

Two Tables from an Arabic Astronomical Handbook

for the Mongol Viceroy of Tibet

E.S. KENNEDY AND JAN HOGENDIJK

This study has no direct connection with Abe Sachs or Assyriology other than the all-pervasive sexagesimal numbers it employs. Nevertheless it is dedicated to the memory of a colleague whose trenchant wit only thinly covered his essential modesty and kindness. We never think of him but with affection and admiration.

INTRODUCTION

In December of 1366 a certain Abū Muḥammad ʿAṭā b. Aḥmad b. Muḥammad Khwāja Ghāzī al-Sanjufīnī completed a *zīj*, a set of astronomical tables, preserved in the unique Paris manuscript BN Arabe 6040. He dedicated it to his patron, Prince Radna, the Mongol viceroy of Tibet, and a direct descendant in the seventh generation from Genghis Khan. The viceregal capital is named in the *zīj*, and Professor Herbert Franke has identified it as the then Ho-chou, in the Chinese province of Kansu, southwest of the modern provincial capital, Lan-chou. As for the author, his name indicates that he or his ancestors came from a village near Samarqand, in Central Asia.

The manuscript is of considerable interest, containing among its forty-two tables information on calendars, lists of fixed stars with their Chinese names written in Arabic characters, numerous Mongol and a few Tibetan glosses, and the usual spherical astronomical and planetary material. From amid the plethora of intriguing questions raised by this curious document, two have been chosen for investigation below.

Among the tables used for eclipse predictions is one based ultimately upon theories originating in India in the seventh century or before. The calculation of the entries by a method which is as elegant mathematically as it is absurd astronomically is described in a document from ninth

century Baghdad. A version of this parallax table turned up in a *zīj* written in Damascus in the fourteenth century. But the Sanjufīnī version is included in a *zīj* from the Caucasus dated two centuries before. Thus, in the course of seven hundred years the theory migrated westward to Europe, thence back eastward, reappearing not far from its place of origin.

The second table enables its user to predict the appearance of the new moon, thus determining, for instance, the end of the Muslim fasting month. There is an enormous literature on the subject, but Sanjufīnī's method appears to be more or less independent, marginally related to that of Thābit ibn Qurra.

The table gives explicitly a terrestrial latitude of $38;10^\circ$, a parameter which is shown below to be embedded in the table itself. The same number appears five times elsewhere in the *zīj*. Now the latitude of Lan-chou is $35;44^\circ$, which differs from the above by about two and a half degrees. A second Mongol provincial capital, Yung-ch'ang fu, is also mentioned in the *zīj*. Its latitude is very close to $38;10^\circ$. Considerations connected with an eclipse prediction in the *zīj* make it very probable that the second city was the base locality for which the tables were computed.

THE PARALLAX TABLE

This takes up the entirety of f. 42v. It is called "A Table of the Equation of Distance for the Middle of the Eclipse" (*Jadwal ta'dīl al-bu'd li wasa' al-kusūf*). A Mongol gloss runs along the right hand side of the page. According to Professor Herbert Franke,¹ this consists of four words, the first two being the usual Mongolian for a lunar eclipse. The last two are Arabic *ta'dīl* and (probably) *bu'd*, transliterated into Mongolian characters. The scribe was evidently astronomically uninformed, for *kusūf* usually denotes a solar eclipse, and this particular table can be used only for solar eclipses.

The page on which it appears has been divided into four main columns, each of which is further split into two subcolumns. The right hand subcolumns are headed "(time) between conjunction and noon," and contain the values of the argument, in hours and minutes: 0;3, 0;6, 0;9, . . . 6;0. The subcolumns on the left are headed "(time) between the eclipse middle and noon." The entries underneath are to seconds of time.

The same table, except for a few differences due to scribal errors, appears as Table L in Part 2, p. 174 of a Byzantine Greek version² of *zīj* written by one 'Abd al-Karīm al-Fahhād al-Shirwānī c. 1175. There it is called in Arabic *Ta'dīl sāt al-ru'ya*, "The Equation of the Hours of Visibility." Every fourth entry has been transcribed and appears as Columns 1 and 3 of our Table 1 below. An application of the table is cited by Neugebauer.³

1. H. Franke, "Mittelmongolische Glossen in einer arabischen astronomischen Handschrift von 1366," to appear in *Oriens*.

2. D. Pingree, *The Astronomical Works of Gregory Chionades*, Vol. 1, *The zīj al-'Alā'ī*, 2 parts (Amsterdam, 1986).

3. O. Neugebauer, *The Astronomical Tables of al-Khwārizmī*, Translation with Commentaries . . . (Copenhagen, 1962), p. 28.

TABLE 1

1		2		3		4		5		6	
<i>t</i>				<i>T</i>				<i>P</i> in hours			
				Sanjufinī- Shirwānī entries		Sanjufinī = <i>T</i> - <i>t</i>		Computed, $\epsilon = 23;51$		Khwārizmī	
hours	degrees	hours		hours		hours		hours		hours	
0;12	3	0;20,53		0; 8,53		0; 8,32		0; 8,38		0; 8,38	
0;24	6	0;41,24		0;17,24		0;16,58		0;17, 7		0;17, 7	
0;36	9	1; 1, 5		0;25, 5		0;25, 9		0;25,23		0;25,23	
0;48	12	1;20,28		0;32,28		0;33, 2		0;33,23		0;33,23	
1; 0	15	1;40, 0		0;40, 0		0;40,31		0;40,54		0;40,54	
1;12	18	1;58,29		0;46,35		0;47,32		0;48, 0		0;48, 0	
1;24	21	2;16, 0		0;53, 0		0;54, 3		0;54,53		0;54,53	
1;36	24	2;36,35		1; 0,36		1; 0, 3		1; 0,25		1; 0,25	
1;48	27	2;54,24		1; 6,24		1; 5,31		1; 6,12		1; 6,12	
2; 0	30	3;11,20		1;11,20		1;10,28		1;11, 5		1;11, 5	
2;12	33	3;27,36		1;15,36		1;14,54		1;15,27		1;15,27	
2;24	36	3;43,28		1;19,28		1;18,49		1;19,32		1;19,32	
2;36	39	3;59,13		1;23,18		1;22,15		1;22,56		1;22,56	
2;48	42	4;14,15		1;26,15		1;25,14		1;25,49		1;25,49	
3; 0	45	4;29,20		1;29,20		1;27,47		1;28,24		1;28,24	
3;12	48	4;43,[4]		1;31, 4		1;29,55		1;30,33		1;30,33	
3;24	51	4;56,37		1;32,37		1;31,40		1;32,16		1;32,16	
3;36	54	5; 9,[4]		1;33, 4		1;33, 3		1;33,41		1;33,41	
3;48	57	5;21,52		1;33,52		1;34, 6		1;34,44		1;34,44	
4; 0	60	5;34,50		1;34,50		1;34,50		1;35,27		1;35,27	
4;12	63	5;47, 6		1;35, 5		1;35,15		1;35,51		1;35,51	
4;24	66	5;59,14		1;35,12		1;35,24		1;36, 0		1;36, 0	
4;36	69	6;11,13		1;35,13		1;35,17		1;35,51		1;35,51	
4;48	72	6;23,24		1;35,24		1;34,55		1;35,29		1;35,29	
5; 0	75	6;35,30		1;35,30		1;34,20		1;35,53		1;35,53	
5;12	78	6;46,10		1;34,10		1;33,32		1;34, 4		1;34, 4	
5;24	81	6;56,32		1;32,32		1;32,32		1;33, 1		1;33, 1	
5;36	84	7;6,[5]5		1;30,55		1;31,22		1;31,49		1;31,49	
5;48	87	7;17, 0		1;29, 0		1;30, 2		1;30,27		1;30,27	
6; 0	90	7;27, 0		1;27, 0		1;28,34		1;28,51		1;28,51	

It can be inferred from Sanjufinī's explanatory text (f. 13v:6) that the function is intended to supply a time for a luni-solar apparent conjunction, corrected for the effect of parallax in longitude. the term *ta'dīl* (equation, correction) appearing in both titles is a misnomer, for the function *T*, the time from noon of the apparent conjunction, appearing in Column 3 has already

had the correction added to it. If t denote the hour angle or time of the true conjunction, the argument in Column 1, then:

$$T - t = P$$

is the correction due to parallax. This function has been plotted as the dotted curve on the graph below, where each dot corresponds to a value of P obtained from the sources. It is clear that whoever calculated the table in the first place did a bad job. The curve is never smooth, and several segments give the impression of having been determined by linear interpolation between fixed points. The plot is practically flat on top. Moreover, whatever theory underlies the computation, it corresponds only crudely to the facts. The function starts from the origin, and it is true that the longitude component of parallax is zero or near it at noon, being larger in the forenoon and afternoon with increased zenith distance. But the P function has a maximum at about $t = 4;24^h$, frequently well before sunset.

A very similar function is displayed in Table 77 of Suter's *Die astronomischen Tafeln des Muhammed ibn Mūsā al-Khwārizmī* (Copenhagen, 1914), the Latin translation of the Majrīfī recension of the *zīj* of al-Khwārizmī (fl. 830), and a second copy is in the *zīj* of Ibn al-Shāfir⁴ (fl. 1350). Entries excerpted from it are shown in Column 6 of our table, and they have been plotted as the continuous curve on the graph. In contrast to the other, they have been calculated with precision. Different but equivalent units are used for the arguments (Column 2), time degrees of daily rotation, whereby fifteen degrees correspond to one hour. Also the Khwārizmī table is more realistic than the Sanjufīnī-Shirwānī version as to daylight length, the argument going to 150° (= 10 hours). Perhaps the more crude table was intended for "unequal hours", defined as a sixth of the time from sunrise to noon. The text says nothing to this effect, but it is specified by Ḥabash,⁵ mentioned just below.

The method of calculating the Khwārizmī table is explained in the *zīj* of Ḥabash al-Hāsib (fl. 830).⁶ In the recursion relation

$$T_n(t) = t + \varepsilon \sin T_{n-1}(t)$$

where ε is the obliquity of the ecliptic, put $T_0 = t$ and calculate successive elements of the convergent sequence $T_1, T_2, T_3, \dots \rightarrow T(t)$ until the desired degree of precision has been attained.⁷ Ḥabash regarded T_5 as sufficient.

The result satisfies the expression

$$(1) \quad T(t) = t + \varepsilon \sin T(t)$$

known as Kepler's equation. Then

$$(2) \quad P(t) = 0;4 \varepsilon \sin T(t)$$

is the desired P for any given t .

The Khwārizmī version of the Table has been calculated with $\varepsilon = 24^\circ$, the standard Indian value, and the method seems to be of Indian origin. But the differences between the two tables suggest that there is a different parameter embedded in the Sanjufīnī version. In order to examine this possibility, a Hewlett-Packard 67 calculator was programmed to compute values of P , the machine pinching off the iteration process when the difference between successive approxima-

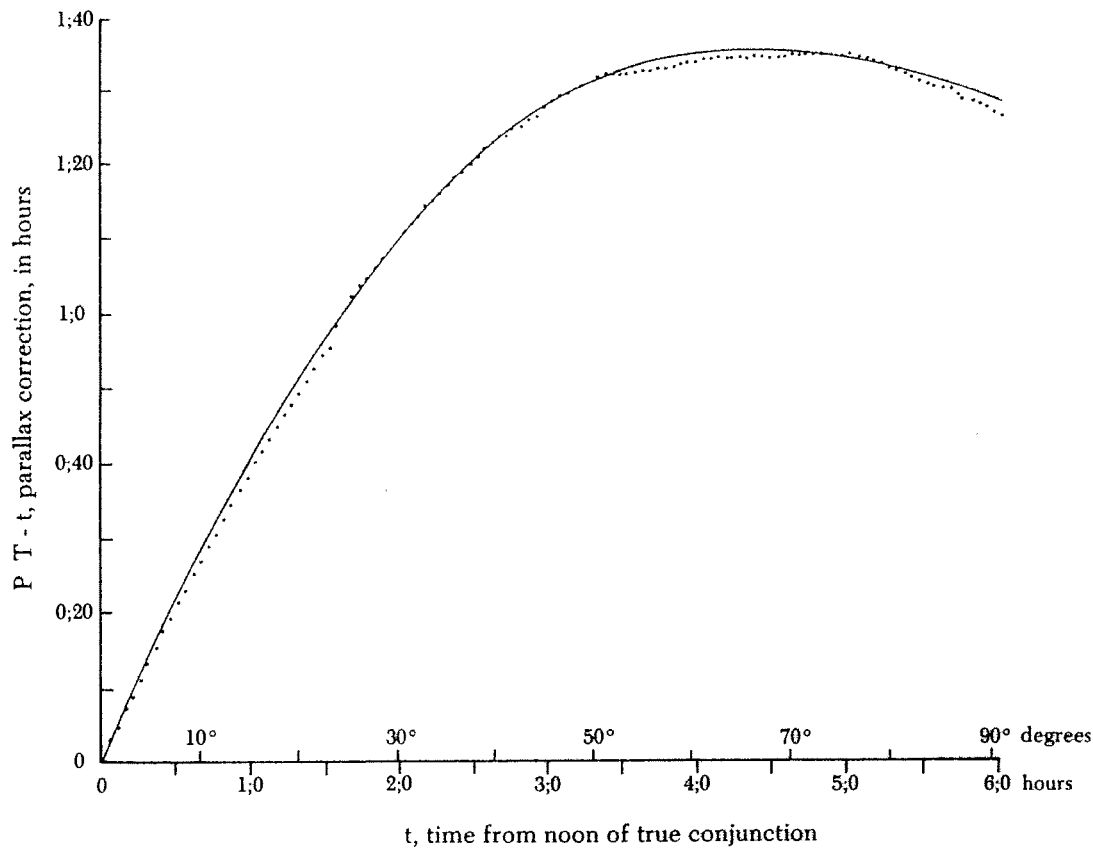
4. Bodleian MS Seld. Arch. A30.

5. Ḥabash, *Al-Zīj al-Ma'rūf bi'-Dimashqī* (Istanbul, MS Yeni Cami 784), f. 212b.

6. Ḥabash, *Al-Zīj al-Ma'rūf bi'-Dimashqī*, ff. 210a-213a; see also Kennedy, "Parallax Theory in Islamic Astronomy," *Isis* 47 (1956) 51, and Kenndy, "Spherical Astronomy in

Zīj," *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften* 2 (1985) 125.

7. Cf. Kennedy and Transue, "A Medieval Iterative Algorithm," *American Mathematical Monthly* 63 (1956) 80-83.



tions became less than 0.0002. Calculations were made for the set of t 's shown on Table 1, for three values of ϵ : (1) 24° , (2) $23;35^\circ$, a common medieval value which is implicit in at least one table of the Sanjufinī *zīj* (f. 40r), and (3) $23;51^\circ$, a Ptolemaic value used along with 24° by Khwārizmī.⁸ The results of calculation using the third value are shown in Column 5 or Table 1, adjoining Sanjufinī's values in the preceding column. The latter are so badly done that, of the total of thirty excerpted, twelve are near none of the three recomputations. Of the remainder, five are close to (1), three to (2) and nine to (3), two of them to within a second of time. It seems probable that Ptolemy's ϵ , which reappears in the section below, underlies the Sanjufinī table.

By way of conclusion, a conjecture as to the rationale of the scheme is presented. The associated parts of the Khwārizmī and Ḥabash *zīj*es are firmly embedded in the Indian astronomical tradition. Professor O. Neugebauer has shown⁹ that in the Indian theory the parallax in longitude is given by the expression

$$(3) \quad P = P_0 \sin a \sin \Delta\lambda,$$

where

$$P_0 = \begin{cases} 0;4 \text{ days, or} \\ 0;4 \times 360 = 24^\circ \text{ of daily rotation, or} \\ 24/15 = 0;4 \times 24 = 1;36 \text{ hours} \end{cases}$$

8. Neugebauer, *Tables*, p. 96.

9. Neugebauer, *Tables*, p. 123.

is the horizontal parallax in the altitude circle. The fact that 24° is the Indian ε is presumably fortuitous. The angle between the ecliptic and the horizon is a and $\Delta\lambda$ is the distance between the conjunction and the highest point of the ecliptic.

To this do two things:

(1) Put $a = 90^\circ$, whence $\sin a = 1$. This assumption flies in the face of reality, making the ecliptic always vertical, whereas only for an observer on the equator is even the mean value of a a right angle.

(2) As argument, replace $\Delta\lambda$ by T , the hour angle of the as yet undetermined apparent conjunction. This step is much less drastic, and in fact Ḥabash speaks repeatedly of the distance from the conjunction to upper midheaven.¹⁰

Expression (3) then becomes

$$(4) \quad \begin{aligned} P &= 24 \sin T, \text{ in time degrees,} \\ &= 0;4 \times 24 \sin T, \text{ in hours,} \end{aligned}$$

which is expression (2). Or perhaps the originator paid no attention to the underlying geometric situation and simply put down (4) as an example of the sinusoidal interpolation common in Indian and early Islamic astronomy. T is defined implicitly by

$$t = T - 24 \sin T,$$

equivalent to (1) above. The technique of solution by iteration was also common in early astronomy.

THE LUNAR VISIBILITY TABLE

On f. 38r of the manuscript of the *zīj* we find a "Table for the Visibility of the Lunar Crescent, for Latitude $38^\circ 10'$," (*Jadwal rā'yat al-ahalla li 'arḍ* 38,10). This also has a marginal gloss in Mongolian which, according to Professor Franke, says "moon visibility thirty-eight latitude," all Mongolian words except that for "latitude" the Arabic *'arḍ*, has been transliterated. Notice that the ten minutes of arc have been dropped. The independent variables are the integer degrees of lunar latitude β between 5° and -5° , and the solar position λ with intervals of 10 degrees, beginning with 0° Aries, and expressed in zodiacal signs plus degrees. The table provides for each combination of these λ and β an entry $f(\lambda, \beta)$, tabulated in degrees and minutes. The entries for $\beta = 5^\circ, 0^\circ$, and -5° are presented below in Table 2. We have omitted the other entries in the table, since they were obtained by linear interpolation between $f(\lambda, 0^\circ)$ and $f(\lambda, 5^\circ)$ (for positive β), or $f(\lambda, 0^\circ)$ and $f(\lambda, -5^\circ)$ (for negative β), but we have used these additional entries in the manuscript in order to check that the entries presented here are free from scribal errors.

The table can be used for the prediction of the moment of first visibility of the lunar crescent, indicating the beginning of the new month in the Islamic calendar. For the first few evenings after luni-solar conjunction, compute the solar longitude λ_1 , the lunar longitude λ_2 , the lunar elongation $\Delta\lambda = \lambda_2 - \lambda_1$, and the lunar latitude β . Find $f(\lambda_1, \beta)$ by means of the table. The crescent will be visible if $\Delta\lambda \geq f(\lambda_1, \beta)$, otherwise it will be invisible. Since the lunar elongation increases by approximately 12° per day, the table implies that the moment of first visibility can occur already 17 hours (for $\lambda = 70^\circ, \beta = 5^\circ$) after conjunction, while the crescent may also remain invisible as long as $2\frac{1}{2}$ days (for $\lambda = 160^\circ, \beta = -5^\circ$).

10. Ḥabash, *Al-Zīj al-Ma'rūf bi'-'Dimashqī*, f. 210a.

TABLE 2

λ	$\beta = 5^\circ$	$\beta = 0^\circ$	$\beta = -5^\circ$	λ	$\beta = 5^\circ$	$\beta = 0^\circ$	$\beta = -5^\circ$
0	8;37	9;39	10;43	180	12;50	19;39	29;18
10	8;30	9;37	10;59	190	12;25	19; 0	28;11
20	8;19	9;38	11;16	200	11;55	18;08	26;45
30	8;07	9;41	11;35	210	11;12	17;03	25;05
40	7;54	9;45	12;09	220	10;37	15;57	23;13
50	7;41	9;44	12;52	230	9;39	14;45	21;20
60	7;28	10;15	13;59	240	9;28	13;33	19;22
70	7;25	10;43	15;18	250	9;18	12;34	17;37
80	7;31	11;25	16;43	260	8;15	11;41	16;03
90	7;56	12;22	18;46	270	8;00	10;57	14;34
100	8;09	13;28	20;29	280	8;01	10;16	13;42
110	9;20	14;48	22;53	290	7;51	10; 0	12;38
120	10; 9	16; 8	25; 4	300	7;56	9;46	12; 0
130	10;53	17;28	27;20	310	8;18	9;31	11;38
140	11;41	18;26	28;37	320	8;15	9;37	11;18
150	12;36	19;21	29;52	330	8;22	9;36	11; 1
160	12;53	19;53	30;31	340	8;30	9;35	10;54
170	12;57	19;53	30;23	350	8;39	9;33	10;50

The manuscript does not explain how the table was computed. We will now analyze the mathematical structure of the table, and we begin with some general remarks.

First visibility of the lunar crescent will take place between one-half and one hour after sunset, when the moon is about to set on the western horizon. The prediction of first visibility is actually a very complicated problem because many factors have to be taken into account, such as the solar depression at the moment of moonset, the lunar elongation, the distance between the moon and the brightest point on the horizon, the apparent size of the moon (which depends on its distance to the earth), weather conditions, and so on. Most of the known medieval Islamic lunar visibility tables are based on one simplified visibility criterion. In what follows we will pay special attention to a special type of lunar visibility criteria, for which we introduce the term "solar criteria." A solar criterion is a criterion which implies that visibility of the lunar crescent can be predicted on the basis of (at most) the terrestrial latitude ϕ of the locality, the solar depression d , and the solar azimuth z at the moment of moonset. The following two criteria, both widely used in Muslim astronomy¹¹ satisfy this definition:

1. The lunar crescent will be visible if and only if the sun is at least a fixed amount c° below the horizon at the time of moonset.
2. The lunar crescent will be visible if and only if the difference in setting time between sun and moon is at least a constant number c of time-degrees (360 time-degrees correspond to a complete daily rotation). Note that the time interval elapsed since sunset is determined by ϕ , d , z .

11. For example, Pingree, *The Astronomical Works of Gregory Chionides*, Vol. 1, *The Zīj al-'Alāḥī*, 2 parts, p. 952.

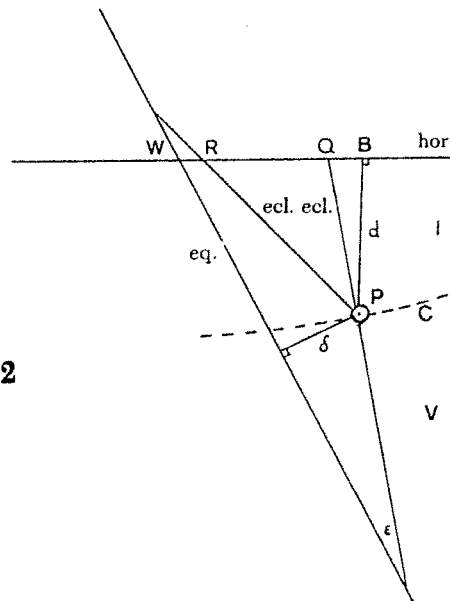


Figure 2

One final remark should be made about solar criteria. Because for a fixed ϕ the visibility is determined by d and z , one can as a general rule distinguish on the celestial sphere below the western horizon an area I , containing the solar positions (d, z) such that the moon will be invisible, and an area V , containing the (d, z) for which the moon will be visible; I and V are separated by a curve C (see Figure 2).

We now return to Table 2, and assume as a working hypothesis that it was accurately computed on the basis of a solar criterion. We consider a point P on the curve C with declination δ less than the obliquity ϵ of the ecliptic (see Figure 2). During a complete revolution of the celestial sphere, P will meet the ecliptic two times, at points with ecliptical longitude λ and $180^\circ - \lambda$. We call the intersections of the ecliptic with the horizon at these two moments Q and R . Then $PQ = f(\lambda, 0)$ and $PR = f(180^\circ - \lambda, 0)$ while angle QPR is determined by λ and ϵ . Assume a value for ϵ . Then, if $\lambda = n \times 10^\circ$ for $0 \leq n \leq 36$, $n \neq 9, 27$, one can read off PQ and PR from the table, and compute angle QPR , so that the spherical triangle PQR can be solved, and the angles between the ecliptic and the horizon at P and Q can be found. In the spherical triangle defined by one of the two positions of the ecliptic, the celestial equator, and the horizon, one knows three elements, so that this triangle can be solved, and the angle between the equator and the horizon can be found. The complement of this angle is the geographical latitude ϕ . One can now compute other interesting parameters, such as the time elapsed since sunset, and the solar depression and azimuth. All these computations are routine applications of spherical trigonometry. However, if angle QPR is small, small errors in $f(\lambda, 0^\circ)$ and $f(180^\circ - \lambda, 0^\circ)$ can produce enormous deviations in the computed values of ϕ . Thus the computed value of ϕ is unreliable if λ is near 90° or 270° .

When this method was carried out for the pairs $f(\lambda, 0^\circ)$ and $f(180^\circ - \lambda, 0^\circ)$ in Table 2, it was found that the resulting geographical latitudes were nearly constant (except for λ close to 90° and 270°). For $\epsilon = 23;50^\circ$ the resulting geographical latitudes ϕ were close to the value $38;10^\circ$, mentioned in

the manuscript, but for $\varepsilon = 23;30^\circ$ they were much nearer to $38;30^\circ$. We therefore inferred that the table was computed for $\varphi = 38;10^\circ$ and for the Ptolemaic value $\varepsilon = 23;51^\circ$. However, neither the solar depression nor the difference in setting time were nearly constant. Thus it seemed that the table was computed on the basis of a solar criterion, different from the two criteria mentioned above.

Once the geographical latitude is known, one can compute the solar depression corresponding to any value of λ , also for the values for which the above method is useless. It turned out that for $50^\circ \leq \lambda \leq 130^\circ$ and $180^\circ \leq \lambda \leq 360^\circ$, the solar depression is very nearly a linear function $g(w)$ of the distance $w = |90^\circ - z|$ between the west point W and the perpendicular projection of the sun (the brightest point B) on the horizon, with extreme values $g(0^\circ) = 9;20^\circ$ and $g(39;20^\circ) = 8;18^\circ$. If we put $g(40^\circ) = 8;18^\circ$, and extend the function linearly, we obtain $g(180^\circ) = 4;40^\circ$, exactly half of $9;20^\circ$ (if $w = 180^\circ$ B is the East point). For $0^\circ \leq \lambda \leq 40^\circ$, and $140^\circ \leq \lambda \leq 180^\circ$, the solar depression was found to be very close to $g(\delta)$, with δ the solar declination.

We note that functions similar to g occur in the lunar visibility theory of Thābit ibn Qurra.¹² However, in the theory of Thābit, the argument of g is the distance between the moon and the brightest point on the horizon. Thābit's approach makes much more sense. Nevertheless, the above-mentioned calculations support the hypothesis that the $f(\lambda, 0^\circ)$ were computed for $\varepsilon = 23;51^\circ$, $\varphi = 38;10^\circ$, on the basis of the following visibility criterion:

Let w be the distance between the west point W and the brightest point B on the horizon, and let g be the linear function of w with $g(0^\circ) = 9;20^\circ$ and $g(180^\circ) = g(0^\circ)/2$.

The lunar crescent will be visible if and only if the solar depression at moonset is at least equal to $g(w)$.

The actual computation of the $f(\lambda, 0^\circ)$ must have been carried out somehow as follows. Beginning with a value λ , one first computes the declination δ . Then one must estimate w . Small errors in w do not have much influence, because g varies only slowly with w . Sanjufīnī seems to have used the bad estimate $w \approx \delta$ for $0^\circ \leq \lambda \leq 40^\circ$ and $130^\circ \leq \lambda \leq 180^\circ$, but for the other values of λ his estimate must have been more accurate. In the recomputed table, $w = \delta$ has been used for $0^\circ \leq \lambda \leq 40^\circ$ and $130^\circ \leq \lambda \leq 180^\circ$, and for the other λ , w has been computed using $9;20^\circ$ as an approximation to the solar depression. From w one derives the real solar depression $g(w)$. By a standard method in Islamic timekeeping, one can compute the hour angle of a point from its declination and altitude above the horizon.¹³ One can apply this method for computing the hour angle h of the point on the celestial sphere diametrically opposite to the sun, and hence the hour angle of the sun itself. Since the right ascension of the sun is known, one can now determine the right ascension of the west point on the horizon, and hence the oblique ascension $\lambda + f(\lambda, 0^\circ)$ of the setting point of the ecliptic. Table 3 has been recomputed by means of the sketched procedure. The agreement is reasonable; discrepancies may be caused by different estimates for w from that which Sanjufīnī may have used.

Other explanations of the mathematical structure of $f(\lambda, 0^\circ)$ are possible, but we prefer the one given because it involves only three parameters (φ , ε , and the maximal solar depression $9;20^\circ$), and because it is related to a criterion used by Thābit ibn Qurra.

12. In R. Morelon, "Fragment arabe du premier livre du Phases de Ptolémée," *Journal for the History of Arabic Science* 5 (1981) 3-22.

13. See for example, Al-Marrākushī, *Jāmi' al-mabādī wa'l-ghāyāt fī 'ilm al-miqāt*, in the facsimile-edition "Comprehensive Collection of Principles and Objectives

in the Science of Timekeeping," (Frankfurt, 1984) vol. 1, or the French translation: J.-J. Sédillot, L.-A. Sédillot, *Traité des instruments astronomiques des Arabes*, Paris 1834 (reprint Frankfurt, 1984, on p. 263 read "Prenez le sinus de la différence"), text, pp. 122-124, transl., pp. 261-265; or Kennedy, "Spherical Astronomy," pp. 43-46.

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 $30^\circ - \lambda$. We call the
Then $PQ = f(\lambda, 0)$
for ε . Then, if $\lambda =$
d compute angle
n the ecliptic and
the two positions
rents, so that this
an be found. The
other interesting
azimuth. All these
ngle QPR is small,
computed values

ole 2, it was found
e to 90° and 270°).
 10° , mentioned in

We have been unable to find a satisfactory mathematical explanation of the entries for $\beta = \pm 5^\circ$. In an accurately computed table, one expects that $f(\lambda, 0^\circ)$ is approximately the mean of $f(\lambda, 5^\circ)$ and $f(\lambda, -5^\circ)$, but this is not so in Table 2. It is conceivable that Sanjufinī (or whoever else compiled the table) computed the $f(\lambda, 0^\circ)$ himself, but plagiarized the $f(\lambda, 5^\circ)$ and $f(\lambda, -5^\circ)$ from another table. Further investigations along these lines were unsuccessful.

TABLE 3: Recomputed values for $f(\lambda, 0^\circ)$

Parentheses contain the errors in minutes, $f(\lambda, 0^\circ)$ minus the recomputed value.

0	9:39	(0)	120	15:53	(+15)	240	13:37	(-4)
10	9:36	(+1)	130	17:01	(+27)	250	12:37	(-3)
20	9:36	(+2)	140	18:43	(-27)	260	11:45	(-4)
30	9:41	(0)	150	19:23	(-2)	270	11:02	(-5)
40	9:51	(-6)	160	19:45	(+8)	280	10:28	(-12)
50	9:42	(+2)	170	19:50	(+3)	290	10:02	(-2)
60	10:05	(+10)	180	19:39	(0)	300	9:45	(+1)
70	10:39	(+4)	190	18:55	(+5)	310	9:35	(-4)
80	11:24	(+1)	200	18:06	(+2)	320	9:31	(+6)
90	12:21	(+1)	210	16:59	(+4)	330	9:31	(+5)
100	13:28	(0)	220	15:50	(+7)	340	9:35	(0)
110	14:40	(+8)	230	14:42	(+3)	350	9:35	(-2)