Al-Kūhī's CONSTRUCTION OF AN EQUILATERAL PENTAGON IN A GIVEN SQUARE*

JAN P. HOGENDIJK**

1. *Introduction*

In the ninth and tenth century A. D. a remarkable revival of the Greek tradition in higher geometry developed in the Islamic world. The foundation for this was laid in the ninth century with the translation into Arabic of EUCLID's *Elements*[1] and *Data*[2] as well as of works of ARCHIMEDES and APOLLONIUS. Diligent study of these classics stimulated fresh geometrical investigations, in which the emphasis gradually shifted from elementary to more advanced levels. By the end of the tenth century A. D. many geometers in the Middle East were interested in problems that are not constructible by means of ruler and compass, such as the trisection of the angle, the construction of the regular heptagon and others. The geometers attempted to solve these problems by means of conic sections, using the theorems in the *Conics*[3] of APOLLONIUS, the only major work on conic sections then available.

[1] The Greek text of the *Elements* is available in the standard edition of HEIBERG, a literal English translation is in HEATH, *Elements*. For the Arabic translation see GAS V, 103 no. 1.

[2] The Greek text was edited by MENGE, a German translation is in THAER. For the Arabic translation see GAS V, 116.

[3] The Greek text of Books I–IV was edited by HEIBERG, a French translation of Books I–VII was published by VER EECKE. There is no satisfactory English translation of the *Conics*. HEATH published an English version of the *Conics* in which he rearranged the propositions and summarized APOLLO-

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One of the most interesting examples of these late tenth-century solutions is AL-KŪHĪ's construction of an equilateral pentagon in a given square, which is printed in this paper for the first time. AL-KŪHĪ's construction is remarkable because it contains a proof of the focus-directrix property of a hyperbola with eccentricity $\varepsilon = 2$. I am not aware of any other occurrence of the focus-directrix property for a central conic ($\varepsilon \neq 1$) in any text by a medieval author. However, the simple case $\varepsilon = 1$ is found frequently[4]. The focus-directrix property was known in antiquity, but it is not mentioned by APOLLONIUS in the *Conics*, and the available evidence does not suggest that the focus-directrix property for the difficult case $\varepsilon \neq 1$ was transmitted to AL-KŪHĪ in any other way (see below, p. 115). Thus it is reasonable to assume that AL-KŪHĪ independently discovered and proved the focus-directrix property of a hyperbola with eccentricity 2. It appears therefore that AL-KŪHĪ went a step further than APOLLONIUS.

Below (section 2) we present an edited Arabic text with an English translation and notes of AL-KŪHĪ's *Treatise on the construction of an equilateral pentagon in a known square* (GAS V, 318 no. 3). Section 1 continues with some biographical data of AL-KŪHĪ, a short history of the problem, some mathematical preliminaries, a discussion of AL-KŪHĪ's solution and an investigation of the possibility of transmission from antiquity.

WAYJAN IBN RUSTAM ABŪ SAHL AL-KŪHĪ (or AL-QŪHĪ) was interested in most of the mathematical sciences, but his fame is mainly based on his work in geometry. Some of his contemporaries considered him, quite rightly, to be the best geometer of his time[5].

NIUS' reasoning in modern notations. For the Arabic translation see GAS V, 139 no. 1. Professor G. J. TOOMER is working on an edition of the Arabic text of Books V–VII (the Greek text of these books is lost).

[4] AL-KŪHĪ proves the focus-directrix property for the parabola ($\varepsilon = 1$) in his *Centers of tangent circles on lines by way of analysis* (GAS V, 319 no. 9), ms. Paris, Bibliothèque Nationale, Fonds Arabe 2457, f. 19b. See also p. 113 and the reference in note 15.

[5] ABU L-JŪD called AL-KŪHĪ *shaykh ʿaṣrihi fī ṣināʿati l-handasa*, i.e. "master of his time in the art of geometry", in a letter to ʿABDALLĀH AL-ḤĀSIB on the methods of AL-KŪHĪ and AL-ṢAGHĀNĪ and on his own method to construct a regular heptagon (GAS V, 354 no. 1), ms. Paris, Bibliothèque Nationale, Fonds Arabe 4821, f. 42b:9. This statement is echoed in AL-SHANNĪ's *Disclosure of the fallacy of Abu l-Jūd* (GAS V, 352,1), ms. Cairo, Dār al-Kutub, Muṣṭafā Fāḍil, Riyāḍa 41m, 130b:7: *wa-khāṣṣatan Abū Sahl*

He was on good terms with the Buwayhid princes ʿAḌUD AL-DAWLA (reigned from 944–983) and his son SHARAF AL-DAWLA (reigned 983–989 A. D.)[6]. In 988 AL-KŪHĪ supervised astronomical observations in an observatory in the garden of the palace of SHARAF AL-DAWLA in Baghdād[7]. On this occasion AL-KŪHĪ met ABU L-WAFĀ' AL-BŪZJĀNĪ, to be mentioned below. AL-KŪHĪ dedicated several of his works to ʿAḌUD AL-DAWLA and SHARAF AL-DAWLA. The *Treatise on the construction of an equilateral pentagon in a known square* is dedicated to "King SHARAF AL-DAWLA" (below, p. 119, (3)), so AL-KŪHĪ must have composed it somewhere between 983–989.

We now turn to the history of the problem of inscribing an equilateral pentagon in a given square. This problem is clearly a modification of the problem of how to inscribe an "equilateral and equiangular" (i. e. regular) pentagon in a given square. This, in turn, may be explained in part as a variation on problems solved in Book IV of EUCLID's *Elements*, such as IV : 11 "In a given circle to inscribe an equilateral and equiangular pentagon" and IV : 8 "In a given square to inscribe a circle"[8].

A regular pentagon can be inscribed in a square in two different ways, as indicated in Figs. 1 and 2. In both cases, only four angular points A, B, C and D can lie on the square. In Fig. 1, the fifth point

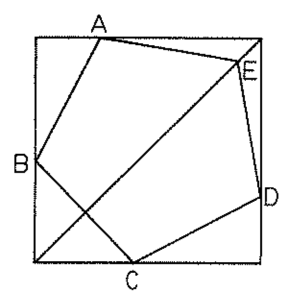


Figure 1

al-Kūhī wa-qad kāna shaykh ʿaṣrihi, i. e. "and especially Al-Kūhī, since he was the master of his time". This corrects HOGENDIJK, *Completion*, p. 113 n. 35.

[6] See DE ZAMBAUR, 212–213, EI[2], vol. I, p. 211–212 (article ʿAḌUD AL-DAWLA), p. 1350–1357 (article BUWAYHIDS).

[7] See IBN AL-QIFṬĪ 351–353. ABU L-WAFĀ' is mentioned on p. 353, line 15–16.

[8] I quote the translation in HEATH, *Elements*, vol. 2, p. 94, 100.

E must lie on the diagonal of the square for reasons of symmetry, and since $< E = 108° > 90°$, E must lie inside the square. If in Fig. 2 F is the midpoint of BC, we have $EF < EB = AD$, which is

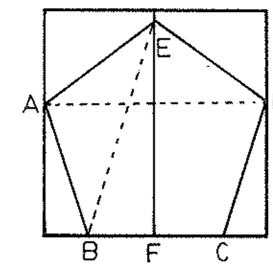


Figure 2

equal to the side of the square, so E is again inside the square. A construction of a regular pentagon inscribed in a square as in Fig. 1 was given by ABU L-WAFĀ' AL-BŪZJĀNĪ (ca. 940–998 A. D.) in Chapter 6 of his *Book on the geometrical constructions necessary for the craftsman* (GAS V, 324 no. 2, see WOEPCKE, *Analyse et extrait*, 337 no. 20). The title of this treatise suggests that possible practical applications (such as the construction of decorative patterns) may have been an additional motivation for studying regular (or perhaps equilateral) pentagons inscribed in a square. However, the importance of such applications should in general not be overestimated, and practical applications are clearly irrelevant in the construction of AL-KŪHĪ as studied below.

The preceding considerations demonstrate that one cannot require a pentagon with all five angular points on a square to be "equilateral and equiangular". One can, however, drop the equiangularity, and the resulting problem of how to construct an equilateral pentagon in a given square admits of two cases as well. This is illustrated in Figs. 3 and 4. The easy case of Fig. 3 was already treated by ABŪ KĀMIL in the second half of the ninth century A. D. by means of algebraical reasoning[9].

[9] In his treatise on the pentagon and decagon. The Arabic text of this treatise is extant in the untitled ms. Istanbul, Kara Mustafa Paşa 67b–78b, see esp. 75b. See also SUTER 27–28, 40–41, which is based on a Hebrew translation. For ABŪ KĀMIL see GAS V, 277–281.

ABŪ KĀMIL assumed that x is the side of an equilateral pentagon inscribed in the way of Fig. 3 in a square of side 10. He then showed

$$x^2 = 200 + 1^1/_2x^2 - 20x - \sqrt{200x^2}$$

and he computed

$$x = 20 + \sqrt{200} - \sqrt{200 + \sqrt{320000}}$$

ABŪ KĀMIL's argument can easily be transformed into a ruler- and compass construction of x.

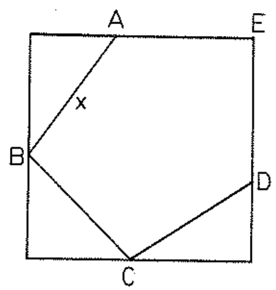


Figure 3

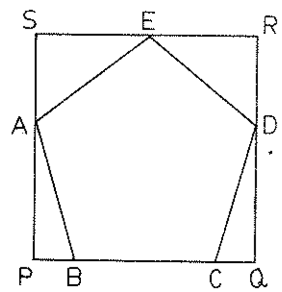


Figure 4

The case of Fig. 4 was solved by AL-KŪHĪ in the treatise edited and translated below. This case is even more interesting than the case of Fig. 3, because it can be shown that an equilateral pentagon ABCDE in a given square PQRS as in Fig. 4 is not constructible by means of ruler and compass, and that its construction is equivalent to the solution of a quartic equation[10].

[10] Let w be half of the side of an equilateral pentagon ABCDE inscribed in a square PQRS with side 1, in the way of Fig. 4. Put y = RD. Then $ED^2 = ER^2 + RD^2$ and $DC^2 = DQ^2 + QC^2$, whence $4w^2 = y^2 + {}^1/_4$ and
$4w^2 = (1 - y)^2 + {}^1/_4(1 - 2w)^2$. So $2y = 1 - w + w^2$, hence
$w^4 - 2w^3 - 13w^2 - 2w + 2 = 0$.
w = 0.3175623 correct to 7 decimals, and $< E = 103°52'$.
$< A = < D = 111°23'$, $< B = < C = 106°41'$ correct to the nearest minute.
We will show that w is not constructible by means of ruler and compass.
Call $f(x) = x^4 - 2x^3 - 13x^2 - 2x + 2$. We have $f(w) = 0$.
We will prove below that
(1) f is irreducible over the rationals.
If the degree of f were not a power of 2, this would be a sufficient proof of the fact that w is not constructible by means of ruler and compass (compare KAPLANSKI p. 9, theorem 4). However, there exist irreducible polynomials

Before proceeding to the solution of AL-KŪHĪ we remark that IBN AL-HAYTHAM also wrote a *Treatise on the construction of a pentagon in a square*[11], which is now lost. It is quite possible that this treatise also contained a construction of the case of Fig. 4.

of degree $4 = 2^2$ having a root that is constructible by means of ruler and compass (example: $g(x) = x^4 + x^2 - 1$). Therefore we will also prove:
(2) the cubic resolvent of f is irreducible over the rationals. It follows from (1) and (2) that the Galois group of f is S_4 or A_4(KAPLANSKI p. 52). Hence the only proper subfield of $\mathbb{Q}(w)$ is $\mathbb{Q}$ itself (KAPLANSKI p. 53, ex. 5, we are working in characteristic zero). So w cannot be constructed by means of ruler and compass (see again KAPLANSKI pp. 8–9).
Proof of (1): Assume that f is reducible over the rationals. Since f has integer coefficients and leading coefficient 1, we can assume that its irreducible factors are also polynomials with integer coefficients and leading coefficient 1 (VAN DER WAERDEN, 71–72). Suppose f has a linear factor $x - \alpha$, with α an integer. Then $f(\alpha) = 0$, so $\alpha(\alpha^3 - 2\alpha^2 - 13\alpha - 2) = -2$, hence α must divide -2, therefore $\alpha = \pm 1$ or $\alpha = \pm 2$. Direct substitution shows that this cannot be the case. So f does not have linear factors.
So $f = f_1 \cdot f_2$, with $f_1(x) = x^2 + px + q$, $f_2(x) = x^2 + rx + s$, p, q, r, s integers. Multiplication of f_1 with f_2 and equation of coefficients shows that p, q, r and s must satisfy (i) $p + r = -2$, (ii) $q + s + pr = -13$, (iii) $ps + qr = -2$, (iv) $qs = 2$. Because of (iv) we have only to investigate the possibilities $q = 1, s = 2$ and $q = -1, s = -2$. Substitution of these values in (i) and (iii) yields $p = 0, r = -2$ if $q = 1, s = 2$; $p = 4, r = -6$ if $q = -1, s = -2$. In both cases we have a contradiction with (ii). So f is irreducible over the rationals.
Proof of (2): Let h be the cubic resolvent of f. Then $h(x) = x^3 + 13x^2 - 4x - 116$ (KAPLANSKI p. 52). If h is reducible over the rationals, it must have a linear factor, because the degree of h is 3. Because h has integer coefficients and leading coefficient 1, it must have an integer root α, as above. α must satisfy $\alpha(\alpha^2 + 13\alpha - 4) = 116$, so α must divide 116. Direct substitution of the twelve possibilities in $\alpha^2(\alpha + 13) = 4(\alpha + 29)$, which is another form of the equation $h(\alpha) = 0$, leads to a contradiction in all cases. So h is irreducible over the rationals.
For sake of completeness we remark that the discriminant of h and f is 767632 (cf. KAPLANSKI, p. 51). This is not the square of an integer, so the Galoisgroup of f is S_4, not A_4 (KAPLANSKI, p. 52).
I am indebted to Professor H. POHLMANN (Brown University) for the reference to KAPLANSKI. This book is very useful for the historian who wishes to decide whether a problem equivalent to an equation of degree 4 is constructible by means of ruler and compass.

[11] *Maqāla fī 'amal mukhammas fī murabba'*, mentioned by IBN ABĪ UṢAYBI'A (vol. 2, p. 98 line 12). The reference occurs in the third list of IBN AL-HAYTHAM's works rendered by IBN ABĪ UṢAYBI'A, that is the list of works composed before the end of 429 H./October, 1038 A.D.

AL-KŪHĪ's *Treatise on the construction of an equilateral pentagon in a known square* is based on Books I–VI of EUCLID's *Elements*, EUCLID's *Data* and parts of Book I–III of the *Conics* of APOLLONIUS. Since the *Conics* is somewhat difficult to fathom for the modern reader, we begin by presenting a survey of the necessary preliminaries for AL-KŪHĪ's construction. The reader should bear in mind that the following survey is an extremely simplified version of some of the theorems in the *Conics*, and does not attempt to be an introduction to APOLLONIUS' work.

(Fig. 5) We consider a double-napped cone with apex A and a plane V which intersects both naps of the cone and which does not contain A. Then the cone and V intersect in two continuous curves $\mathcal{H}_1$ and $\mathcal{H}_2$. In modern terminology $\mathcal{H}_1$ and $\mathcal{H}_2$ form together a hyperbola, but APOLLONIUS called each of the curves $\mathcal{H}_1$ and $\mathcal{H}_2$ a hyperbola, and the pair ($\mathcal{H}_1$, $\mathcal{H}_2$) two *opposite sections*. The Arabic geometers often used the term hyperbola (*qiṭ' zā'id*) not for $\mathcal{H}_1$ but for the part of the plane V contained by the corresponding nap (one of the shaded parts of Fig. 5). $\mathcal{H}_1$ itself was often called the *boundary* (*muḥīṭ*) of the hyperbola (compare p. 124, (23)).

Every hyperbola has exactly one *transverse axis* (*sahm, sahm mujānib*), that is a straight line l which bisects all chords PP' perpendicular to it; thus PQ = QP' in Fig. 6. The halves of the chords are called *ordinates* (*khuṭūṭ al-tartīb*). Let the axis intersect the hyperbola in R and the opposite section in S, and let C be the midpoint of RS. C is called the *centre* (*markaz*), R a *vertex* (*ra's*), RS the *transverse axis* (*sahm, sahm mujānib*; the two terms are used for the line l as well as the segment RS). RS is also called a *transverse diameter* (*quṭr mujānib*) or simply *diameter* (*quṭr*).

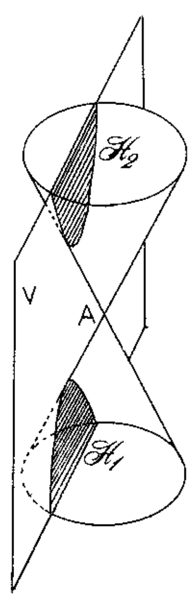


Figure 5

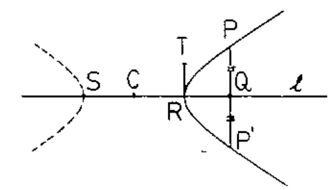


Figure 6

APOLLONIUS proves that the ratio $PQ^2 : (QR \cdot QS)$ is the same for all ordinates PQ. If RT is a segment drawn perpendicular to RS such that RT : RS is equal to this ratio, we have

$$PQ^2 : QR \cdot QS = RT : RS \qquad (I)$$

for every ordinate PQ. Segment RT is called the *erect side* or *latus rectum* (*ḍil' qā'im*). Putting PQ = y, QR = x, RS = d, RT = p we can rewrite (I) as

$$y^2 = px + (p/d)x^2 \qquad (II)$$

(II) is the Cartesian equation of the hyperbola in a system of coordinates with origin R, x-axis RQ extended, y-axis RT extended in Fig. 6.

In Conics I, 54 APOLLONIUS shows how one can "find" a hyperbola with given transverse axis RS, vertex R and erect side RT, that is to say, he constructs a cone which intersects the original plane in the desired hyperbola. AL-KŪHĪ wrote a treatise on an instrument called the *perfect compass*, with which any desired conic section could be drawn (GAS V, 317, 1). The instrument does not seem to have had much use. This was to be expected, because most constructions by means of conics belonged to the realm of unapplied mathematics; the geometer only performed the "constructions" in his imagination.

A few other theorems from the *Conics* will be discussed below in the appropriate place in AL-KŪHĪ's argument. We emphasize that APOLLONIUS' approach to the theory of conic sections is much more general than the above summary might suggest. The concept "axis" plays only a subordinate role in APOLLONIUS' arrangement and theorem (I) is proved to be true in a much more general situation. Thus the summary presents a very distorted view of the *Conics*.

In the following paraphrase of AL-KŪHĪ's construction references like (23) will be to the numbers of the sentences into which I have divided the text and translation below (p. 118–142).

AL-KŪHĪ's solution is divided in an analysis (i. e. solution backward)[12] and a synthesis, in Greek style. Both the analysis and the synthesis are subdivided into three propositions (1–3, 4–6). Propositions 1, 2, 5 and 6 are trivial, the real work is done in propositions 3 and 4.

In proposition 1 ((5) – (8)) AL-KŪHĪ says that the construction of an equilateral pentagon EZTKL in a square ABGD can be reduced to the construction of two lines TZ, ZE in rectangle EBGM in Fig. 7 such that EZ = ZT = 2TM.

In proposition 2 ((9) – (13)) he studies this problem in a different notation (Fig. 8). Suppose we have constructed two lines AE and EZ in a rectangle ABGD (which is half of the original square) such that AE = EZ = 2ZD. AL-KŪHĪ draws EKT//AB and DK//ZE to meet in K. Clearly AE = EZ = DK and EK = ZD. Thus the initial problem can be reduced to the problem of how to construct in the rectangle ABGD three lines AE, EK, KD such that AE = 2EK = KD, EK//AB.

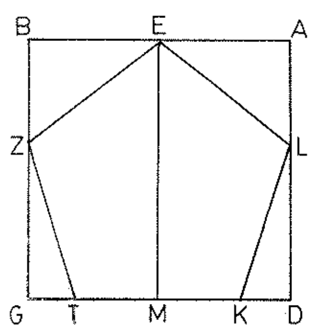


Figure 7

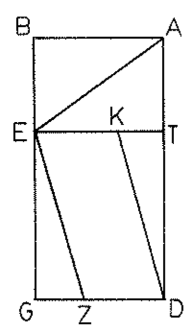


Figure 8

[12] See the classical description in PAPPUS' *Collection* VII. Greek text in ed. HULTSCH, 634–636, English translation in HEATH, HGM II, 400–401.

This problem is solved in proposition 3 ((14) – (45)). The notations are similar to those used in proposition 2, but AL-KŪHĪ reflects the figure.

The idea of the solution is quite simple. We have to find a point K as in Fig. 9 in such a way that if KE is the perpendicular drawn onto BG, we have EA = 2KE and KD = 2KE.

Since EA = 2KE, K must belong to the locus $\mathscr{H}$ of all points X such that the perpendicular distance XX' to the given line BG is half of the distance X'A between the projection X' and the given point A (Fig. 10). AL-KŪHĪ shows that $\mathscr{H}$ is a hyperbola.

Again, since KD = 2KE, K must belong to the locus $\mathscr{F}$ of all points Y such that the distance YD between Y and the given point D is twice the distance YY' between Y and the given line BG (Fig. 11). The modern reader will immediately realize that $\mathscr{F}$ is a hyperbola with focus D, directrix BG and eccentricity 2. AL-KŪHĪ proves in (24) – (41) that $\mathscr{F}$ is a hyperbola. The two hyperbolas $\mathscr{H}$ and $\mathscr{F}$ are completely determined by A, BG and D, so point K can be found as a point of intersection.

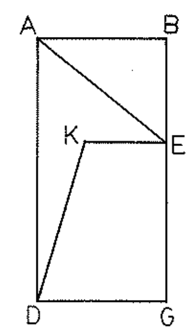


Figure 9

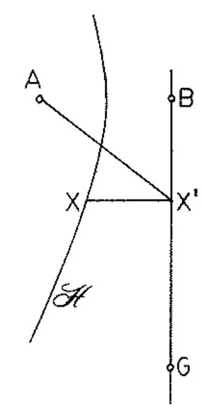


Figure 10

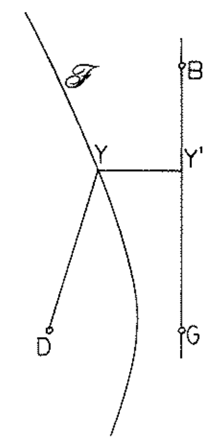


Figure 11

The details of AL-KŪHĪ's proofs are as follows. First ((16) – (23), Fig. 12) AL-KŪHĪ deals with $\mathscr{H}$. We have AE = 2EK, so $AE^2 = 4EK^2$. Drop a perpendicular KL onto AB. Then $AE^2 = BE^2 + AB^2 = LK^2 + AB^2$ and EK = BL, hence $KL^2 + AB^2 = 4BL^2$, thus $KL^2 = 4BL^2 - AB^2$. AL-KŪHĪ bisects AB in N and he chooses S on AB extended such that BS = $^1/_2$AB. Then AB = 2BN = 2BS, so

$4BL^2 - AB^2 = 4(BL^2 - BN^2) = 4(BL + BN)(BL - BN) = 4LS \cdot LN$. Hence $KL^2 = 4LS \cdot LN$, therefore

$$KL^2 : (LN \cdot LS) = 4SN : SN.$$

This is the canonical form of the fundamental property (I), p. 107 of the hyperbola $\mathscr{H}$ with vertex N, transverse axis segment SN and erect side equal to 4SN. K is therefore on $\mathscr{H}$, and $\mathscr{H}$ can be "found" by *Conics* I:54, because we know N and S.

The proof that K is on a hyperbola $\mathscr{F}$ ((24) – (41)) is less obvious (Fig. 13). We assume $KD = 2KE$ and we drop a perpendicular KM onto GD. Let KM extended intersect AG in O. Since $AD = 2DG$ we have $MO = 2MG$, but also $DK = 2KE = 2MG$, thus $MO = DK$ (27).

Therefore $OM^2 = DK^2 = DM^2 + MK^2$, whence $DM^2 = OM^2 - MK^2$. Define W as the point on KM extended such that $MW = MK$. Then $OM^2 - MK^2 = (OM - MK)(OM + MK) = OK \cdot OW$, therefore $DM^2 = OK \cdot OW$ (31). (Thus we have $(OK \cdot OW) : OA^2 = DM^2 : OA^2$ (III)).

Let C be the point on AG such that $CF = FD$, where CF is the perpendicular drawn onto GD. (C can be found as the point of intersection of AG and the bisector of angle ADG.) Then $CF = 2FG$, because $AD = 2DG$ and $AD//CF$. Therefore $FD = 2FG$, that is to say $GD = 3FG$. F is therefore a known point (33). Since $AD//OM//CF$, $DM^2 : OA^2 = DF^2 : CA^2 = FC^2 : CA^2$, and (III) becomes $(OK \cdot OW) : OA^2 = FC^2 : CA^2$ (34).

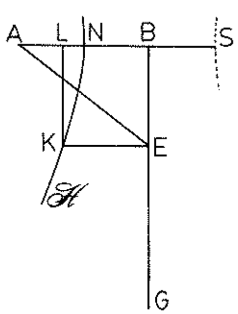


Figure 12

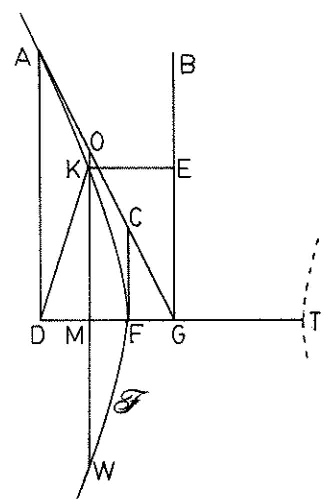


Figure 13

Conics III:16 is the following theorem: If AC and CF are tangent to a conic section at A, F and if $OKW//CF$ intersects AC in O and the conic in K and W, then $(OK \cdot OW) : OA^2 = FC^2 : CA^2$.

AL-KŪHĪ uses the converse of this theorem, and he concludes from (34) that K and W must be on a conic section $\mathscr{F}$ which is tangent to FC at F and to AC at A[13] (35).

The next problem is to specify the type of this conic section, and to find the information which is necessary for "finding" the conic by *Conics* I:52–59. Here we will amplify the reasoning of AL-KŪHĪ, who limits himself to the mere essentials (36–39).

Since the chord KW is perpendicular to FM and bisected by FM, and since KW is parallel to the tangent FC at F, MF must be an axis of the conic (by *Conics* II:7, and the definition of axis).

In order to establish the type of the conic AL-KŪHĪ studies the tangent AG, the ordinate AD and the vertex F. Since $GF = \frac{1}{2}GD$, the conic cannot be a parabola, for in that case we would have $GF = FD$ (*Conics* I:35). So the conic must be a hyperbola, an ellipse or possibly a circle. Therefore by *Conics* I:36 $GF : FD = GT_1 : T_1D$, if T_1 is the other endpoint of the (transverse of major) axis beginning in F (Fig. 14). In the case of the equilateral pentagon (Fig. 13) we have $GF : FD = 1:2$, and if T is on DG extended such that $TG = GD$, also $GT : TD = 1:2$. So $T_1 = T$. Since G is between F and T, the conic must be a hyperbola (compare with Fig. 14), with transverse axis FT.

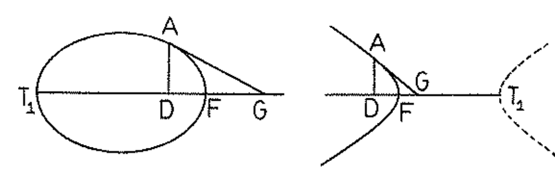


Figure 14

Finally (40) AL-KŪHĪ determines the erect side p corresponding to the transverse axis TF. By the fundamental property (I), p. 107, we have $AD^2 : (DF \cdot DT) = p : TF$. Since $AD = DT$ and $DF = \frac{2}{3}DG = \frac{1}{3}AD$, it follows that $p = 3TF$. T and F are known points, so that we have sufficient information to "find" $\mathscr{F}$ by

[13] Using modern theorems one can prove in the following way that GA is tangent to the hyperbola. Because the focus D is the pole of the directrix BG, and because AD is perpendicular to the axis GD, G must be the pole of line AD, so GA must be tangent to the hyperbola at A.

means of *Conics* I:54. Hence K can be found as a point of intersection of $\mathscr{H}$ and $\mathscr{F}$, and the problem is solved.

The synthesis (propositions 4–6, (46)–(99)) is exactly the analysis in reverse, so a detailed discussion is not necessary here.

The most outstanding feature of AL-KŪHĪ's solution is clearly his proof of the focus-directrix property for a hyperbola with eccentricity $\varepsilon = 2$. In modern terms one could say that he proved that in the case $\varepsilon = 2$ the focus-directrix property is equivalent to a Cartesian equation of a hyperbola such as (II), p. 107. We have only described his analysis, in which he assumes the property and proves the "equation", but in the synthesis ((61) – (81)) he argues the reverse. Thus he gave a complete proof of the equivalency.

A few trivial changes in the proof suffice to make it general for an arbitrary eccentricity $\varepsilon > 0$. Let D be a given point, BG a given line and suppose that K is a point such that the perpendicular distance KE to BG is ε times the distance KD, as in Fig. 13. Again we suppose that DG is perpendicular to BG, and we draw a perpendicular through D to DG. If we define A as the point on the latter perpendicular such that $AD = \varepsilon DG$, points O, C and F can be defined as above. We have $MO = DK$, $CF = FD$ and $FD = \varepsilon FG$. Repeating the previously shown method one sees that K is on the conic section through A and F such that F is the vertex, DF the axis, and AC is tangent at A.

If $\varepsilon = 1$, $GF = FD$ so the conic will be a parabola. If $\varepsilon \neq 1$ we define point T on GD extended ($\varepsilon < 1$) or DG extended ($\varepsilon > 1$) such that $GT:TD = GF:FD = 1:\varepsilon$. As above, the conic is seen to be an ellipse ($\varepsilon < 1$) or a hyperbola ($\varepsilon > 1$) with transverse axis TF. The corresponding erect side $|1 - \varepsilon^2|$TF can be found as above (because $|1 - \varepsilon^2| \neq 1$ for $0 < \varepsilon < 1$, the conic cannot be a circle).

Thus AL-KŪHĪ could easily have proven the general focus-directrix property if he had wanted to. However, it should be understood that the focus-directrix property did not play an important role in Arabic geometry. There existed no special terms for "focus" and "directrix"[14], and there is no evidence that AL-KŪHĪ considered the property KD = 2KE and the conic $\mathscr{F}$ somehow more important or more interesting than the property EA = 2KE and the conic $\mathscr{H}$.

We will now investigate the possibility of a transmission from Greek into Arabic. In the *Conics*, APOLLONIUS does not mention the focus-directrix property. He mentions the foci of the ellipse and the hyperbola in *Conics* III:45–52 (in a different context), but the focus of the parabola and the directrix do not occur in the *Conics*. The focus-directrix property of the parabola ($\varepsilon \neq 1$) is proven in DIOCLES' *On Burning Mirrors*, which was translated into Arabic[15]. The only ancient traces of the property for $\varepsilon \neq 1$ are in the *Collection* of PAPPUS OF ALEXANDRIA (ca. 320 A.D.). Book VII contains two separate proofs of the focus-directrix property, one for the easy case $\varepsilon = 1$, and one for the more difficult case $\varepsilon \neq 1$[16]. These proofs are given as a lemma to the *Surface-Loci* of EUCLID, a work now lost. In *Collection* IV: 44 PAPPUS renders a trisection of an arc in which the focus-directrix property of a hyperbola with $\varepsilon = 2$ is assumed, but not proven[17].

As far as is known, Book VII of the *Collection* was never transmitted to Arabic geometry. We further note that AL-KŪHĪ's proof is very different from PAPPUS' proof (for $\varepsilon \neq 1$), and totally unrelated to the proofs of the simpler case $\varepsilon = 1$ given by PAPPUS and DIOCLES. There is no trace of EUCLID's *Surface-Loci* in Arabic geometry. Only the possibility of a transmission connected with *Collection* IV: 44 requires a somewhat closer examination.

In *Collection* IV: 36–44 PAPPUS presents three trisections of the angle, all taken over from other sources. The first trisection (*Collection* IV: 36–42) is also extant in an Arabic text attributed to AḤMAD IBN MŪSĀ[18]. This text is so close to the Greek text in *Collection* IV: 37, 38, 39 that one must assume that AḤMAD IBN MŪSĀ actually translated the text attributed to him from the Greek.

[14] The term *focus* was introduced by KEPLER in 1604 (*Ad Vitellionem Paralipomena* IV, 1 = *Werke*, II, 91). I have not been able to establish the first occurrence of the term "directrix" (of a conic section) in the sense of polar of a focus. DE LA HIRE used the term *directrix* in 1673 in a different sense (FLADT, 44).

[15] Proposition 5 in ed. TOOMER (p. 66–70); see also propositions 4, 10 and TOOMER's introduction, p. 15–17.

[16] PAPPUS, *Collection* VII, ed HULTSCH, 1004–1014, trl. VER EECKE 794–800, summary in HEATH, HGM II, 120–121.

[17] For the text of *Collection* IV: 36–44 see ed. HULTSCH, 272–284; French translation in VER EECKE, 210–221, summary in HEATH, HGM I, 235–237 (IV: 36–42), 241–243 (IV: 43, 44).

[18] GAS VII: 404 (addition to GAS V, 252), edited and translated in HOGENDIJK, *Trisections*.

Hence AḤMAD's text is either dependent on PAPPUS' *Collection* (possibly through some condensed work of late antiquity) or on PAPPUS' source. In the former case one could argue that the hypothetical condensed work might also have contained *Collection* IV: 44, and that this trisection was thus available to AL-KŪHĪ in some form.

Assuming this to be true, it is likely that AL-KŪHĪ also had access to the trisection in *Collection* IV: 43. This trisection is essentially the same as that in *Collection* IV: 44 because the same curves are used, but in *Collection* IV: 43 the focus-directrix property is not assumed, and thus the trisection in *Collection* IV: 43 was of course much more satisfactory to any reader who used the *Conics* as a standard reference work (such as the compiler of our hypothetical condensed work). AL-KŪHĪ could now have used *Collection* IV:43, 44 to find a proof of the focus-directrix property in the case $\varepsilon = 2$. In Fig. 13 he might have remarked that KD = 2KE has $<$ KDT $= 2 <$ KTD as a consequence (this is seen by *Collection* IV:44). From $<$ KDT $= 2 <$ KTD it follows immediately by *Collection* IV:43 that K is on the hyperbola $\mathcal{F}$, hence AL-KŪHĪ could simply have copied *Collection* IV:43. However, AL-KŪHĪ's actual approach to the problem is very different from this. This is evident from the fact that AL-KŪHĪ's proof can easily be generalized to the case of arbitrary ε, whereas the fact that $\varepsilon = 2$ is crucial in the proof which one can construct from *Collection* IV:43, 44. Hence the possibility that AL-KŪHĪ had access to a condensed work based on *Collection* IV:43, 44 (or to the *Collection* itself) can be excluded[19].

One could still argue that AL-KŪHĪ had access to the trisection in *Collection* IV:44 in a source which is not dependent on the *Collection*. AL-KŪHĪ published a trisection of the angle[20] which exhibits a superficial similarity to *Collection* IV:44 (or 43), because AL-KŪHĪ also employs a circle and a hyperbola. However, a closer examination shows that the two trisections are unrelated. It suffices to remark that AL-KŪHĪ uses an equilateral hyperbola (thus $\varepsilon = \sqrt{2}$) and does not use the focus-directrix property; we recall that $\varepsilon = 2$ for the hyperbola in *Collection* IV:43, 44. Further, if the trisection in *Collection* IV:44 would have been available in some form in the 10th century, one would expect to find references or traces of it in 10th-century works on the trisection, for example the collection of trisections composed by AL-SIJZĪ[21]. However, I have found no evidence for this. Thus there is also no evidence that the focus-directrix property of the hyperbola was transmitted to AL-KŪHĪ through a source related to PAPPUS' *Collection* IV:43, 44.

Of course one could adopt the axiom that any trace of creativity in Arabic geometry must be due to transmission directly from the Greek, and thus one could postulate an unknown Greek source from which AL-KŪHĪ took the focus-directrix property. This assumption does not seem to be fruitful in this case. AL-KŪHĪ's other writings (such as his additions[22] to ARCHIMEDES' *On the Sphere and Cylinder* II) bear ample witness of his geometrical ability. Unless evidence to the contrary turns up we have every reason to assume that AL-KŪHĪ independently rediscovered and proved the focus-directrix property for the case $\varepsilon = 2$.

2. *Text and translation*

AL-KŪHĪ's *Treatise on the construction of an equilateral pentagon in a known square* survives in two slightly different versions, which will be denoted as I and II.

The original version I is extant in the manuscript

P: Paris Bibliothèque Nationale, Fonds Arabe 4821, 29b–33b, dated Tuesday 15 Ramaḍān 544 H./January 16, 1161 A.D. by the scribe, AL-ḤUSAYN IBN MUḤAMMAD IBN ʻALĪ (the Christian date should actually be January 17, 1150).

Version I is also extant in the manuscript Teheran, Dānišgāh 1751, 68b–70b, written in 1250 H./1835 A.D. This manuscript was unfortunately not available to me. It must contain I because it contains the preface (3–4 in my edition, below, p. 118). The preface

[19] This agrees with my conclusion (in *Trisections*, p. 430) that the Arabic text attributed to Aḥmad ibn Mūsā is not dependent on PAPPUS' Collection.

[20] GAS V, 318,6, see the edition and English translation in SAYILI.

[21] GAS V, 331,7; see the French summary in WOEPCKE, *Algèbre*, 117–125.

[22] GAS V, 320,25. AL-KŪHĪ's additions have been preserved thanks to the fact that NAṢĪR AL-DĪN AL-ṬŪSĪ appended them to his own edition of ARCHIMEDES' *On the Sphere and Cylinder* II. So AL-KŪHĪ's additions are also extant in all or most of the manuscripts mentioned by SEZGIN in GAS V, 129 b. A French summary of the additions is available in WOEPCKE, *Algèbre*, 103–114; the Arabic text has been printed in the Hyderabad edition of AL-ṬŪSĪ's *Rasāʼil* (vol. 2, p. 115 : 8 – 127 : 15).

is printed in full in the catalogue (DANECHE-PAJOUH, vol. 8, p. 276–7); it is literally the same as in P.

Version I was revised by someone who made a considerable number of (mainly stylistic) changes. The result was version II. The reviser omitted the preface, which contained the dedication to Sharaf al-Dawla. A number of inadequacies in II show that the text was not revised by AL-KŪHĪ himself. In (63) – (64) AL-KŪHĪ shows that a conic section labeled "FKA" actually passes through the point A. The reviser did not understand that anything had to be proven, and he therefore deleted a sentence in (63) and a word in (64) which he considered superfluous. What is left is practically devoid of meaning. An omission in (41) shows that the reviser was not sufficiently familiar with the intricacies connected with the concept "known" (the Greek "given") and its subdivisions. See my notes to the translation for all details.

Version II survives in the following four manuscripts:

A0: Istanbul, Aya Sofya 4830, 168b–171a, written in 627 H./1230 A.D. in Damascus (the date is on f. 86a of the manuscript)

A2: Istanbul, Aya Sofia 4832, 121b–123b, written in or before 568 H./1173 A.D. (the date is part of an owner's mark on f. 1a)

D: Damascus, Ẓāhiriyya, ʿāmm 5648, 188b–191b, written in 1305 H./1888 A.D. (the date is on f. 221b)

C: Cairo, Dār al-Kutub, Muṣṭafā Fāḍil, Riyāḍa 40m, 203b–205b, written by MUṢṬAFĀ ṢIDQĪ, dated Sunday 21 Dhu ʾl-Qaʿda 1159 H./Dec. 4, 1746 A.D.

For manuscript C see KING, p. 442 no. 18. All other relevant references to manuscript catalogues can be found in GAS V, 318 no. 3, 403.

My edition is based on the manuscripts P, A0, A2, D and C. I have edited and translated version I (using A0, A2, D and C to correct obvious scribal errors in P), and I have noted the variant readings of II (i.e. the readings common to A0, A2, D and C) at the bottom of each page of the Arabic text.

The few remaining variants in the individual manuscripts have been printed at the end of the Arabic text as one lot. This procedure has been followed in order to facilitate comparision between I and II.

D and C are provided with all diacritical marks. These are often

omitted in A0, A2 and P. Trivial variants involving diacritical marks and variants of a strictly orthographical nature have not been noted.

I have divided the text and translation into 100 sentences, numbered 1, . . ., 100. The sentence numbers have been used for references in the apparatus and in the notes to the translation. Letters indicating points in geometrical figures have been transcribed according to the system proposed in HERMELINK and KENNEDY.

The manuscript A0 contains a number of marginal remarks by MUḤAMMAD IBN SARTĀQ AL-MARĀGHĪ, written in the end of Ṣafar 728 H./middle of January 1328 A. D. (the entire date is given on f. 165a). Some of the remarks are barely legible or have disappeared in part because the margin of the manuscript was cut off. I have added the Arabic text of the remarks that I could read on p. 142–143 at the end of the text of AL-KŪHĪ. English translations of the remarks can be found in the notes to the appropriate passages in the text of AL-KŪHĪ.

Arabic text

In the Arabic text and the apparatus the following abbreviations have been used in round brackets:

أ "recto", i.e. front side of a page
ب "verso", i.e. back side
س ms. P, Paris B.N., Fonds Arabe 4821, 29b–33b
ك ms A0, Aya Sofya 4830, 168b–171a
ل ms. A2, Aya Sofya 4832, 121b–123b
د ms D, Damascus, Ẓāhiriyya, ʿāmm 5648, 188b–191b
ق ms. C, Cairo, Dār al-Kutub, Muṣṭafā Fāḍil, Riyāḍa 40m, 203b–205b

The following symbols have also been used:

+ added (or: added by)
— omitted (or: omitted by)

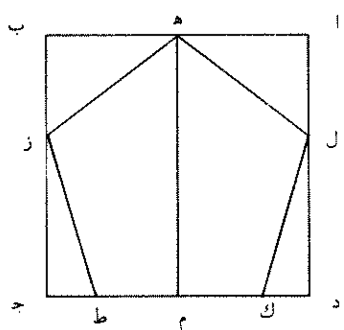
References such as (٢٢) are to the numbers of the sentences into which I have divided the text.

(١) بسم الله الرحمان الرحيم

(٢) رسالة أبي سهل ويجن بن رستم القوهي في عمل مخمس متساوي الأضلاع في مربع معلوم

(٣) بعد فراغنا من عمل المسبع المتساوي الأضلاع في الدائرة نبدأ باستخراج شكل أحسن وأبعد وأغمض وأصعب استخراجاً من عمل المسبع تقرباً إلى الملك شرف الدولة وهو عمل مخمس متساوي الأضلاع في مربع معلوم
(٤) واتفق بدولته وإقباله لعبده ويجن بن رستم وذلك بحسن نظره وجميل رأيه لكافّة الناظرين في هذا العلم وأرجو أن يقع موقعاً بحسب مقداره وبعده

آ (٥) نريد أن نعمل في مربع ا ب جـ د مخمساً متساوي الأضلاع وليس بمتساوي الزوايا وتكون الزوايا الخمس على أضلاع المربع كما في الصورة



(٣-٤): –
(٥) الزوايا الخمس: زواياه الخمس

(1) In the name of God, the merciful, the compassionate.

(2) Treatise by ABŪ SAHL WAYJAN IBN RUSTAM AL-QŪHĪ on the construction of an equilateral pentagon in a known square.

(3) After having finished the construction of the equilateral heptagon in the circle, we begin the derivation of a proposition that is more beautiful, more profound, more abstruse and more difficult to derive than the construction of the heptagon, in order to seek the favour of King SHARAF AL-DAWLA. It is the construction of the equilateral pentagon in a known square. (4) It (the solution) occurred, thanks to his power and his prosperity, to his servant WAYJAN IBN RUSTAM, and that (happened) as a consequence of his (i.e. SHARAF AL-DAWLA's) genuine concern and favorable opinion of all students of this science. I hope it will be received with due appreciation of its magnitude and profundity.

1. (5) We wish to inscribe in square ABGD an equilateral but non-equiangular pentagon, such that the five angles are on the sides of the square, as in the figure (Fig. 15).

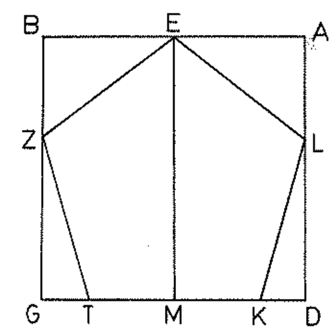
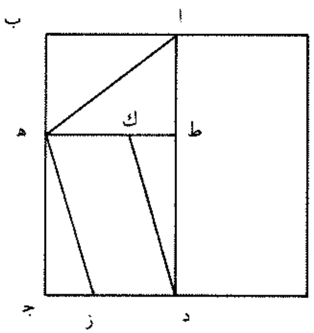


Figure 15

(2) The Arabic technical term *maʿlūm* "known" is used as an equivalent of the Greek term δεδομένος "given".
(3) For AL-KŪHĪ's constructions of the heptagon see SAMPLONIUS and HOGENDIJK, *Heptagon* pp. 102–103.

(٦) فعلى التحليل ننزل أن في مربع اب جد مخمس ه ز ط ك ل متساوي الأضلاع وخط هم عمود من نقطة ه فهو يقسم خط ج د بنصفين وكذلك خط طك بنصفين فخط ك ط ضعف خط طم (٧) وكل واحد من خطي هز زط (س ٣٠ أ) مساو لخط طك فكل واحد من خطي هز زط ضعف خط طم (٨) فينبغي أن نخرج في نصف مربع اب جد خطين كخطي هز زط حتى يكون كل واحد منهما ضعف خط طم



ب (٩) فنريد أن نخرج في سطح اب جد الذي هو نصف المربع المعلوم خطين كخطي اه هز حتى يكون كل واحد منهما ضعف خط زد

(١٠) فعلى التحليل ننزل أن كل واحد من خطي اه هز ضعف خط زد (١١) ونجعل خط هط موازياً (ك ١٦٩ أ) لخط اب وخط دك موازياً لخط هز فخط دك مساو لخط هز (١٢) وخط هك مساو لخط زد لأن سطح هكدز متوازي

(٦) من نقطة ه: على خط دج ‖ جد: دج ‖ وكذلك خط طك: وخط طك أيضاً ‖ ك ط: طك
(٨) وذلك ما أردنا أن نبين +
(١٢) زد: دز

(6) Analysis: We assume that the equilateral pentagon EZTKL has been inscribed in square ABGD. Line EM is a perpendicular (drawn) from point E, so it bisects GD, and similarly TK, so line KT is twice line TM. (7) But each of the lines EZ, ZT is equal to TK, so each of the lines EZ, ZT is twice line TM.
(8) Thus we are required to draw in half of the square ABGD two lines EZ, ZT such that each of them is twice line TM.

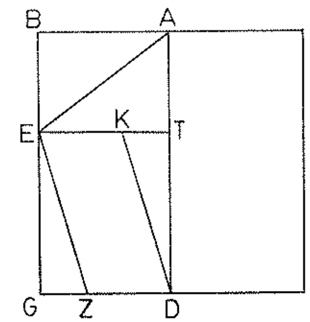


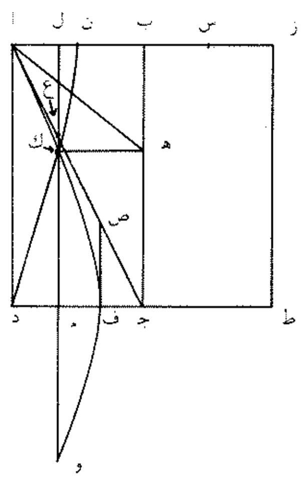
Figure 16

2. (9) Thus we wish to draw in rectangle ABGD, which is half of the known square, two lines AE, EZ such that each of them is twice line ZD (Fig. 16).
(10) Analysis: We assume that each of the lines AE, EZ is twice line ZD. (11) We draw line ET parallel to line AB, and line DK parallel to line EZ. Then line DK is equal to line EZ. (12) But line

(6) Version II has "line EM is a perpendicular onto line DG" instead of "line EM is a perpendicular (drawn) from point E" in Version I. In the Greek and Arabic geometrical literature points are often defined in part by the figure. Thus AL-KŪHĪ does not clearly define M in the text, but Fig. 15 shows that he had the point of intersection of line GD and the perpendicular through E in mind.
(8) MUḤAMMAD IBN SARTĀQ (see p. 117) adds: "Remark. In the third proposition the subject of consideration and study is lines EL, LK, and in the second proposition lines EZ, ZT. This procedure (is followed) because of some kind of (cosmetic) advantage (?), for there is no difference in considering what . . ." (the rest is illegible). "In the third proposition" should actually be "in the first proposition".
(9) I have translated *saṭḥ* as "rectangle" when possible, but *saṭḥ* can have the more general meaning of any plane figure (thus: "area" in (12), (87)). In passages such as (21) and (30), "rectangle" is the only possible translation.

الأضلاع فكل واحد (د ١٨٩ أ) من خطي اه دك ضعف خط هك (١٣) فينبغي أن يكون في سطح اب جد الذى هو نصف المربع المعلوم إخراج ثلاثة خطوط مثل دك كه ها كما في الصورة حتى يكون كل واحد من خطي اه دك ضعف خط هك وخط هك مواز لخط اب

جـ (١٤) فنريد أن نخرج في سطح اب جد الذي هو نصف المربع المعلوم خطين كخطي اه دك حتى يكون كل واحد منهما ضعف خط هك و هك (س ٣٠ ب) مواز لخط اب
(١٥) فعلى التحليل ننزل أن كل واحد من خطي اه دك ضعف خط هك وأن خط لكم مواز لخط اد ونصل خط اجـ



(١٣) يكون في: نخرج في ‖ إخراج: – ‖ كما في الصورة: – ‖ وخط هك مواز: ويكون خط هك موازيا

(١٤) و هك مواز: ويكون هك موازيا

EK is equal to line ZD because area EKDZ is a parallelogram. So each of the lines AE, DK is twice line EK.
(13) Thus one is required to draw in rectangle ABGD, which is half of the known square, three lines DK, KE, EA as in the figure, such that each of the lines AE, DK is twice line EK, and (such that) line EK is parallel to line AB.

3. (14) Thus we wish to draw in rectangle ABGD, which is half of the known square, two lines AE, DK such that each of them is twice line EK, and (such that) EK is parallel to line AB (Fig. 17).
(15) Analysis: We assume that each of the lines AE, DK is twice line EK, and that line LKM is parallel to line AD. We join line AG.

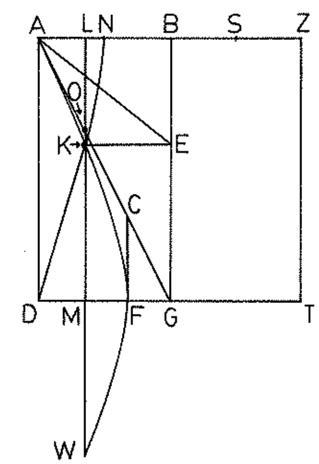


Figure 17

(13) Version II omits "as in the figure".
(14) See the introduction, pp. 109–112 for a summary of proposition 3.

(١٦) فلأن خط (ق ٢٠٤ أ) اه ضعف خط ه‍ك فمربع خط اه أربعة أمثال مربع خط هك (١٧) ولكن مربع خط اه مساو لمربعي خطي اب ب‍ه لأن زاوية اب‍ه قائمة (١٨) وخط ب‍ه مساو لخط كل (١٩) فمربعا خطي اب كل مساويان لأربعة أمثال مربع خط هك أعني مربع خط بل (٢٠) ومربع اب أربعة أمثال مربع بن إذا كان (ل ١٢٢ أ) اب مقسوماً بنصفين على نقطة ن (٢١) فيبقى مربع كل أربعة أمثال سطح سل في لن إذا كان خط سب مساوياً لخط بن (٢٢) فنقطة ك على محيط القطع الزائد الذى سهمه خط سن المعلوم القدر والوضع لأن كل واحد من نقطتي س ن معلومة ورأسه نقطة ن المعلومة وضلعه القائم أربعة أمثال خط سن (٢٣) فهو معلوم أيضاً فمحيط ذلك القطع وهو نك معلوم الوضع (٢٤) وأيضاً لأن خط دك ضعف خط ك‍ه وخط ك‍ه مساو لخط مج فخط دك ضعف خط جم (٢٥) وخط عم أيضاً ضعف خط جم لأن نسبة عم إلى مج (د ١٨٩ ب) كنسبة اد إلى دج و اد ضعف دج (٢٦) فكل واحد من خطي (س ٣١ أ) دك عم ضعف خط مج (٢٧) فلهذا خط دك مساو لخط عم (٢٨) فمربع خط دك مساو لمربع خط مع (٢٩) ولكن مربع خط دك مساو لمربعي خطي دم مك (٣٠) ومربع خط مع مساو لسطح وع في عك مع مربع كم إذا كان خط وم

(١٦) فمربع: يكون مربع
(٢٢) واحد: واحدة
(٢٣) نك: قطع نك
(٢٤) لخط: يكون خط ‖ جم: مج
(٢٥) وخط عم أيضاً ضعف خط جم: –
(٢٧) فلهذا خط دك مساو: فلذلك يكون خط دك مساوياً
(٢٨) مع: عم
(٣٠) مع: عم ‖ مك: كم

(16) Because line AE is twice line EK, the square of line AE is four times the square of line EK. (17) But the square of AE is equal to the (sum of the) squares of lines AB, BE, because angle ABE is a right angle. (18) And line BE is equal to line KL. (19) So the (sum of the) squares of lines AB, KL is equal to four times the square of line EK, that is the square of line BL. (20) And the square of AB is four times the square of BN, if AB is bisected in point N. (21) So, by subtraction, the square of KL is four times the rectangle SL by LN, if line SB is equal to line BN.
(22) So point K is on the boundary of the hyperbola with axis line SN – which is known in size and position because each of the points S, N is known – and with vertex the known point N, and with erect side four times line SN. (23) So it (the hyperbola) is also known, so the boundary of this (conic) section, that is NK, is also known in position.
(24) Again, because line DK is twice line KE, and line KE is equal to line MG, line DK is twice line GM. (25) But line OM is also twice line GM, because the ratio of OM to MG is equal to the ratio of AD to DG, and (because) AD is twice DG. (26) So each of the lines DK, OM is twice line MG. (27) Therefore line DK is equal to line OM. (28) So the square of DK is equal to the square of line MO.
(29) But the square of line DK is equal to the (sum of the) squares of the lines DM, MK. (30) And the square of line MO is equal to the rectangle WO by OK plus the square of KM, if line WM is equal to line MK.

(21) $BL^2 = BN^2 + SL \cdot LN$ by *Elements* II:6.
(22) By the converse of *Conics* I:21. MUḤAMMAD IBN SARTĀQ (see p. 142) adds: "This is known from proposition 4 of section 1 of class 3 of species 4 of genus 1 of the two genera of the mathematical sciences of my book *The Perfection*. This species is about sections of cones and cylinders. Concerning the proofs (?) for the hyperbola the books on conic sections are also known to the mathematician."
See for more details on the huge geometrical compilation called *The Perfection*: HOGENDIJK, *Discovery*.
(23) Converse of *Conics* I:54.
(25) Version II omits erroneously: "But line OM is also twice line GM."
(30) Since WM = MK, $MO^2 = MK^2 + WO \cdot OK$ by *Elements* II: 6.

مساوياً لخط مك (٣١) فإذا ألقينا مربع كم المشترك يبقى مربع دم مساوياً لسطح وع في عك (٣٢) وإن جعلنا خط دف مساوياً (ك ١٦٩ ب) لخط فص يكون خط دف ضعف خط فج لأن صف ضعف خط فج (٣٣) فنقطة ف تكون معلومة (٣٤) ونسبة مربع دم أعني سطح وع في عك إلى مربع عا كنسبة مربع دف أعني مربع فص إلى مربع صا (٣٥) فلهذا نقطة ك على محيط قطع ما مماس لخطي اص صف على نقطتي ا ف ويجوز على نقطة و

(٣٦) لأن خط دف ضعف خط فج فخط دج أعني طج ثلاثة أمثال خط جف (٣٧) وكذلك خط طد ثلاثة أمثال خط دف (٣٨) فنسبة طد إلى دف كنسبة طج إلى جف (٣٩) فلهذا القطع هو قطع زائد وسهمه خط طف

(٤٠) وأيضاً لأن نسبة مربع اد إلى سطح طد في دف كنسبة (ق ٢٠٤ ب) طف الذى هو القطر المجانب إلى ضلعه القائم ومربع اد ثلاثة أمثال سطح طد (د ١٩٠ أ) في دف لأن خطي اد دط متساويان فالضلع القائم له ثلاثة أمثال قطر طف (٤١) فمحيط قطع اكف معلوم لأن سهمه وهو طف معلوم الوضع والقدر ورأسه نقطة ف وهي معلومة وضلعه القائم معلوم لأنه ثلاثة أمثال قطر (س ٣١ ب) طف المعلوم

(٣٥) فلهذا: فلهذا تكون

(٣٦) لأن خط: لأن ‖ خط فج: فج

(٣٩) القطع هو قطع زائد: يكون القطع قطعا زائدا

(٤٠) دف كنسبة: طف كنسبة

(٤١) لأن سهمه وهو طف معلوم: –

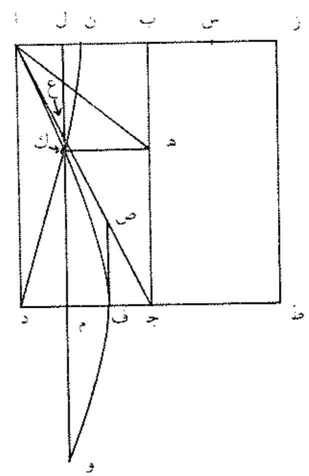
(31) So if we subtract the common square of KM, the square of DM is equal to the rectangle WO by OK.
(32) If we make line DF equal to line FC, line DF will be twice line FG, because CF is twice line FG. (33) So point F is known.
(34) But the ratio of the square of DM, that is the rectangle WO by OK, to the square of OA is equal to the ratio of the square of DF, that is the square of FC, to the square of CA. (35) Point K is therefore on the boundary of a certain (conic) section which is tangent to lines AC, CF at points A, F, and which passes through point W.
(36) Because line DF is twice line FG, line DG, that is TG, is three times line GF. (37) Similarly, line TD is three times line DF. (38) So the ratio of TD to DF is equal to the ratio of TG to GF. (39) The (conic) section is therefore a hyperbola, and its (transverse) axis is line TF.
(40) Again, since the ratio of the square of AD to the rectangle TD by DF is equal to the ratio of TF, which is the transverse diameter, to its erect side, and (since) the square of AD is three times the rectangle TD by DF, because lines AD and DT are equal, therefore the erect side corresponding to it (TF) is three times diameter TF.
(41) So the boundary of (conic) section AKF is known, since its (transverse) axis TF is known in position and in size, and (since) its vertex is the known point F, and (since) its erect side is known, because it is three times the known diameter TF.

(32) FC is supposed to be perpendicular to DF. C can be found by bisecting angle ADG. MUḤAMMAD IBN SARTĀQ (see p. 142) did not realize this, because he wrote on f. 169a of A0:
"Points C and O are where diagonal AG intersects parallels KM and FC, and FC must be twice FG. Contradictory."
He then changed (33) into:
"If we make line GC half of CA, then point C falls with respect to O on the side of G, because BN, which is half of AB, is less than BL, which is equal to GM. We have drawn CF parallel to AD and KM."
(34) Because AD//OM//CF.
(35) By the converse of *Conics* III:16, see the introduction. AL-KŪHĪ says that K is on a "certain" conic because it has still to be determined whether the conic is an ellipse, a parabola or a hyperbola.
(39) By *Conics* I:35–36, II:7, see the introduction.
(40) Actually $AD^2/(TD \cdot DF) = P/TF$ (for the erect side p) by *Conics* I:21. AL-KŪHĪ wrote TF/p by an oversight; the conclusion $p = 3TF$ is correct. The author of version II made things worse by "emending" the text to $AD^2/(TD \cdot TF) = P/TF$.
(41) By the converse of *Conics* I:54. Version II omits "since its (transverse)

(٤٢) فنقطة ك معلومة لأنها على الفصل المشترك لخطين معلومي الوضع أعني قطعين زائدين (٤٣) فخط ك ه معلوم الوضع (٤٤) فنقطة ه معلومة (٤٥) فخط ا ه معلوم وذلك ما أردنا أن نبين (ل ١٢٢ ب)

د (٤٦) تركيب ذلك إن سطح ا ب ج د نصف مربع ا ز ط د ونريد أن نخرج في سطح ا ب ج د خطين على ما وصفنا في التحليل

(٤٧) فنقسم كل واحد من خطي ا ب ب ز بنصفين على نقطتي س ن ونجعل خط د ف ضعف خط ف ج (٤٩) ونعمل على سهم ن س قطعاً زائداً يكون ضلعه القائم مساوياً لضعف خط ا ز حتى يكون أربعة أمثال خط ن س الذى هو القطر المجانب له



(٤٢) على : — ‖ قطعين زائدين : القطعين الزائدين

(٤٥) نبين : نعلم

(٤٦) إن : —

(42) So point K is known, because it is common to two lines known in position, that is, two hyperbolas. (43) So line KE is known in position. (44) So point E is known. (45) So line AE is known. That is what we wanted to prove.

4. (46) Synthesis of this: Rectangle ABGD is half of the square AZTD. We wish to draw in rectangle ABGD two lines as we described in the analysis.
(47) So we bisect each of the lines AB, BZ in points S, N. (Fig. 18).
(48) We make line DF twice line FG. (49) We construct on axis NS a hyperbola with erect side equal to twice line AZ, in order that it be four times line NS, which is the corresponding transverse dia-

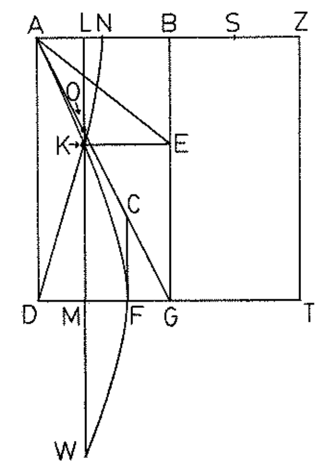


Figure 18

axis TF is known". The result: "So the boundary of (conic) section AKF is known in position and size" is incorrect, because the concept "known in size" is meaningless with regard to the "boundary" of a potentially infinite conic section. The reviser of the text was clearly unaware of this terminological subtlety. We also note that the length of a finite section of the conic in question (a hyperbola) could not be determined by means of ancient and medieval techniques. Thus version II does not make sense in any way.

(42) Version II omits *ʿalā*. Without this word, the text has to be translated as: "So point K is known because it is the common intersection of two lines known in position." Actually, the two (single-branch) hyperbolas intersect in two points, as AL-KŪHĪ probably realized.

وليكن القطع نك (٥٠) ونجعل أيضاً على سهم فط قطعاً زائداً يكون ضلعه القائم مساوياً (ك ١٧٠ أ) لضعف خط دط حتى يكون ثلاثة أمثال خط فط الذى هو القطر المجانب له وليكن القطع فكا (٥١) ونخرج من نقطة ك التي هي تقاطع القطعين خطاً موازياً لخط اب وليكن كه (٥٢) ونصل خطي دك اه

(٥٣) فأقول إن كل واحد من خطي اه دك (س ٣٢ أ) ضعف خط هك

(٥٤) برهانه إنا نصل خط اج ونجعل خط لكم موازياً لخط اد وكذلك (د ١٩٠ ب) خط فص حتى يكون مماساً للقطع

(٥٥) فلأن الضلع القائم لقطع نك أربعة أمثال قطره المجانب وهو نس فمربع كل أربعة أمثال سطح سل في لن (٥٦) وكذلك مربع اب أربعة أمثال مربع بن (٥٧) فمربعا كل اب جميعاً مساو لأربعة أمثال سطح سل في لن مع مربع بن أعني مربع بل (٥٨) فمربعا كل با أعني مربعي هب با أربعة أمثال مربع بل أعني مربع كه (٥٩) فمربعا هب با أعني مربع اه لأن زاوية هبا قائمة مساو لأربعة أمثال مربع هك (٦٠) فلهذا خط اه ضعف خط هك

(٦١) وأيضاً لأن خط دف ثلثا خط دج فيكون هو (ق ٢٠٥ أ) ثلث خط طد لأن دط ضعف خط دج (٦٢) فمربع اد أعني ضلع المربع المعلوم ثلاثة أمثال سطح طد في دف (٦٣) ولكن مربع دا أعني الذي ينتهي إلى القطع ثلاثة أمثال سطح طد في

(٥٤) برهانه: برهان ذلك
(٥٥) فمربع: يكون مربع
(٥٧) مساو: مساويان
(٥٨) كل با: كل اب ‖ هب با: هب اب
(٥٩) فمربعا: فمربعا خطي ‖ با: اب ‖ هبا: ابه
(٦٠) فلهذا: فلذلك يكون
(٦١) فيكون هو: يكون خط دف ‖ لأن دط: وذلك لأن خط طد
(٦٣) ولكن مربع دا أعني الذى ينتهي إلى القطع ثلاثة أمثال سطح طد في دف: –

meter. And let it be hyperbola NK. (50) We also make on axis FT a hyperbola with erect side twice line DT, in order that it be three times line FT, which is the corresponding transverse diameter. Let it be (conic) section FKA.

(51) We draw from point K, which is the intersection of the two (conic) sections, a line parallel to line AB. Let it be KE. (52) We join lines DK, AE. (53) I say that each of the lines AE, DK is twice line EK.

(54) Proof: We join line AG. We make line LKM parallel to line AD, and similarly line FC, in order that it be tangent to the (conic) section.

(55) Because the erect side of (conic) section NK is four times its transverse diameter NS, the square of KL is four times the rectangle SL by LN. (56) Similarly, the square of AB is four times the square of BN. (57) So the sum of the squares of KL, AB is equal to four times the rectangle SL by LN plus the square of BN, that is the square of BL. (58) So the (sum of the) squares of KL, BA, that is the (sum of the) squares of EB, BA, is four times the square of BL, that is the square of KE. (59) So the (sum of the) squares of EB, BA, – that is the square of AE, because angle EBA is a right angle – is equal to four times the square of EK. (60) Therefore line AE is twice line EK.

(61) Again, since line DF is two-thirds of line DG, it is one-third of line TD, because DT is twice line DG. (62) So the square of AD, that is the side of the known square, is three times the rectangle TD by DF. (63) But the square of DA, that is the line which ends at the (conic) section, is three times the rectangle TD by DF, be-

(50) *Conics* I:54. Strictly speaking the conic should be called "FK", not "FKA", but in (64) it will be proven that the conic passes "precisely" through A.
(54) Since FC is perpendicular to the axis, it is a tangent by *Conics* I:17.
(55) *Conics* I:21.
(57) *Elements* II:6.
(63) *Conics* I:21. The qualification "that is the line which ends at the conic section" is absolutely necessary, because it has not yet been proven that the conic passes through A. The reviser of the text did not understand that this has to be proven. He therefore deleted "But the square DA, that is the line which ends at the conic section, is three times the rectangle TD by DF". Thus (63) – (64) in version II are not at all clear.

دف لأن ضلعه القائم ثلاثة أمثال قطره المجانب (٦٤) فلهذا قطع فك يجوز على نقطة آ بعينه

(٦٥) وأيضاً لأن خط دج أعني خط طج ثلاثة أمثال خط جف وخط طد ثلاثة أمثال خط دف فنسبة خط طد إلى خط دف كنسبة خط طج إلى خط جف (٦٦) فلهذا خط جا مماس لقطع فكا على نقطة آ

(٦٧) وأيضاً لأن نسبة صف إلى فج (س ٣٢ ب) كنسبة اد إلى دج و اد ضعف دج فخط صف ضعف خط فج (٦٨) وكذلك خط دف ضعف خط فج (٦٩) فخط دف مساو لخط (ل ١٢٣ أ) فص (٧٠) ونسبة مربع فص إلى مربع صا كنسبة سطح وع في عك إلى مربع عا لأن كل واحد من خطي اص صف مماس للقطع على نقطتي آ ف (٧١) ونسبة مربع اص إلى مربع دف كنسبة مربع اع إلى مربع (د ١٩١ أ) دم لتوازي الخطوط في مثلث اجد (٧٢) فبالمساواة نسبة مربع فص إلى مربع فد كنسبة سطح وع في عك إلى مربع دم (٧٣) ومربع صف مساو لمربع فد لأنهما متساويان (٧٤) فسطح وع في عك مساو لمربع دم (٧٥) وإذا جعلنا مربع (ك ١٧٠ ب) مك مشتركاً بينهما يكون سطح وع في عك

(٦٤) قطع فك يجوز: يجوز قطع فك ‖ بعينه: –
(٦٥) فنسبة: يكون نسبة
(٦٦) فلهذا خط جا مماس: فلذلك يكون خط اج مماساً
(٦٨) خط فج: فج
(٦٩) فص: صف
(٧٠) فص: صف ‖ كل واحد من: – ‖ مماس: مماسان
(٧١) لتوازى: وذلك لتوازى
(٧٣) لأنهما: لأن خطى صف فد

cause its erect side is three times its transverse diameter. (64) Therefore (conic) section FK passes precisely through point A.

(65) Again, because line DG, that is line TG, is three times line GF, and (because) line TD is three times line DF, the ratio of line TD to line DF is equal to the ratio of line TG to line GF. (66) Therefore line GA is tangent to (conic) section FKA at point A.

(67) Again, because the ratio of CF to FG is equal to the ratio of AD to DG, and (because) AD is twice DG, therefore line CF is twice line FG. (68) Similarly, line DF is twice line FG. (69) So line DF is equal to line FC.

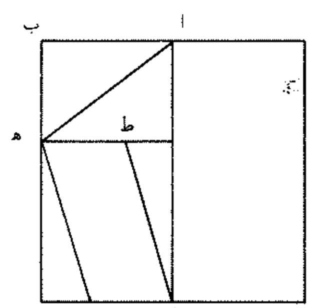
(70) The ratio of the square of FC to the square of CA is equal to the ratio of the rectangle WO by OK to the square of OA, because each of the lines AC, CF is tangent to the (conic) section at points A, F (respectively). (71) But the ratio of the square of AC to the square of DF is equal to the ratio of the square of AO to the square of DM, because the lines (AD, OM, CF) in triangle AGD are parallel. (72) So, *ex aequali*, the ratio of the square of FC to the square of FD is equal to the ratio of the rectangle WO by OK to the square of DM.

(73) But the square of CF is equal to the square of FD, because they (CF, FD) are equal. (74) So the rectangle WO by OK is equal to the square of DM. (75) Taking the square of MK in common, the rectangle WO by OK plus the square of KM, that is the square of

(64) Converse of *Conics* I:21. Version II omits "precisely", compare with the preceding note.
(66) *Conics* I:34.
(70) *Conics* III:16.
(72) *Elements* V:21.
(75) *Elements* II:6, I:47.

مع مربع كم أعنى مربع عم مساوياً لمربع دم مع مربع مك أعني مربع دك (٧٦) فمربع خط مع مساو لمربع خط دك (٧٧) فخط دك مساو لخط عم (٧٨) ولكن خط عم ضعف خط مج لأن اد ضعف خط دج (٧٩) فخط دك ضعف خط مج (س ٣٣ أ) (٨٠) وخط مج مساو لخط كه (٨١) فخط دك ضعف خط كه (٨٢) فكل واحد من خطي اه دك ضعف خط كه

ه (٨٣) نريد أن نخرج في سطح اب جد الذى هو نصف المربع المعلوم خطين كخطى اه هز حتى يكون كل واحد منهما ضعف خط زد (٨٤) فنجعل كل واحد من خطي اه دط ضعف خط هط وخط هط مواز لخط اب كما بينا قبل (٨٥) ونخرج خط هز موازياً لخط دط (٨٦) فأقول إن كل واحد من خطي اه هز ضعف خط زد (٨٧) برهانه إن خط هز مساو لخط دط وخط زد مساو لخط هط لأن سطح هطدز متوازى الأضلاع (٨٨) وكل واحد من خطي اه دط ضعف خط هط (٨٩) فكل واحد من خطي اه هز (د ١٩١ ب) ضعف خط زد وذلك ما أردنا أن نبين (ق ٢٠٥ ب)



(٧٥) كم: مك
(٧٦) مع: عم
(٧٨) خط دج: دج
(٨٢) وذلك ما أردنا أن نبين: +

OM, is equal to the square of DM plus the square of MK, that is the square of DK. (76) So the square of line MO is equal to the square of line DK. (77) So line DK is equal to line OM. (78) But line OM is twice line MG, because AD is twice line DG. (79) So line DK is twice line MG. (80) But line MG is equal to line KE. (81) So line DK is twice line KE. (82) So each of the lines AE, DK is twice line KE.

5. (83) We wish to draw in rectangle ABGD, which is half of the known square, two lines AE, EZ such that each of them is twice line ZD (Fig. 19).
(84) So we make each of the lines AE, DT twice line ET, and line ET parallel to line AB, as we explained above. (85) We draw line EZ parallel to line DT. (86) I say that each of the lines AE, EZ is twice line ZD.
(87) Proof: Line EZ is equal to line DT, and line ZD is equal to line ET, because area ETDZ is a parallelogram. (88) But each of the lines AE, DT is twice line ET. (89) So each of the lines AE, EZ is twice line ZD. That is what we wanted to prove.

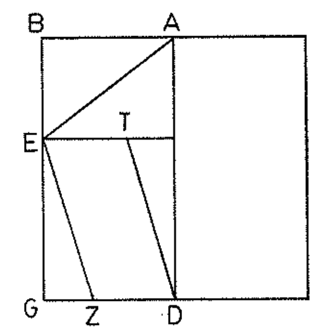
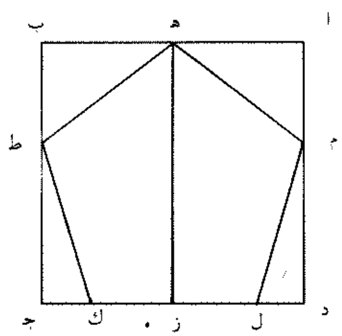


Figure 19

و (٩٠) نريد أن نعمل في مربع اب جد المعلوم مخمساً متساوى الاضلاع وليس بمتساوى الزوايا (٩١) فنقسم كل واحد من خطي اب جد بنصفين على نقطتي ه ز ونصل خط هز (٩٢) ونخرج في سطح ب جزه (س ٣٣ ب) الذى هو نصف مربع اب جد خطين وليكن هط طك حتى يكون كل واحد منهما ضعف خط زك كما بينا قبل (٩٣) ونجعل خط زل مساوياً لخط زك وخط دم مساوياً لخط جط ونصل خطي لم مه (ك ١٧١ أ) (٩٤) فأقول إن مخمس هطكلم المعمول في مربع اب جد متساوى الاضلاع

(٩٥) برهانه إن كل واحد من خطوط هط طك كل ضعف خط كز فهي إذن متساوية (٩٦) ولأن كل واحد (ل ١٢٣ ب) من خطي لد دم مساو لكل واحد من خطي كج جط وزاوية لدم مساوية لزاوية كجط لأن كل واحدة منهما قائمة فقاعدة مل مساوية لقاعدة كط (٩٧) وبهذا التدبير خط هم مساو لخط هط (٩٨) وكل واحد من خطوط هط طك كل لم مه متساوية



(٩٢) وليكن: وها ‖ كما بينا: وذلك مما بيناه
(٩٣) جط: طج
(٩٧) خط هم مساو: يكون خط هم مساوياً
(٩٨) وكل واحد من خطوط: فخطوط

6. (90) We wish to inscribe in the known square ABGD an equilateral but non-equiangular pentagon (Fig. 20).
(91) So we bisect each of the lines AB, GD in points E, Z. We join line EZ. (92) We draw in rectangle BGZE, which is half of square ABGD, two lines, let them be ET, TK, such that each of them is twice line ZK, as we explained above. (93) We make line ZL equal to line ZK, and line DM equal to line GT. We join lines LM, ME. (94) I say that pentagon ETKLM, inscribed in square ABGD, is equilateral.
(95) Proof: Each of the lines ET, TK, KL is twice line KZ, so they (ET, TK, KL) are equal. (96) Since each of the lines LD, DM is equal to each of the lines KG, GT, and (since) angle LDM is equal to angle KGT, because both angles are right, therefore base ML is equal to base KT. (97) By the same reasoning, line EM is equal to line ET. (98) (Thus,) all lines ET, TK, KL, LM, ME are equal.

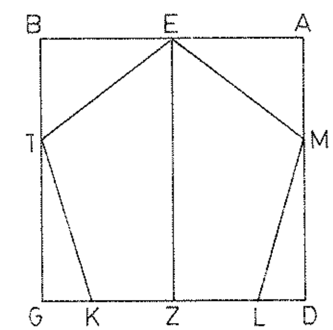


Figure 20

(٩٩) فمخمس هطكلم متساوي الأضلاع وهو معمول في مربع ابجد المعلوم وذلك ما أردنا أن نبين (١٠٠) تمت الرسالة.

(99) So pentagon ETKLM is equilateral, and inscribed in the known square ABGD. That is what we wanted to prove.
(100) The treatise is finished.

(99) MUḤAMMAD IBN SARTĀQ (see p. 142) adds: "Since line AB is equal to MT, and line EL is greater than line EZ, angle MET is less than angle LME. So pentagon ETKLM is not equiangular. As to whether line EL is longer than MK or not, this requires (closer) consideration.
MUḤAMMAD IBN SARTĀQ AL-MARĀGHĪ, who is in need of God most high, read this treatise also, – excellent is the knowledge in it – and he corrected it, in the end of Ṣafar of the year 728 (Hijra, i. e. the middle of January, 1328). The proofs of the analysis concerning the hyperbola are from (the theory of) conic sections, and they can be known from the first parts of the books on conic sections, and especially from my book of the perfection in mathematics. End."
(100) The manuscripts close with different religious formulas. In P the name of the scribe is mentioned, and P and C contain a date, see p. 116.

Variants in the individual manuscripts

(١) استعنت بالله + (ك)، على الله توكيلي وهو حسبني (ل)

(٢) أبي سهل : الشيخ أبي سهل (ك) ‖ ويجن بن رستم : - (ل) ‖ رستم : وستم (س، ك) ‖ رسالة في عمل مخمس متساوي الأضلاع في مربع معلوم لأبي سهل القوهي (ق، د)

(٤) رستم: وستم (س)

(٥) قال: + (ق، د)

(٦) في: - (ك) ‖ هز ط ك ل: ز هـ ط ك ل (ك، ل) ‖ خط جد: خط د جـ (ل، د، ق) د جـ (ك)

(٧) مساو لخط ط ك فكل واحد من خطي هز ز ط: - (س) ‖ فكل واحد من خطي هز ز ط مساو لخط ط ك: (مكررة في ك)

(٩) فنريد: نريد (ك)

(١٠) واحد: (مكررة في ك)

(١٣) ك هـ: هـ ك (ك)

(١٥) هـ ك: هـ ط (ك، ل)

(١٧) مساو لمربعي خطي اب ب هـ لأن زاوية اب هـ قائمة وخط ب هـ: (مكررة في س)

(١٩) أعني مربع: أعني موضع مربع (س)

(٢١) س ل: س ك (س)

(٢٢) س ن: - (س)

(٢٥) ضعف د جـ: ضعف اجـ (ل)

(٣١) يبق: بق (ق)

(٣٥) ما: - (ك)

(٣٦) جـ ف: ص ف (س)

(٤٠) إلى سطح ط د: إلى سطح طه (س) ‖ د ف لأن: ط ف لأن (ل) ‖ له: - (ك)

(٤١) ا ك ف: ا ك (ك)

(٤٦) ا ز ط د: ا ز ط (س) ا ط د (ل)

(٥٠) سهم ف ط: سهم قط (ك) ‖ ف ك ا: ف ك (ق، د)

(٥٢) د ك ا هـ: ا هـ د ك (ك)

(٥٥) س ل: س ك (س)

(٥٧) مع مربع ب ن: وكذلك مربع ا س (س)

(٥٨، ٥٩) أعني مربعي هـ ب ا ب أربعة أمثال مربع ب ل أعني مربع ك هـ فمربعا خطي هـ ب ا ب: (مكررة في ك)

(٦٠) ضعف: ضعف ضعف (س)

(٦١) فيكون : ويكون (س)

(٦٥) د جـ: د جـ ا (س) ‖ أمثال خط د ف: أمثال خط ر ب (س) ‖ فنسبة خط (س): يكون نسبة خط (ق، ك، ل) يكون نسبة (د)

(٧٥) يكون: كان (ك) ‖ مساويا: مساو (س)

(٨٢) فكل: وكل (س)

(٨٤) مواز: موازيا (د، ق)

(٨٧) برهانه (س، ك): برهان ذلك (ل، د، ق) ‖ لخط د ط وخط ز د مساو: - (س) ‖ هـ ط د ز: هـ ك د ز (س)

(٨٩) فكل: وكل (س)

(٩٣) ز ل: ز ا (س)

(٩٥، ٩٦) ك ل ضعف خط ك ز فهى إذن متساوية ولأن كل واحد من خطي: - (س)

(١٠٠) والحمد لله رب العالمين + (ل)

ولله الحمد والمنة + (ك)

بعون الله وتوفيقه والحمد لله رب العالمين + (د)

بخط الحسين بن محمد بن على في جـ يه ط ثمد وهو يو كنون الثاني غقسا

والحمد لله دائما والصلاة على النبي محمد وآله + (س)

والصلاة والسلام على محمد وعلى آله وصحبه أجمعين

وقد استقرت سفينة القلم على جودى الزبر والرقم في يوم الأحد الحادى والعشرين من ذى القعدة سنة تسع وخمسين ومائة بعد الألف من هجرة من أنزل عليه القرآن حرفا بعد حرف عليه أفضل التحية + (ق)

Marginal remarks by MUḤAMMAD IBN SARTĀQ AL-MARĀGHĪ (in ms. A0)

حواشي محمد بن سرتاق المراغي (في مخطوط ك)

(ك، ١٦٨ ب) حاشية في الشكل الثالث المعتبر المبحوث عنه هو خطا هل لك وفي الشكل الثاني هو خطا هز هط وهذا تصرف لصنف حنه (حنة؟) فإنه لا فرق بين الاعتبار في التي . . .

(ك، ١٦٩ أ، جملة ٢٢) هذا يعلم من شكل د من فصل آ من صنف جـ من نوع د من جنس آ من جنسي التعالم الرياضية من كتابي الإكمال والصنف هذا في قطوع المخروطات والأساطن وتعلم للرياضي كتب المخروطات أيضاً في براهين(؟) القطع الزائد

(ك، ١٦٩ أ، جملة ٣٢) نقط ص ع هي حيث يقاطع قطر اجـ موازى كم ويجب أن يكون ف ص مثلي ف ج خلف

(ك، ١٦٩ أ – ب: شطب محمد بن سرتاق في جملة ٣٢ الكلمات التالية:
« دف مساوياً لخط ف ص» و «لأن ص ف ضعف خط ف جـ»)

(ك، ١٦٩ ب أضاف محمد بن سرتاق إلى جملة ٣٢) د ص نصف ص ا فتقع نقطة ص من ع إلى جانب جـ لأن ب ن الذى هو نصف اب أصغر من ب ل الذى هو مثل جم وأخرجنا موازى ص ف ل اد و كم

ك، ١٧١ أ، جملة ١٠٠) ولأن خط اب كخط م ط وخط هل أعظم من خط هز فزاوية م ه ط أصغر من زاوية ل م ه فمخمس ه ط ك ل م ليس بمتساوى الزوايا وأما أن خط هل أطول من خط م ك أم لا فيحتاج إلى نظر

وطالع هذه الرسالة أيضاً وأفضل فيها العلم وحققها الفقير إلى الله تعالى محمد بن سرتاق المراغي في آخر صفر سنة ثمان وعشرين و . . . (سبعمائة في حاشيته في ك، ١٦٥ أ) وبراهين التحليل من القطع الزائد من المخروطات وتعلم من أوائل كتب المخروطات خصوصاً من كتابي الإكمال في الرياضي والسلام

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