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Book 1 of Jan de Witt's Elementa Curvarum Linearum and the Conics of Apollonius

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1. Introduction

To most people in Holland, Jan de Witt (1625–1672) is known as a statesman who played an outstanding role in the history of the Dutch republic. It is much less known that he was also a talented mathematician. Christiaan Huygens wrote: "Could he have spared all his strength for mathematical works, he would have surpassed us all" [7, p. 466]. De Witt authored two mathematical treatises. His *Waerdye van Lyf-renten naer proportie van Los-renten* (1671) is a work in Dutch on the mathematical theory of life annuities, inspired by political and fundraising problems. His other mathematical treatise is a work in Latin entitled *Elementa Curvarum Linearum* (Elements of Curved Lines [16]), in two books. De Witt wrote most of this treatise in the years 1646–1649, before the beginning of his political career. The work did not appear until 1660,[1] and it was reprinted in 1683 and in 1695 [14].

The present paper was written on the occasion of the symposium, held on April 3, 1997 in Wageningen, celebrating the publication of a facsimile edition of the Latin text of Book 1 of the *Elementa Curvarum Linearum* with Dutch translation and commentary by Professor Albert Grootendorst [15]. A Dutch translation with commentary of Book 2 will be published by Professor Grootendorst in the near future.

2. The role of Book 1 of Jan de Witt's *Elementa Curvarum Linearum* in the history of mathematics.

The purpose of Book 1 of the *Elementa Curvarum Linearum* can be described as follows. Until the middle of in the 17th century, the basic text on conic sections was the first part of the *Conics* of Apollonius (ca. 200 B.C.) [12, 1, 13, 10]. The *Conics* consisted of eight Books, of which the first four had been published in a Latin translation by Commandino in Bologna in 1566. In the *Conics*, Apollonius defines a conic section as the intersection of a (right or oblique) cone and a plane. He then presents a complicated theory of conic sections without any algebraic symbolism, using proportions and the Greek theory of application of areas [10].

[1] The date on the first edition is 1659 but De Witt received the first copies in February 1660.

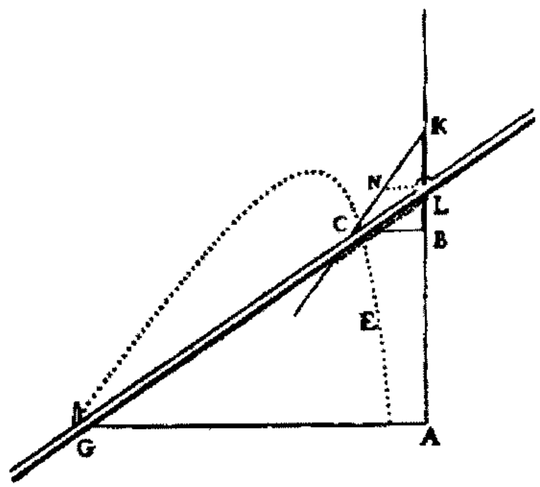

FIGURE 1. See also [5, p. 321]

In 1637, René Descartes showed in his *Géométrie* [5] how algebra and algebraic symbols could be used for the solution of geometric problems, and he established a connection between algebraic equations in x and y and curves. In the *Géométrie*, there is an interesting difference between circles and conic sections on one hand, and higher-degree curves on the other hand. For circles and conic sections, Descartes relied on the Greek geometrical works of Euclid and Apollonius. Thus when Descartes derived an algebraical equation $y^2 = p \cdot x$, he concluded on the basis of the *Conics* of Apollonius that the corresponding curve was a parabola [5, p. 328-329]. For higher-degree curves, Descartes sought to provide his own definitions by means of (idealized) instruments consisting of various rulers. Descartes felt that an algebraic equation in x and y was not acceptable as a definition of a curve; it had to be shown that such a curve could be drawn by an instrument [3].

Of course, it was well-known since antiquity that conic sections could also be drawn by instruments. In the *Géométrie*, Descartes presented the following construction of a hyperbola [5, p. 321]:

In Figure 1, G and A are fixed points and a perpendicular to AG is drawn through A. The ruler GL pivots around G and intersects the perpendicular at a variable point L. Point L pushes a segment LK of fixed length along the perpendicular, and from the variable point K the straight line KN is drawn in a fixed direction. Line KN intersects line GL at point C, which describes a curve. Descartes finds the algebraic equation of the curve and he shows (on the basis of the *Conics*) that it is a hyperbola.

The Dutch mathematician Frans van Schooten junior (1615–1660), a friend of Jan de Witt, presented several constructions of conic sections in his *Organic description of conic sections*, published in 1646. This work contains the following construction of an ellipse [11, p. 11]:

(Figure 2) Let m_1 and m_2 be two intersecting straight lines in the plane, and let P, Q and R be three fixed points on a ruler. If the ruler is moved in such a way that P is always on m_1 and Q is always on m_2, point R will describe an ellipse (or a circle in some special cases).

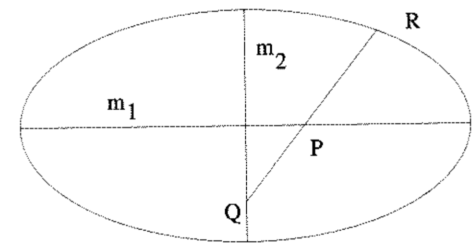

FIGURE 2.

The idea of Book 1 of the *Elementa Curvarum Linearum* is to use constructions of this type as *definitions* of the conic sections. For example, De Witt uses the two above-mentioned constructions as his definitions of the hyperbola and the ellipse. Thus he does not define the conic sections as plane sections of the cone; he mentions the words cone and conic section only in one passage towards the end of Book 1 of the *Elementa Curvarum Linearum* [15, pp. 182–185]. However, he still used the Apollonian terms parabola, hyperbola and ellipse.

In Book 1 of the *Elementa Curvarum Linearum*, De Witt proves most of the basic theorems on conic sections of Books I and II of the *Conics* of Apollonius (see the list below). His proofs are simpler than those of Apollonius. De Witt believed that his own approach is also more natural because his type of definition of conic sections can be extended to curves of higher degree. In this paper we will not be concerned with Book 2 of the *Elementa Curvarum Linearum*, on the relationship between conic sections and algebraic equations in x and y. Summaries of Book 2 can be found in [15, pp. 9-10] and [8, pp. 115-125].

The two Books of De Witt's *Elementa Curvarum Linearum* were published in 1660 as an appendix to Frans van Schooten's Latin edition of the *Géométrie* of Descartes [6]. Thus the reader of the *Géométrie* could use the work of De Witt, instead of that of Apollonius, to understand what Descartes had said about conic sections. However, Book 2 of the *Elementa Curvarum Linearum* probably interested more readers than Book 1, for the following reason.

In the *Géométrie*, Descartes argued that all curves with algebraic equations could be constructed by geometrical instruments of a kind which he considered legitimate [3, p. 331-2]. The mathematicians generally accepted his conclusion and stopped worrying about the details of the geometrical constructions of the curves [4, p. 344]. Thus Book 1 of the *Elementa Curvarum Linearum* and the *Conics* of Apollonius were no longer of much interest to mathematicians in the end of the 17th century. Certain simple second-degree algebraic equations in x

and y became acceptable as definitions of the parabola, hyperbola and ellipse, and the mathematicians became interested in reducing a given second-degree equation in x and y to a simple form. This is the problem which De Witt discussed and solved in Book 2 of the *Elementa Curvarum Linearum*.

De Witt was not influenced by the projective theory of conic sections in Desargues' *Broullion Project* of 1639. In the preface to his *Organic Description* Frans van Schooten jr. mentions Desargues' work [11, p. 30, fn 29], but he does not seem to have read or understood it. What Desargues and De Witt had in common was their motivation to simplify or redo parts of the *Conics* of Apollonius.

3. Jan de Witt on Apollonius

The *Elementa Linearum Curvarum* was published in a period when De Witt was very busy with political work, and the publication process was overseen by Frans van Schooten jr. De Witt first sent Book 2 to van Schooten and Van Schooten returned it with his corrections in the beginning of October 1658. On October 8, De Witt sent Book 1 to Van Schooten with an accompanying letter, in which he describes the relation of Book 1 to the *Conics* of Apollonius in the following words.

> "Naer mijn beste onthoudt vervatt het eerste capittel in twee propositiën ende eenige corollaria, daeruut vloeyende, 't gene in Apollonio meer als 20 propositiën beslaet ende daeronder verscheyden soo veel opmerckingen requireren als een van dese twee. Het tweede capittel in weynich propositiën, 't gene in Apollonio door meer dan 50 is verspreyt ende daeronder eenige soovele hooftbreeckens vereysschen als 't verschreven geheele capittel, daerinne ick niet weet, dat eenige difficile propositie werdt bevonden. Het derde capittel in ses off seven propositiën, 't gene Apollonius in ontrent de 30 propositiën voorstelt." [17, pp. 425-6]

A paraphrase of this passage follows:

> "As far as I remember, the first chapter contains in two propositions and a few corollaries what Apollonius treats in more than 20 propositions, some of which require as many notes as either of these two. The second chapter (contains), in few propositions, what is scattered in Apollonius over more than 50 (propositions), some of which require as much mental effort as the whole above-mentioned chapter, in which there is not a single difficult proposition that I know of. The third chapter (contains) in six or seven propositions what Apollonius presents in around 30 propositions."

To check these statements I have compiled the following list in which I compare the contents of Book 1 of the *Elementa Curvarum Linearum* with the *Conics*. The notation II:14 refers to (part of) proposition 14 of Book II of the *Conics*. A notation such as ≈ II:16 means that the proposition or corollary in question corresponds approximately to *Conics* II:16. The notation - means that the theorem or construction in question is not found in the *Conics*.

Chapter 1 of *Elementa Curvarum Linearum I*: The Parabola

De Witt, *Elementa* I	Apollonius, *Conics*
prop. 1	I:11
cor. 1	I:26
cor. 2	- (necessary for cor. 4)
cor. 3	- (necessary for cor. 4)
cor. 4	I:27
cor. 5	I, def. 4
cor. 6	- (necessary for cor. 9)
cor. 7	I:20
cor. 8	I:17
cor. 9	I:32, II:15
cor. 10	I:51,52
prop. 2	I:46,49
cor. 1	II:44
cor. 2	I:35,33
cor. 3	II:49
cor. 4	- (necessary for cor. 5)
cor. 5	II:51

Thus the first chapter corresponds to around 15 propositions in the *Conics*, and De Witt's number 20 is somewhat optimistic.

Chapter 2 of *Elementa Curvarum Linearum I*: The Hyperbola

De Witt, *Elementa* I	Apollonius, *Conics*
prop. 3	II:12
cor. 1	II:14
cor. 2	II:2, -
prop. 4	II:8,3
prop. 5	≈ II:10,13
cor. 1	II:11
cor. 2	II:8,16
cor. 3, 4	-
cor. 5	cf. I: definitions 5,6
cor. 6	-
cor. 7	II:44,45
prop. 6	II:2
cor. 1	II:10
cor. 2, 3	≈ I:17, 32
prop. 7	II:31

prop. 8	≈ I:55
prop. 9	-
prop. 10	≈ I:21
cor. 1	≈ II:1
cor. 2	I:12
cor. 3	I:21
prop. 11	I:37
prop. 12	I:38
cor.	II: 49

The second chapter corresponds to about twenty propositions in the *Conics*. If we add *Conics* II:1,4 and I:32, 47, 48, 50, 51, which are implicit in this chapter, the total is about 25 propositions, so De Witt's number 50 is too high.

Chapter 3 of *Elementa Curvarum Linearum I* : The Ellipse

De Witt, *Elementa* I	Apollonius, *Conics*
prop. 13	≈ I:13
cor. 1	I:15
cor. 2	part of I:17
cor. 3	≈ I:21
cor. 4	cf. I: 7
cor. 5	- (cf. I:22)
cor. 6	I:13
cor. 7	cf. I:56-58
prop. 14	I:47, 50
cor. 1	cf. I:58, II:47
cor. 2	I:30
cor. 3	cf. I:15
cor. 4	I:32
prop. 15	cf. II:6
cor. 1	cf. II:6
cor. 2	II:27, 44, 47
cor. 3	-
prop. 16	-
cor.	part of II:49
prop. 17	I:32
cor.	-
prop. 18	I:37
cor.	part of II:49

Chapter 3 corresponds to 22 propositions in the *Conics* so the number 30 given by De Witt is again optimistic.

Apollonius often deals with the parabola, hyperbola and ellipse in a single proposition, so one proposition of the *Conics* may appear in two or three of my lists. The contents of the three Chapters of the *Elementa Curvarum Linearum* corresponds to 50 propositions of the *Conics* and not to 20 + 50 + 30 as De Witt suggests.

Thus De Witt exaggerated the difference between Book 1 of his *Elementa Curvarum Linearum* and the *Conics*.

Book 1 of the *Elementa Curvarum Linearum* does not contain the theorems of Book III of the *Conics* on poles and polar of conics and on the foci of the hyperbola and ellipse. De Witt treated the foci in Book 2. The question arises wheher De Witt was influenced by Books V–VII of the *Conics*, which have come down to us through the Arabic, and which were published in 1710. Arabic manuscripts of these Books were known to the mathematician and Arabist Jacobus Golius (1596–1667), with whom De Witt had good relations [17, pp. 434, 437]. In his summary of Chapter 3 of the *Elementa Curvarum Linearum* [8, p. 112], P. Van Geer mentions two "theorems of Apollonius, " which concern the ellipse, and which Apollonius proved in *Conics* VII:12, 31 [2, pp. 412–414, 450–452].[2] Van Geer must have mentioned these theorems because they can be proved by extending an argument in De Witt's Chapter 3, Theorem XIII, but De Witt does not mention the two "theorems of Apollonius" themselves in spite of their elegance. Thus there was probably no influence of the Arabic Books V–VII of the *Conics* on De Witt.

4. Similarities and differences between the *Conics* and Book 1 of the *Elementa Curvarum Linearum*

One of the basic problems in Apollonius' theory of conic sections is the determination of all diameters of a conic section and of a fundamental property of a diameter and the corresponding ordinates. De Witt also discusses this problem and it will be illuminating to compare the approaches of the two mathematicians. The following concepts of Apollonius are necessary to explain this problem (Figure 3). Let $\mathcal{K}$ be an arbitrary curve.

A *chord* of $\mathcal{K}$ is any straight segment with both endpoints on $\mathcal{K}$.

A *diameter* of $\mathcal{K}$ is any straight line ℓ for which there exists a straight line m such that ℓ bisects all chords of $\mathcal{K}$ parallel to m. The halves of the bisected chords are called the *ordinates* corresponding to ℓ.

The diameter is called an *axis* of $\mathcal{K}$ if its ordinates are perpendicular to it.

In the beginning of Book 1 of the *Conics*, Apollonius defines a conic section as a plane section of a cone. Using this cone, he proves that every conic section has one diameter ℓ_0. This diameter ℓ_0 does not have to be an axis, but it always intersects the conic at at least one point P_0. For the parabola, this is

[2] The first "theorem of Apollonius" states that the parallelogram formed by the four tangents to the ellipse parallel to two conjugate diameters has the same area as the parallelogram formed by the tangents parallel to the axes, and the second "theorem of Apollonius" is to the effect that the sum of the squares of any two conjugate diameters (i.e. the squares of the parts of these diameters inside the ellipse) is equal to the sum of the squares of the axes. Two diameters are called conjugate if the ordinates of the first diameter are parallel to the second diameter and vice versa.

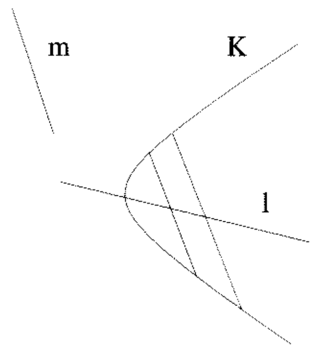

FIGURE 3.

the only point of intersection. For the ellipse and hyperbola,[3] ℓ_0 intersects the conic in a second point P_0'. We call the midpoint C of P_0P_0' the *centre* of the hyperbola or ellipse.

Apollonius proves the following statements in Book 1 of the *Conics* (Figure 4, compare [10]):

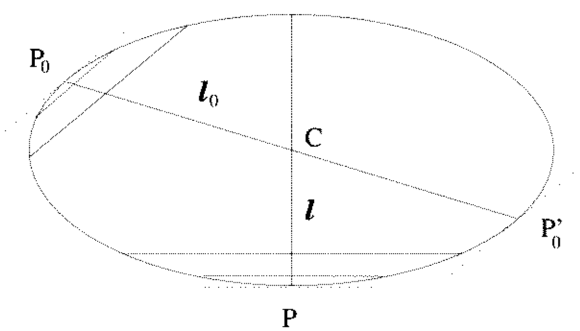

FIGURE 4.

- The "original diameter" ℓ_0 and its ordinates satisfy what I will call the *Apollonian fundamental property.* Apollonius states this property as an equality between the square of an ordinate and a rectangle, but it can be interpreted in modern algebraic symbols as the equation of the conic in a (not necessarily rectangular) coordinate system of which the diameter ℓ_0 and the tangent at P_0 (see below) are the coordinate axes. This equation is of the form $y^2 = px$, $y^2 = px + \frac{p}{d}x^2$, or $y^2 = px - \frac{p}{d}x^2$ [12, pp. 182-184]. In the first case, Apollonius calls the conic section a parabola, in the second case a hyperbola and in the third case an ellipse.[4]

[3] Apollonius called the two branches of a modern hyperbola two "opposite hyperbolas." This difference is terminological interest only, because Apollonius deals with the two "opposite hyperbolas" as one mathematical object.

[4] The name parabola is explained by the rectangle px applied to, or constructed along

- The conic has a tangent at P_0 (and one at P_0') parallel to the ordinates of ℓ_0.
- The conic has a tangent at any other point P.
- The conic has a diameter ℓ through any point P. For the parabola, ℓ and ℓ_0 are parallel. For the ellipse and hyperbola, ℓ passes through C. The ordinates of ℓ are parallel to the tangent at P.
- The new diameter ℓ and its ordinates satisfy the Apollonian fundamental property of the same form as the property of the original diameter and its ordinates, but for different constants. For the parabola, for example, the new fundamental property can be interpreted as $y'^2 = p'x'$, where y' is the length of any ordinate of ℓ, x' the part of ℓ between the ordinate and P, and the constant p' depends on p, ℓ_0, P_0, ℓ, P.

Apollonius notes that all properties which had been proved for the original diameter are also true for any other diameter. Thus the concept of original diameter loses much of its meaning.

I now turn to some similarities and differences between the *Conics* and Book 1 of the *Elementa Curvarum Linearum.*

As has been mentioned above, De Witt defines the conics by means of idealized instruments in the plane, not by means of the cone. However, for the parabola, he immediately proves the Apollonian fundamental property for one diameter ℓ_0 (Cap. 1, Th. 1, and cor. 5, [15, pp. 52-61]). He then proves that all lines ℓ parallel to ℓ_0 are diameters and that the ordinates of these new diameters satisfy the Apollonian fundamental property of the parabola (Cap. 1, Th. 2, [15, pp. 68–73]). His proof is somewhat simpler than, but not essentially different from, the proof by Apollonius.

De Witt defines the hyperbola by means of the construction of Figure 1, and he then proves an identity equivalent to the modern equation $xy = c$ of the hyperbola in a coordinate system with the asymptotes as coordinate axes (Cap. II, Th. 3, [15, p. 88–92]). By means of the asymptotes he finds the centre of the hyperbola and he proves that all straight lines through the centre, except the asymptotes, are diameters (Cap. II, Th. 5, Cor. 4–5, [15, pp. 104–107]), and that their ordinates satisfy the Apollonian fundamental property of the hyperbola (Cap. II, Th. 9, Cor. 4–5, [15, pp. 122–125]). Apollonius based his theory of asymptotes, in Book 2 of the *Conics*, on his theory of diameters of the hyperbola in Book 1 of the *Conics*. De Witt turned Apollonius' approach the other way around and simplified the theory in this way.

De Witt defines the ellipse by Van Schooten's construction (Figure 2). He first proves that one of the lines m_1 or m_2 used in the construction of the ellipse is a diameter and that the corresponding ordinates satisfy the Apollonian fundamental property of the ellipse (Cap. III, Th. XII and cor. 4, [15, pp.

(Greek: paraballomenon) a fixed segment of length p. The names hyperbola ("excess") and ellipse ("defect") are related to the terms $+\frac{p}{d}x^2$ and $-\frac{p}{d}x^2$ by which the square y^2 exceeds or falls short of the rectangle px.

138-151]). To show the existence of the other diameters, De Witt considers an ellipse which has been constructed by means of lines m_1 and m_2 which intersect at point C, and he chooses another arbitrary line m_3 through C (Figure 5). He then shows how a new line m_4 and a new ruler with points P', Q' and R' can be constructed in such a way that all points on the new ellipse, constructed from m_3, m_4, P', Q', R' lie on the old ellipse, constructed from m_1, m_2, P, Q, R. (Cap. III, Th. XIII, [15, pp. 154-165]).

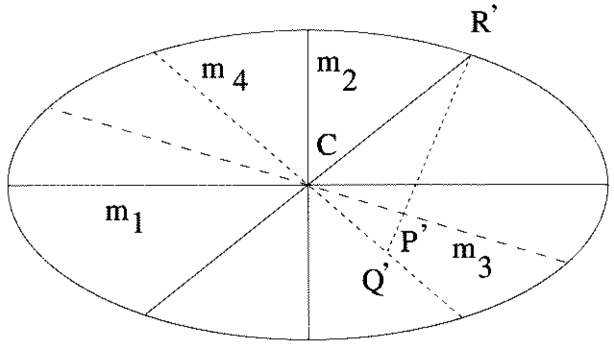

FIGURE 5.

For sake of simplicity, he assumes that m_1 and m_2 make a right angle, but he notes that this assumption does not limit the validity of the argument (Cap. III, Th. XIII, cor. 1, [15, pp. 164-167]). De Witt's proof does not involve the Apollonian fundamental property of the ellipse, and his proof is not inspired by the *Conics*. From a mathematical point of view, Chapter III, Theorem XIII is among the most interesting parts of Book 1 of the *Elementa Curvarum Linearum*, and a witness of the mathematical talents of Jan de Witt.

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