree per about 71 years and 7 months. The vernal equinox now moves from the stellar constellation Pisces in the direction of the stellar constellation Aquarius. As a result of the precession of the equinoxes, the celestial longitude of the fixed stars increases by one degree per 71 years and 7 months. The celestial longitude of a star is defined as the arc from the vernal point to the perpendicular projection of the star on the ecliptic, measured in the direction of the apparent solar motion.

The increase of the celestial longitude of stars was discovered by the Greek astronomer Hipparchus (ca. 150 BC). who compared star measurements of his own time with measurements made by Greek astronomers two and three centuries before him. Thus Hipparchus discovered the phenomenon of the precession of the equinoxes. The ancient and medieval astronomers explained this phenomenon using models very different from the modern ones used today. I now describe the model used by Ptolemy, which influenced medieval Islamic astronomy deeply. Ptolemy considered that the earth was stationary and surrounded by a number of concentric spheres (or strictly speaking, spherical shells). They are, in ascending order, the sphere of the moon, Venus, Mercury, Sun, Mars, Jupiter, Saturn, the sphere of the fixed stars, and finally, the ninth sphere. Ptolemy considered that the ninth sphere rotated around the earth once a day around the celestial axis (the line between celestial North pole and South pole). The ninth sphere contained in itself a fixed circle in the same plane as the equator of the earth. The eighth sphere was the sphere of the fixed stars, which contained a circle in the same plane as the ecliptic. In the view of Ptolemy and his Islamic successors, the eighth sphere rotated with a very slow motion with respect to the ninth sphere. Thus in ancient and medieval astronomy, the "fixed stars" moved very slowly with respect to the equator, whereas in modern astronomy, the equator moves very slowly with respect to the fixed stars.

According to Hipparchus (followed by Ptolemy) the motion of the equator and ecliptic was $1\ ^o$ per 100 years, but this value was corrected in the third century H./ninth century A.D. when the astronomers of Caliph al-Ma'mūn determined it as $1\ ^o$ per 66 years, or, according to other sources, $66\frac{2}{3}$ years. Ḥabash al-Ḥāsib corrected this value to 1 degree in 70 years, which is very close to the modern one.

Certain astronomers in antiquity followed Theon of Alexandria who believed that the precession of the equinoxes proceeded at a variable speed. Some Islamic astronomers worked out sophisticated cosmological models to explain this supposed variation, and also the very slight decrease in the obliquity of the ecliptic from $23^o51'$ in the *Almagest* of Ptolemy and $23^o35'$ observed in Baghdād during the time of Caliph al-Ma'mūn (early third/ninth century). Ptolemy's value for the obliquity of the ecliptic is too high, but the decrease in the obliquity which was observed by the medieval Islamic astronomers is a scientific fact which has been verified in modern astronomy (and which is due to the gravitational attraction of the other planets on the earth). This subject of study reached its high point in the work of the Islamic Spanish astronomer al-Zarqālluh (fifth/eleventh century).

In modern times, many nonsensical ideas have been related to the precession of the equinoxes by mystics and astrologers who do not understand the mathematical nature of this concept. They argue that the vernal point has been in Pisces for 2000 years between the birth of Christ and the year A.D. 2000, and it now moves into Aquarius, thus inaugurating a new era in the history of mankind. These astrologers usually confuse the celestial constellations Pisces and Aquarius with the zodiacal signs Pisces and Aquarius. The ancient Babylonian astronomers in the fifth century B.C. subdivided the circle of the ecliptic into 12 equal sectors of 30 degrees, called the zodiacal signs.

These zodiacal signs have received the names Aries, Taurus, etc. until Pisces from the Greeks, but it would be much better if they had simply been named I, II, ... XII, since they are purely mathematical subdivisions of the ecliptic. The *names* Aries ... Pisces of the sectors I, ..., XII are explained by the fact that in Greek antiquity, the mathematical thirty-degree arcs coincided more or less with the celestial constellations Aries, Taurus, ... Aquarius, Pisces. Now, 2000 years later, the sectors I, ..., XII coincide more or less with the celestial constellations Pisces, Aries ,,,, Aquarius. The celestial constellations themselves consist of figures which have been rather arbitrarily formed from clear stars in the sky. The boundaries between the constellations have been arbitrarily fixed by the International astronomical Union in 1927 in parts of the celestial sky which seem rather "empty" of clear stars. From a scientific point of view, no special importance can be attached to the moment when the vernal point crosses the boundary between the constellation Pisces and the constellation Aquarius according to the decisions of the International Astronomical Union (or any other institution). In the ecliptic, the vernal point is always by definition the boundary between the zodiacal (i.e. astrological) sign Pisces and the zodiacal sign Aries, so the vernal point can never be in the zodiacal sign Aquarius.

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Equation of time.

In ancient times, the time of day was measured by the position of the sun in the sky; thus time could be indicated by a sundial. In this way one can define the length of one day as the period between two culminations of the sun in the south. The problem with this definition is that the length of the day according to this definition is not constant, because of the following two reasons:

- The sun does not move in the celestial equator but in the ecliptic, which is inclined at an angle of about $23\frac{1}{2}$ degrees toward the equator;
- The motion of the sun in the ecliptic is not uniform since the orbit of the earth around the sun is not a perfect circle but an ellipse. The motion is slowest when the sun is at its apogee (near the end of spring), fastest when it is in its perigee (near the end of fall).

The effect of these two deviations from uniformity is only a few seconds per day, but the cumulative effect over several months can be more than half an hour. This means that the culmination of the sun in the south (which defines the local solar time) is half an hour earlier or later than expected on the basis of a uniform length of day. The equation of time can also be noticed in modern times by the fact that in the period after the winter solstice, sunrise is for a long time at approximately the same hour, although the sunset is becoming considerably later. (A modern clock or watch always indicates mean solar time plus or minus a constant which depends on your locality.)

The cumulative effect of the equation of time was already noticed in ancient astronomy in connection with the analysis of lunar and solar eclipses. In order to compensate this effect, one has to define the astronomical day in a way which does not involve the true sun. In modern astronomy, the *mean sun* is defined as a point which moves on the celestial equator and coincides with the true sun once per year, when the true sun is at the vernal equinox. The motion of the mean sun is uniform but approximnately the same as the (slightly irregular) motion of the true sun. The mean solar time is now defined by means of the mean sun, and the "equation of time" is the difference between the hour angles of the true sun and the mean sun (that is, the difference of time between their culminations at the local meridian). In ancient and medieval astronomy, the mean sun was located in the ecliptic (at the centre of the epicycle of the true sun) and the mean sun was not used for the definition of the equation of time. The length of the mean day was computed, and the equation of time was arbitrarily defined to be zero at a particular moment of some year (the "epoch"). At all other moments the equation of time could then be computed. The zero point was often chosen in such a way that the equation of time is never negative (see van Dalen, On Ptolemy's Table, p. 103).

Tables for the equation of time have been computed since the time of Ptolemy (ca. A.D. 150). The computation requires simple geometry of the sphere and trigonometrical functions. The final formula depends on the latitude of the locality on earth, the obliquity of the ecliptic, the parameters of the solar model used (eccentricity, apogee). Many medieval Islamic astronomers added tables for the equation of time to their astronomical handbooks (Zījes), often slightly modifying the definition to facilitate computational procedures. The tables by al-Khwārizmī, Kūshyār ibn Labbān and al-Kāshī have been studied by modern historians. The mathematical analysis of these tables and the reconstruction of the parameters is an interesting mathematical problem.

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