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INHALT DES VIERTEN BANDES

AUFSÄTZE

above, even the selection of passages from the *Enneads* may have been in part dictated by the interests of a Hermetic school[62].

There are other traces of the influence of Hermetism on Arabic philosophy and the subject requires further research. I only mention one instance which seems to have been entirely overlooked in spite of the amount of work expended in the field. The "Longer Version" of the *Theology of Aristotle* has some intriguing references to a doctrine of the word and of the will of God, shared (or taken up) by Ismāʿīlī theology[63]. Zimmermann seems to regard the *Sirr al-Khalīqa* as the source or at least the earliest known occurence of these elements in the Islamic world and tries to explain the importance of the "Word" in the *Theology of Aristotle* as a misinterpretation of the Stoic λόγοι σπερματικοί. While the confusion may have played its rôle, I think it is simpler to assume a direct influence of Hermetic conceptions. Also the pervasiveness of the term *rūḥānī* in our texts with a positive valuation (= incorporeal) which corresponds neither to the Stoic nor to the medical usage common in Greek is best explained as a reflex of Hermetic-Gnostic terminology[64].

[62] This assertion would naturally require to be substantiated by a detailed study of the *Theology*. Cf. e. g. p. 51,4–5 Dieterici: *afāḍa ʿalā nafsi-hi* (scil. *al-marʾ al-ṣāliḥ*) *al-nūru al-awwalu min nūri-hi*; 118,11–119,5; etc.

[63] Zimmermann, *op. cit.* 196 ff. after Pines. For the word of God, cf. *Corpus Hermeticum* I,6; 8; 10; 11; IV,1; XII,14; for his will: I,8, 31; IV,1; V,7; XIII,2. The "doffing metaphor" (Zimmermann 138–141; al-Kindī's *Risāla* 274,3; 8; 275,12 etc.) also belongs here.

[64] Cf. Endress, *Proclus Arabus* 128–131. In *Theol. Ar.* 32 Dieterici, *jawāhir rūḥāniyya* = Gr. ἀσώματα. Quite often, *rūḥānī* will be an equivalent of *ʿaqlī* or *ilāhī*.

THE TREATISE ON GEOMETRY IN AL-HINDĪ'S *PROBLEMS OF PHILOSOPHY*

JAN P. HOGENDIJK*

1. *Introduction and summary.*

The Institute for the History of Arabic-Islamic Science in Frankfurt published in 1985 a facsimile-edition of the *Problems of Philosophy* (*Jumal al-Falsafa*) of Muḥammad ibn ʿAlī ibn ʿAbdallāh al-Hindī. The facsimile was made from a manuscript written by Al-Hindī himself in 529 H./A. D. 1135. Before the publication of this facsimile neither the author nor his work were known. The *Problems of Philosophy* consists of seven treatises (*maqālāt*) on arithmetic, geometry, astronomy, music, logic, natural philosophy and theology. All treatises are in the form of questions and answers. Al-Hindī says that his work contains "doctrines of the learned" and "opinions of the wise philosophers"[1], but he does not mention his sources explicitly.

In this paper we will analyze the treatise on geometry on pages 16–44[2] of the facsimile edition. In Section 2 we discuss the nature of this treatise and we translate the beginning of it. Al-Hindī compiled his treatise from a large number of sources, of which I have only been able to identify the three that have been listed in Section 3. His most important sources seem to have been two 10th-century encyclopedic works written by non-mathematicians, namely the letters of the Ikhwān al-Ṣafāʾ ("Brethren of Purity")[3] and the *Keys of the Sciences* by Abū ʿAbdallāh

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[1] *aqwāl al-ʿulamāʾ, ārāʾ al-falāsifa al-ḥukamāʾ*, see p. 2, lines 7–8 of the facsimile-edition.

[2] Pages 33–34 in the facsimile-edition should be placed between pages 16 and 17. Compare footnotes 7 and 35.

[3] On the Ikhwān al-Ṣafāʾ see EI², III, pp. 1071–1076. The Arabic text of

al-Khwārizmī[4]. Section 4 is about material in al-Hindī's treatise that derives ultimately from the Greek geometers Heron of Alexandria[5] and Geminus[6], who lived in the first century A. D. This material has not been found in other Arabic sources, and one wonders how it was transmitted. In Section 5 we will show that al-Hindī had no advanced mathematical training, although he knew the geometry necessary for the study of astronomy. The final section is devoted to traces of contemporary discussions in Arabic-Islamic mathematics in al-Hindī's treatise. The most interesting of these traces is a statement that contradicts Euclid's parallel postulate.

2. *A survey of Al-Hindī's treatise on geometry.*

Al-Hindī's treatise is in five parts[7], on lines and angles, on surfaces, on solids, on mensuration and on units of measurement respectively. We will mainly be concerned with the first three parts, containing a total of 185 questions and answers. Most of these concern the definition of terms and the classification of figures (for example triangles and quadrilaterals), but in some questions a property of a figure is asked for, and the answer is then a simple theorem. No proofs are given.

Al-Hindī first defines several basic concepts such as lines and angles, surfaces, triangles, quadrilaterals, polygons, and different figures bounded by two circular arcs or by one circular arc and one or two straight lines. He then discusses spherical, cylindrical and conical surfaces. Thereupon he defines several solids, such as regular polyhedra, parallelepipeds, pyramids, prisms, the sphere, the cone, the cylinder, and sections or segments of these. The fourth part on mensuration contains rules for the determination of the surface areas of different figures, without proofs but with numerical examples. The rules and examples are given in the form of 39 questions and answers. The highlights of this section are a correct formula for the surface area of a segment of a circle[8] and Heron's formula for the area of a triangle in terms of its sides[9]. The last part on units of measurement consists of 13 questions and answers.

Al-Hindī seems to have written his geometrical treatise not as an introduction to geometrical problem-solving, but rather as a preliminary for the study of the subsequent treatises on astronomy etc, with the treatise on theology as the final goal. It seems to me that his treatise on geometry could not be very succesful from a didactical point of view, because the explanations are very concise and not accompanied by any figures. In order to give the reader an idea of the plan of the work a translation of the first thirteen questions and answers will now be presented. The questions have been numbered for future reference.

"In the name of God, the merciful, the compassionate. The second treatise of the *Problems of Philosophy* on geometry.

(1) What is *jūmaṭriyā*[10]? *Jūmaṭriyā* is geometry.

(2) What is geometry? It is an art dealing with magnitudes and the knowledge of their nature, their kinds and their properties, and the quantitative relationship (*qadr*) between those (magnitudes) that are of the same kind[11].

(3) What are the magnitudes? They are the (things) that possess distances[12], and it is said that they (the distances) are the quantity of each type of them (i. e. of magnitudes) with respect to the (magnitudes) of the same kind[13].

the letter (*risāla*) on geometry is in Dieterici, vol. 14, pp. 292-301 and in Ikhwân al-Ṣafā' pp. 49-72, a German translation is in Dieterici vol. 3, pp. 23-45.

[4] Abū 'Abdallāh Muḥammad ibn Aḥmad ibn Yūsuf al-Khwārizmī (ca. A. D. 950) is not the same person as the famous mathematician Muḥammad ibn Mūsā al-Khwārizmī who wrote on Hindu-Arabic numbers and on algebra. For the Arabic text of the section on geometry in the *Keys of the Sciences* see al-Khwārizmī pp. 202-209, a German translation is in Wiedemann.

[5] On Heron of Alexandria see DSB vol. 6, pp. 310-315.

[6] On Geminus see DSB vol. 5, pp. 344-347; for his date see Neugebauer vol. 2, pp. 579-580.

[7] Here are the detailed references. First part: pp. 16, 33, 34, 17, 18 lines 1-2; second part: pp. 18 line 3-27 line 5; third part: p. 27 line 6-p. 32 plus p. 35 line 1; fourth part: pp. 35 line 2-42 line 16; fifth part: p. 42 line 17-p. 44.

[8] p. 35, line 14 - p. 37 line 7.

[9] p. 40, lines 2-12.

[10] The same transcription of the Greek γεωμετρία occurs in the *Keys of the Sciences* (al-Khwārizmī 202:8).

[11] Only magnitudes of the same kind can have a ratio according to Book V of Euclid's *Elements* (see Heath, *A history of Greek mathematics*, vol. 2, pp. 384-385).

[12] I always translate *bu'd* as "distance" for sake of consistency, even though "dimension" might be clearer in some cases.

[13] Here the "distance" seems to be defined as a ratio. However, length, breadth and depth are magnitudes, not ratios.

(4) How many are the magnitudes? Three: lines, surfaces and solids.

(5) How many are the distances? Three: length, breadth and depth.

(6) What is a line? It is a magnitude having one distance – length – only.

(7) Where is it found? Abstractly: by way of the (human) mind and imagination, not by way of the senses. By way of the senses: it is in a surface because it is the end of it (the surface). Thus, if the breadth is taken away from a surface, only the length remains, and that is a line.

(8) What is the end of a line? Its end is two points.

(9) What is a point? It is a thing without distance, that is to say, without length and without breadth and without depth.

(10) Where is the point found? In the (human) mind it is found abstractly, and for the senses it is found in the line, because the line is length without breadth. Thus if the length is taken away from it, its end remains, and that end is a point. So the point possesses neither length nor breadth nor depth by necessity.

(11) How many are the types of principal (i. e. general) lines? Two, composite (*murakkab*) and incomposite (*ghayr murakkab*)[14].

(12) How many are the types of incomposite lines? Three, straight, circular (*muqawwas*) and curved (*munḥanī*).

(13) What is a straight line? (13.1) It is (a line), the distance of which is equal to the distance between the two endpoints. (13.2) And it is said to be (a line) which is stretched rectilinearly between the two endpoints. (13.3) And it is said to be the shortest line joining two points. (13.4) And it is said that it is (a line) such that all points that are assumed on it are in one direction. (13.5) And it is said to be (a line) such that if its end (i. e. the two endpoints) are fixed and if it is turned around, it will not be moved from its position. (13.6) And it is said to be (a line) such that its parts can fit on each other from all sides."[15]

Notations such as 20:4 will henceforth be used as references to page 20, line 4 in the facsimile-edition.

[14] Question 16 runs: "what is a composite line? It is composed of two or more than two incomposite lines of one or more kinds" (16:21, 33:1).

[15] 16:1–18. See sections 3 and 4 of this paper for a further discussion of some of the quoted passages.

3. *Three sources.*

The problem of finding Al-Hindī's sources is complicated because many texts on elementary geometry define simple geometrical concepts in similar ways. Thus the fact that Al-Hindī's treatise and another Arabic text contain a few similar definitions does not mean that Al-Hindī used that text. However, Al-Hindī's treatise contains numerous passages that resemble the letter on geometry of the Ikhwān al-Ṣafā'[3] and the section on geometry in the *Keys of the Sciences* of Al-Khwārizmī.[4] We will now discuss some striking similarities which indicate that Al-Hindī actually had these two texts in front of him when he wrote his own treatise.

We begin with the *Keys of the Sciences*. Al-Hindī's question 40 runs: "How many are the straight lines that meet (other lines)? They are nine kinds: side, arm, base, diagonal, perpendicular, chord, arrow, sine, versed sine" (34:12–14). These concepts are then further defined in questions 41–49 (34:14–17:5). Al-Khwārizmī defines the same concepts in the same order, and in almost the same words (al-Khwārizmī pp. 205–206).

In his 158th question Al-Hindī asks:

"What is a torus (*ḥalqa*)? It is a solid with one round surface with a space in the middle in which a sphere can fit" (30:11–12). This vague definition is literally the same as that in Al-Khwārizmī p. 209:4–6[16].

Questions 6–10 in the quotation in section 2 resemble the following passage in the *Keys of the Sciences*:

"The line is a magnitude with one distance – length – only. It can only be seen together with the surface, because it is its end. Abstractly, it can be perceived by the imagination only. The two ends of the lines are points. The point is a thing without distance in length, and without breadth and depth. It cannot be perceived by the senses without the line, because it is its end. Abstract-

[16] Al-Hindī found this definition not sufficiently precise, for he added a passage that he probably took from another source: "(159) How is the torus produced? From two tangent circles such that one is perpendicular to the other, if the perpendicular circle is turned on the circumference of the other circle until the circumference returns to the point where it began to move" (30:12–14).

ly, it can only be perceived by the imagination" (al-Khwārizmī pp. 203–204).

The letter on geometry of the Ikhwān al-Ṣafā' was clearly the most important source of Al-Hindī's treatise. The whole section on units of measurement (42:17–44:8) is virtually identical with part of this letter (ed. Dieterici 299:19–301:6).

The following similarity is also remarkable. Al-Hindī says in (17:5–6):

"(50) How many are the positions of circular lines? Two, parallel and intersecting (above the line he writes: meeting). (51) What are parallel circular lines? They are drawn about one centre. (52) What are meeting circular lines? They are (circular lines) containing an angle."

(50) and (51) are strange because circles with different centres do not always meet or intersect. When writing (50) and (51), Al-Hindī probably looked at the following passage of the Ikhwān al-Ṣafā':

"Parallel circular lines are circular lines with one centre, intersecting circular lines are (circular lines) with different centres" (ed. Dieterici 295:21–296:1).

He then realized that the Ikhwān al-Ṣafā' incorrectly defined intersecting circles, and wrote (52), which is in itself a correct statement. However, he overlooked that (50) should have been modified as well.

On one occasion Al-Hindī asked a question motivated by the Ikhwān al-Ṣafā', and then forgot to answer that question. He says (italics mine):

"(139) How many are the kinds of solids that are contained by quadrilaterals (i. e. rectangles) *and how many quadrilaterals contain them?* They (the kinds) are four. Among them is one with three equal distances, I mean length, breadth and depth, and it is called cube, and it belongs to the figures inscribed in a sphere; among them is one with two equal distances and a third distance which is shorter, and it is called brick (*libnah*); and among them is one with two equal distances and a third distance which is greater than the two above-mentioned (distances), it is called beam (*tīr*)[17] and among them is the (kind) with different distances, called tablet-shaped (*lauḥī*)" (28:19–29:3).

[17] The word *tīr* (beam used in building) could also be read *bi'r* (well,

This is all Al-Hindī has to say about these parallelepipeds, but the Ikhwān al-Ṣafā' also discuss the number and kind of the rectangular faces in each case (ed. Dieterici, p. 298).

A third source of Al-Hindī is, not surprisingly, Euclid's *Elements*. In 21:14, Al-Hindī lists the five kinds of quadrilaterals defined in the *Elements*[18], and in this case he explicitly refers to Euclid.

Al-Hindī also used a variety of other sources that I have not been able to identify. Except for the section on mensuration these unidentified sources seem to have been of secondary importance[19].

4. *Material of Greek origin.*

One of the historically interesting aspects of Al-Hindī's treatise is the fact that it contains material of Greek origin not hitherto found in Arabic sources. An example is the answer to question 135:

spring). Al-Khwārizmī mentions both possibilities, but he says that *bi'r* is more appropriate than *tīr* (al-Khwārizmī p. 208:8–9). The Ikhwān al-Ṣafā' use *bi'r* (see for example Dieterici, 298:8). Al-Bīrūnī, on the other hand, only gives *tīr* (*Introduction to Astrology* pp. 29, 30). The three types of parallelepipeds occur already in the *Definitions* of Heron, where they are called πλινθίς (square brick), δοκός (beam) and σφηνίσκος (wedge) (*Definitions* 112, 113 and 114, see Heron pp. 68–71). Thus it appears that *tīr* was the original Arabic term and that *bi'r* is a later corruption.

[18] *Elements*, definition 22 of Book 1, see Heath, *Elements*, vol. 1, p. 154.

[19] Al-Hindī appended to his treatise on geometry on p. 44 the following list of books which he does not seem to have used for this treatise. He writes in smaller handwriting (compare 112:4):

"Arrangement of (the works) in an autograph manuscript of Isḥāq ibn Ḥunayn that are read after (the *Elements* of) Euclid. The *Book of Optics* of Euclid, one treatise; the *Phaenomena* of Euclid, one treatise; the *Nights and Days* of Theodosius, two treatises; the *Spheres* of Theodosius, three treatises; the *Inhabited (world)* of Theodosius, one treatise; the *Ascensions* of Hypsicles, one treatise; the *Moving Sphere* of Autolycus, one treatise; *Risings and Settings* of Autolycus, two treatises; the *Distances and Sizes of the Stars* of Aristarchus, one treatise; the total is thirteen treatises". The thirteen ancient books are the core of the well-known collection of "middle books (*mutawassiṭāt*)" that should be read after Euclid's *Elements* and before the *Almagest* of Ptolemy. For the mathematical translations of Isḥāq ibn Ḥunayn (830–910) see GAS V, 272–273.

"Is it possible that a sphere circumscribes another solid even though its faces are equilateral and equiangular (figures) of two kinds?

Archimedes mentioned that a sphere can circumscribe two solids, contained by triangles and squares, each of them having fourteen faces; one of them is contained by eight equilateral and equiangular triangles, and six equilateral and equiangular quadrilaterals, so that this figure is composed of air and earth, and the other, in opposition to that (first solid), is contained by six equilateral and equiangular triangles and eight equilateral and equiangular quadrilaterals" (28:2–6).

This answer resembles the following passage in the *Definitions* of Heron of Alexandria:

"Euclid proved in the thirteenth Book of the *Elements* how these five solids (i. e. the five regular polyhedra) can be inscribed in a sphere, but he only thinks of the Platonic solids. Archimedes says that there are a total of thirteen solids that can be inscribed in a sphere, adding eight to the above-mentioned five. (He says that) Plato also knew one of them, the solid with fourteen faces, of which there are two sorts, one consisting of eight triangles and six squares, of earth and air, and already known to some of the ancients, the other again made up of eight squares and six triangles, which seems to be more difficult".[20]

The resemblance between the two passages is striking because of the rather nonsensical nature of the contents. We know from Pappus of Alexandria (ca. A. D. 250) that Archimedes discovered 13 (not 8) semi-regular polyhedra[21]; a semiregular polyhedron is a (convex) solid with congruent solid angles, whose faces are regular polygons, not all having the same number of sides. Such a polyhedron can always be inscribed in a sphere. There is a semiregular polyhedron with eight triangular and six square faces[22], but a polyhedron with eight square faces and six triangular faces

[20] Heron pp. 64–67, translation adapted from Heath, *A history of Greek mathematics* vol. 1, p. 295. There Heath also discusses the Platonic doctrine which associates the cube with earth and the octahedron with water.

[21] Heath, *A history of Greek mathematics* vol. 2, p. 98. Further information on the semi-regular polyhedra and beautiful drawings can be found in Williams pp. 72–79.

[22] Such a polyhedron was constructed by Thābit ibn Qurra, see Bessel-Hagen and Spies.

can never be inscribed in a sphere[23], hence it is not surprising that Heron found this second polyhedron "more difficult". Thus the alleged quotation of Archimedes is certainly not genuine.

An Arabic translation of the *Definitions* of Heron is not extant, and no reference to such a translation occurs in the known Arabic literature[24]. The two quoted passages support the thesis of Sonja Brentjes that the *Definitions* of Heron were transmitted into Arabic[25]. Heron's *Definitions* also contain the equivalents of Al-Hindī's definitions 13.5 and 13.6 of a straight line[26], but Al-Hindī need not have taken these from Heron because they were also transmitted by Al-Nayrīzī who took them from Simplicius.[27]

The answer to question 11 quoted in section 2 is also of Greek origin. Proclus tells us in his commentary on Book 1 of Euclid's *Elements*[28] that composite (σύνθετον) and incomposite (ἀσύνθετον) lines were first discussed by Geminus in a large geometrical work which is now lost. A translation of the work of Geminus has not been found, but parts of it may have been transmitted into Arabic[29].

Finally we draw attention to Al-Hindī's definition 13.4 of a straight line. This definition is not found in any Greek source, but it occurs in the *Introduction to Geometry* of Al-Sijzī, who at-

[23] Consider a polyhedron with 8 square faces and 6 equilateral triangular faces. The polyhedron has 14 faces and $(32 + 18)/2 = 25$ edges, so by Euler's theorem it must have $25 - 14 + 2 = 13$ angular points. Assume that all angular points lie on a sphere. Suppose that at one angular point only three faces come together. In all cases one easily shows that the surrounding sphere is too small to contain the remaining 11 faces, hence a contradiction follows. Thus at least 4 faces come together at each angular point. All angular points are therefore endpoints of at least 4 edges. Because there are 25 edges with 2 endpoints each, it follows that the number of angular points is at most $50/4 < 13$. Contradiction.

[24] GAS V, 153.

[25] Brentjes p. 261.

[26] Heron p. 18 lines 1–5.

[27] Curtze p. 7, lines 25, 6.

[28] Proclus, tr. Morrow, p. 90; see also Heath, *Euclid's Elements*, Vol. 1, p. 153, 165–169.

[29] Incomposite and composite lines are not mentioned in the (Latin translation of the) commentary of al-Nayrīzī to Euclid's *Elements*. This commentary contains several statements attributed to a certain Aghānis, who may or may not be identical to Geminus.

tributes it to the ancients[30]. It is unlikely that Al-Hindī copied 13.4 from the *Introduction to Geometry*, because Al-Sijzī gives a number of additional definitions of a straight line that do not occur in Al-Hindī's treatise.

5. *Al-Hindī as a geometer.*

Al-Hindī derived the material in his treatise from other sources, but the systematical arrangement of his treatise indicates that he had a working knowledge of elementary geometry. At the end of the treatise on astronomy he added his own discussion of the rising and setting of ecliptical signs at localities close to the north pole (p. 112). Thus we can assume that he had mastered the geometry necessary for the study of astronomy. The following question and answer shows that he was aware of the existence of more advanced geometry:

"(58) How many are the types of curved lines? they are infinitely many, but five kinds are used in geometry: the three conic sections, called the sufficient, exceeding and deficient section (i. e. the parabola, hyperbola and ellipse), and the line called spiralic[31], and the line called dome-shaped[32]" (17:21–18:2).

[30] GAS V, 333, 20; ms. Chester Beatty 3652, f. 2b:9: "The ancients defined it (as follows): some of them said that it is such that all points that are assumed on it are in one direction" (*wa-ḥaddahu* (sc. *al-khaṭṭ al-mustaqīm*) *al-qudamā', minhum man qāla innahu huwa lladhī kullu l-nuqaṭi llatī tufraḍu 'alayhi fa-hiya fī samtin wāḥidin*)

[31] *lawlabī*. Here the source of Al-Hindī may have been the commentary of Al-Nayrīzī on the *Elements*. The Arabic text of the relevant part of this commentary is lost, but the Latin translation mentions "linee, que sunt *leulavie*" (lines that are spiral-shaped) without giving further information (Curtze p. 6 line 21). Al-Sijzī mentions spiral-shaped (*lawlabī*) lines in his *Introduction to Geometry* (f. 2b, see footnote 30), and it appears from his description that the curve in question is the cylindrical helix, not the spiral of Archimedes. According to Proclus' commentary to the *Elements* (which is not known to have been translated into Arabic) the curve was studied in antiquity by Apollonius of Perga and by Geminus (Proclus pp. 85, 196).

[32] *nīmkhā'ījī*, from Persian *nīmkhāyeh*, a cupola, dome. I do not know what curve this is. I know of only one other occurrence of this term in a mathematical text, namely in the following passage in the Cairo edition (not in Dieterici's edition) of the *Letters* of the Ikhwān al-Ṣafā': "among the figures there is a (figure) called egg-shaped (*bayḍī*), and one called crescent-

Further on, Al-Hindī says that the intersection of a cylinder with a plane is called a "curved line" (35:1), but he does not say that the curved line is an ellipse. In 31:5–10 he gives the definition of a right cone of Euclid and the general definition of a cone of Apollonius, and in 31:13–16 he states that there cannot be two different circles of equal size on one cone. This is correct for a right cone, but on an oblique cone there can be different circles of equal size, as Apollonius shows in *Conics* I, proposition 5[33]. Thus Al-Hindī was not familiar with the theory of conic sections. Nothing in the treatise indicates that Al-Hindī was familiar with any other advanced topic in mathematics. Therefore we conclude that Al-Hindī knew the geometry necessary for the study of astronomy, but little more than that.

6. *Traces of contemporary mathematics in Al-Hindī's treatise.*

Most of the material that has been discussed thus far, including that derived from the *Keys of the Sciences* and the geometrical letter of the Ikhwān al-Ṣafā' is ultimately of Greek origin. The concepts "sine" and "versed sine" which Al-Hindī took from the *Keys of the Sciences* are Indian. We now cite some passages from Al-Hindī's treatise that involve mathematical developments in the Arabic-Islamic tradition.

1. Question 99 runs: "What is the property of the heptagon inscribed in a circle? It is that if in the circle an isosceles triangle is inscribed on its side (i. e. having the side of the heptagon as its base), each of the angles at the base, which are subtended by the equal sides, is three times the remaining angle. Thus it follows that if we can construct an isosceles triangle, such that each of the angles at the base is three times the remaining angle, we can construct the heptagon in the circle" (23:21–24:4). Abu l-Jūd Muḥammad ibn al-Layth had constructed the heptagon in this

shaped, and one called the pine-apple shaped cone, and one called myrobalan-shaped, and one called dome-shaped (*nīmkhā'ījī*, misspelled as *nīmkhānījī*), and one called drum-shaped, and one called olive-shaped" (Ikhwān al-Ṣafā' pp. 56–57). I thank Dr. Remke Kruk for helpful suggestions concerning the interpretation of *nīmkhā'ījī*.

[33] See Heath, *Apollonius*, pp. 2–3.

way in the late tenth century A. D., and he made similar statements on several occasions.[34]

2. In the beginning of the treatise there is a long marginal note, extending over several pages[35] and consisting of a lengthy discussion of the concept of a point. I have not been able to identify the treatise from which the note was taken; the note is somewhat similar to but much longer than the discussion of a point in Ibn al-Haytham's *Commentary on the Premisses* of Euclid.[36]

3. Question 59, which begins on 18:2 and continues in the margin, involves the hyperbola and its asymptote. The text runs:

"How is it possible that if some point is assumed on the cone[37] and if there is drawn a line to the line of the (conic) section[38], that it approaches it, but does not meet it, and whenever their extensions increase, they approach each other but never meet? That is (possible), if a point is marked on one of the asymptotes of the (conic) section, and if there is drawn a line parallel to the other (asymptote). Then, no matter how small the distance is between it and the common point (of the two asymptotes), that line will meet the conic section. Further, whenever the line of the (conic) section and the asymptote are more distant from the common point of the two (asymptotes), they will approach each other, and the distance between them will be less, but they do not meet.

This may also be possible (*wa-qad yakūnu dhālika*) for two parallel lines if one imagines the distance between them divided indefinitely. Then they also approach but do not meet."

[34] See Hogendijk, Heptagon, pp. 249, 254, 271 note 1.

[35] The marginal remark should be read in the following order: p. 16 right side, top of page; p. 33 top of page, left side, bottom of page; p. 16 bottom of page; p. 34 right side, top of page; p. 17 top of page, left side, bottom of page; p. 34 bottom of page; p. 18 top of page; p. 19 top of page, left side. The remark was written by Al-Hindī, compare page 112 line 4.

[36] GAS V, 370, 28; see Sude pp. 27–30.

[37] This should probably be: ⟨section of the⟩ cone.

[38] In ancient and Arabic-Islamic geometry the (conic) section is sometimes considered to be part of the plane inside the cone that produces the section, and therefore the "line of the section" is the curve that we would call the conic section. Al-Hindī does not say anywhere that the conic section in question is a hyperbola, nor does he explain what an asymptote is.

The last sentence is unclear but interesting. If one assumes that two parallel (that is: non-intersecting) lines approach one another, it follows that through a given point one can draw more than one line parallel to a given line. This is inconsistent with Euclid's famous parallel postulate. A fragment of a text of Al-Bīrūnī (972–1048) also contains a hint to the effect that some of his contemporaries doubted the truth of the parallel-postulate[39]. Thus Al-Hindī's treatise contains an echo of discussions that belong to the prehistory of non-Euclidean geometry.

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GAS see F. Sezgin.

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[39] The text was first identified by P. Bulgakov. For a Russian translation see Rozenfeld and Yuschkevitch, pp. 63–66. For a summary see Hogendijk, Rearranging, pp. 158–159.

In a marginal remark to his treatise on astronomy Al-Hindī renders five true statements that may seem paradoxical to persons without sufficient training in the mathematical sciences. The first of these statements is "to imagine two straight lines which approach one another continuously but do not meet" (p. 100, lines 14–15 in the margin). A similar statement is discussed in the fragment of Al-Bīrūnī (the "straight lines" are segments).

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DAS KAPITEL ZUR ZAHLENTHEORIE IN DEN *PROBLEMEN DER PHILOSOPHIE VON* AL-HINDĪ

SONJA BRENTJES*

Die Abhandlung *Probleme der Philosophie* von Muḥammad b. ʿAlī b. ʿAbdallāh al-Hindī, die 529 H/1135 geschrieben wurde, heute als Autograph vorliegt und 1985 vom Institut für Geschichte der Arabisch-Islamischen Wissenschaften in Frankfurt a. M. als Faksimile-Edition herausgegeben worden ist[1], bereichert die Geschichte der Mathematik des islamischen Mittelalters um ein weiteres enzyklopädisches Werk. Sein Autor war bislang unbekannt. Das erste Kapitel der aus sieben Teilen bestehenden philosophischen Enzyklopädie[2] ist der Zahlentheorie gewidmet. Es soll in dem vorliegenden Artikel nach seinem Inhalt und seinen Quellen näher untersucht werden. Daraus werden Einschätzungen über die Bedeutung des Autors dieser Enzyklopädie für die Geschichte der Zahlentheorie abgeleitet.

1. *Inhalt der allgemeinen Einführung in die Philosophie durch al-Hindī*

Der äußeren Form nach ist al-Hindī's Abhandlung in Fragen und Antworten gegliedert. „Die erste *Maqāla* zur Arithmetik" umfaßt in der Frankfurter Faksimile-Edition die Seiten 4 bis 14. Sie wendet sich, nach einer kurzen Einleitung, zunächst grundlegenden Fragen zu, die die Einteilung und Definition der philosophischen Wissenschaften betreffen. Als erstes fragt al-Hindī, was

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[1] al-Hindī, Muḥammad ibn ʿAlī ibn ʿAbdallāh. *Problems of Philosophy (Jumal al-falsafa)*, ed. F. Sezgin. Frankfurt a. M. 1985. (Publications of the Institute for the History of Arabic-Islamic Science, Series C, Vol. 19)

[2] Eine Untersuchung des zweiten Kapitels dieser Enzyklopädie, das die Geometrie behandelt, gibt J. P. Hogendijk o. S. 19–32.