# Mathematics in Medieval Islamic Spain

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## 1. Introduction

From the seventh to the eleventh century, a large part of present-day Spain and Portugal belonged to the Islamic world. I will use the term "Islamic Spain" to indicate the part of the Iberian peninsula that was under Muslim rule. The term Islamic Spain is not strictly correct, because Spain did not exist in the early Middle Ages, but the important medieval scientific centers (Córdoba, Zaragoza, Toledo) are all in present-day Spain. Until recently the general view has been that Islamic Spain was important in the history of mathematics only because of its role in the transmission of mathematics from Arabic to Latin. In the last 15 years this view has changed as a result of the investigation of unpublished manuscript sources. We now know that there were creative mathematicians in Islamic Spain in the eleventh century. In this paper I will try to give you an impression of their work. No previous acquaintance with the history of mathematics will be assumed, and I will begin with some general remarks on the historical context.

## 2. The historical context

Before 300 b.c. the ancient Greeks developed geometry as a deductive system. Greek mathematics flourished until the third century a.d. and then declined. The Romans were not interested in theoretical mathematics, and the tradition remained dormant until it was revived by the Muslims. Around a.d. 800, the caliphs made Baghdad into the scientific capital of the world. They had Arabic translations made of many Greek texts on mathematics and astronomy (including Euclid's *Elements*, the works of Archimedes, Apollonius, etc.), and also of Sanskrit works from India. This is the beginning of Arabic science, i.e. science written in Arabic. When we use a term such as "Arabic mathematics", we should bear in mind that large contributions were made by non-Arabs, notably the Iranians.

Islamic Spain is far away from Baghdad, and it took a while for science to reach the area. In the tenth century there was much interest in learning in Islamic Spain, and there was a library with more than 400,000 books in the capital, Córdoba. At that time mathematics and astronomy were studied on a rudimentary level, necessary for practical applications, such as astrology and timekeeping. Around the year 1000 the interest in theoretical mathematics and astronomy deepened. The eleventh century is the golden age of Islamic Spanish science. After the



eleventh century, the size of Islamic Spain was reduced and there was a decline in the scientific activity.

The Christians reconquered a large part of present-day Spain in the eleventh and twelfth centuries: Toledo was captured in 1085, Zaragoza in 1118. The end of Islamic domination of these areas did not mean the complete end of the scientific tradition, and in the twelfth century, many Arabic manuscripts from Spain were translated into Latin. Not many people realize how important this event was for the history of mathematics. In the early Middle Ages mathematics was practically nonexistent in western Europe; a few facts from elementary mathematics were mentioned in Latin texts and encyclopedias, but hardly anybody knew how to prove a theorem. By means of the twelfth century Latin translations, the Christians came in contact with mathematics as a deductive science. The creative period of Islamic Spanish mathematics occurred just before the takeover by the Christians of most of Spain and the translation movement.

## 3. Sources and the state of research

Some texts by eleventh-century mathematicians and astronomers have come down to us in the original Arabic. These are not the manuscripts written by the authors themselves, but are always later copies, made in the Islamic world or in medieval Christian Spain, in places where Arabic astronomy was studied (such as the court of Alfonso the Wise in the thirteenth century). A few texts survive only in a medieval Latin or Hebrew translation. One can also find traces of eleventh-century mathematics in texts by later authors (from Islamic or Christian Spain or North Africa). Even so, the evidence we have is very incomplete, and the history of mathematics in the eleventh century has to be patched together from various bits and pieces of information. Not all relevant sources have been studied, and many manuscripts still await edition and translation. There are various researchers all over the world who are working on these materials. The most important center for the study of Islamic Spanish science is the Department of Arabic Philology in the University of Barcelona under the direction of Julio Samsó. The center publishes a series of editions of sources, and Samsó has recently published the first reliable survey of science in Islamic Spain, including the results of research up until 1992 [5].

## 4. General remarks

In the history of mathematics in Islamic Spain, a distinction can be made between arithmetic and algebra on the one hand, and geometry and trigonometry on the other hand. In arithmetic and algebra, Islamic Spain seems to have lagged behind the East. The mathematicians of the eleventh century worked with Eastern texts that had become obsolete in the East, such as the arithmetic and the algebra of al-Khwārizmī (ca. 830). Recent advances in the East were unknown in Islamic Spain. For geometry and trigonometry the situation was different. The Islamic Spanish mathematicians were working on the same level as their Eastern Islamic colleagues, and they were aware of many recent developments in the East.

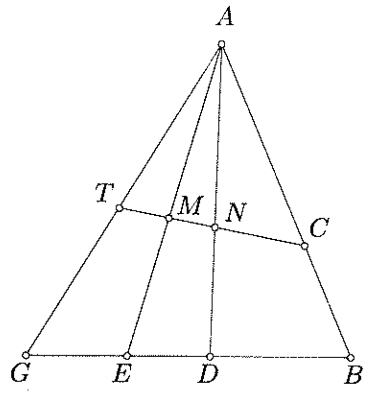


Figure 1

I will not be giving you a list of all mathematicians with their known works and contributions to geometry and trigonometry, because I believe such lists are boring for the nonspecialist.[1] Instead, I will discuss two concrete examples in some detail.

## 5. Al-Mu'taman and his "Book of Perfection"

My first example is al-Mu'taman ibn Hūd, the king of the kingdom of Zaragoza (in northeastern Spain), who died in 1085. The kingdom of Zaragoza was one of the so-called petty kingdoms into which Islamic Spain had disintegrated in the eleventh century. Al-Mu'taman was also a mathematician, who wrote a very long mathematical work, entitled *The Book of Perfection* (in Arabic: Istikmāl) [1]. For a long time this work was believed to be lost, but numerous fragments have recently turned up in four anonymous Arabic manuscripts [2]. The work is in a very poor state of preservation because the most important manuscript (in Copenhagen) was damaged and many leaves are missing. As a result, there are 11 gaps in the text we have. [2] In the *Book of Perfection* al-Mu'taman presents the essentials of mathematics, philosophically arranged, with all the proofs. The work resembles, to some extent, the *Éléments de Mathématique* of N. Bourbaki. Al-Mu'taman adapted most of the material in his *Book of Perfection* from existing works by Euclid, Archimedes, Apollonius, and other mathematicians from antiquity or the eastern Islamic world. However, some theorems are "new", in the sense that they are not found anywhere else in the ancient and medieval literature we know. The following two examples still play a role in modern geometry:

[1]Thus, I will not be talking about al-Zarqāllu's new variant of the *astrolabe*, which has received a great deal of attention in the literature. Like the standard astrolabe, this variant is based on stereographic projection, but the pole of projection is shifted from the north pole to the vernal point.

[2]See note added in proof.



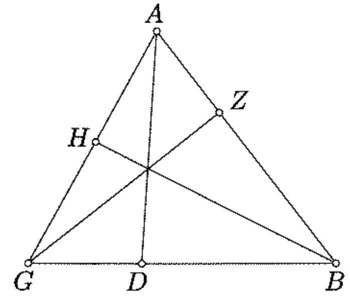


Figure 2

(1) (Figure 1) In "proposition 16 of section 3 of species 1 of species 3" al-Mu'taman considers a triangle $ABG$ with points $D$ and $E$ on its base, and a transversal $TMNC$ that intersects $AG$ at $T$, $AE$ at $M$, $AD$ at $N$, and $AB$ at $C$.[3]

He proves $(TC : CN) \cdot (NM : MT) = (GB : BD) \cdot (DE : EG)$. In modern terms this is the oldest extant statement of the perspective invariance of cross-ratios in a rather general situation. (There is an ancient Greek proof, in the *Mathematical Collection* of Pappus of Alexandria, for the special case where $B$ and $C$ coincide.)

(2) The so-called theorem of Ceva (Figure 2) is stated and proved in the margin of the *Book of Perfection*, as "proposition 18 of section 3 of species 1 of species 3".[4] Consider a triangle $ABG$ with points $D$ on side $BG$, $H$ on side $AG$, and $Z$ on side $AB$. Then lines $AD$, $BH$, and $GZ$ intersect at one point if and only if $(BZ : ZA) \cdot (AH : HG) = (BD : DG)$. (This is proved by two applications of the theorem of Menelaus.) The theorem has been named after Giovanni Ceva, who stated it in 1678, but it should now perhaps be renamed the "theorem of al-Mu'taman."

In the extant parts of the *Book of Perfection*, al-Mu'taman never distinguishes between his own contributions and theorems that he adapted from other sources. Cross-ratios occur very rarely in Arabic mathematics, but we know that they were used extensively in Euclid's lost work on *Porisms*, which was transmitted to Arabic in some form. I therefore believe that al-Mu'taman took his cross-ratio theorem from an Arabic translation of a lost Greek work (if this is true, we get new information on Greek mathematics from an eleventh-century Islamic Spanish source). I do not know whether the so-called theorem of Ceva was a contribution by al-Mu'taman or not.

[3]In ancient Greek and medieval mathematics, there was no real number concept. The line segment is a basic concept; a line segment does not have a length, but it is a (positive) length. Line segments can be compared, and there existed a theory of ratios between line segments. Ratios could be ordered, and a ratio between two line segments could be compared to a ratio between two integers. All straight lines were bounded; a straight line could be produced indefinitely, but an infinite straight line did not exist.

[4]Because the theorem has a proposition number, it belonged to the original text, and was first left out by an oversight.

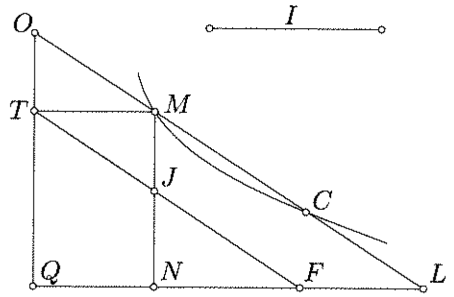


Figure 3

There are only a few rather complicated propositions that we can attribute to al-Mu'taman with some certainty. I will now discuss one example, namely his simplification of a construction by Ibn al-Haytham (a geometer who worked in Egypt in the early eleventh century), which is based on an ancient Greek construction. The following explanation will take some time but it will be worthwhile because it gives an idea of al-Mu'taman's ability as a mathematician. I will try to give the general ideas and avoid the details of the proofs, which involve proportions and similar triangles. My notations are those of Ibn al-Haytham [4, pp. 315–318].

The ancient Greek construction is as follows (Figure 3). We are given a rectangle $TQNM$ and a straight segment $I$. We want to construct a straight segment $TJF$ that intersects $MN$ at $J$ and $QN$ extended at $F$ in such a way that $FJ = I$. Solution: Draw a hyperbola through $M$ with asymptotes $TQ$ and $QN$. (In antiquity and the Middle Ages, a hyperbola was always a single-branch hyperbola.) Draw a circle with center $M$ and radius $I$. Let them intersect at $C$. Draw $TF \parallel MC$. This is the desired line.

The proof is easy: Extend $MC$ to meet $QT$ at $O$ and $QN$ at $L$. By parallelograms, $TJ = OM$ and $TF = ML$, so $FJ = ML - OM$. By a property of the hyperbola: $OM{=}CL$. Hence $FJ = MC = I$. I note that the problem cannot be solved by ruler and compass.

This construction was used in rather a confusing way by the Egyptian geometer Ibn al-Haytham around 1040 in his work on *Optics*, in a series of a preliminaries for the study of reflection in circular mirrors.

Ibn al-Haytham considers a circle with a given diameter $BG$ and on this circle a given point $A$ (Figure 4). A straight segment $EK$ is also given. He wants to construct a straight line through $A$ that intersects the circle at $H$ and the diameter at $D$ in such a way that $DH = EK$. Point $D$ is assumed to be outside the circle.

The basic idea is as follows. First suppose that $DH$ is arbitrary. Ibn al-Haytham draws line $GZ$ parallel to $BA$, meeting $DH$ at $Z$. He proves $AZ : DG = BG : DH$ (I omit the details).

Therefore $DH{=}EK$ if and only if $AZ : DG = BG : EK$. Unfortunately, we do not know the length of $DG$, but we know the angles at $G$: $\angle DGZ = \angle GBA$, and $\angle ZGA = \angle GAB = 90°$.



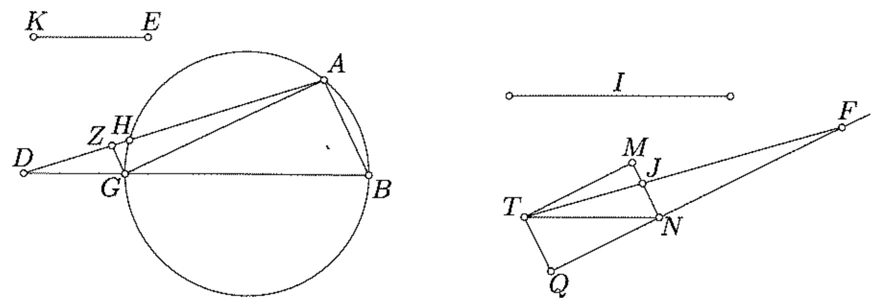


Figure 4

Ibn al-Haytham now uses one of his favorite techniques, which consists of constructing an auxiliary figure similar to the original one. Choose a segment $NT$ of arbitrary known length, and make rectangle $TMNQ$ such that $\angle TNM = \angle DGZ$.

Choose $I$ such that $I : NT = BG : EK$, and find (by the Greek construction) $TJF$ such that $FJ = I$. Then construct $D$ such that $\angle GAD = \angle NFT$. Then $GAZD$ is similar to $NFJT$. Thus $AZ : DG = FJ : NT = I : NT = BG : EK$. Because $AZ : DG = BG : DH$, we have $DH = EK$ as required. Note that the known segment $NT$ in the auxiliary figure corresponds to the unknown segment $GD$ in the original figure.

Al-Mu'taman simplified and generalized the construction of Ibn al-Haytham, on the basis of the following three ideas:

(1) Because in Figure 4 $NT$ is an arbitrary segment, we can identify it with a known segment in the original figure, such as $BG$. If this is done, the auxiliary figure and the original figure coincide in a nice way, and the construction is as follows (Figure 5). Complete rectangle $GABQ$, draw the hyperbola with center $A$ and asymptotes $GQ$ and $QB$. Choose $I$ such that $I : BG = BG : EK$ and find $C$ as a point of intersection of the hyperbola and the circle with center $A$ and radius $I$. Extend $CA$ to intersect circle $ABG$ again at $H$ and line $BG$ at $D$. Only a small number of proportions and similar triangles[5] are needed to prove $AC : BG = BG : DH$. Because $AC{=}I$ and $I : BG = BG : EK$ we have $DH{=}EK$ as required.

(2) Thus far we have been discussing the case where point $D$ is outside the circle. Al-Mu'taman saw that one can solve the similar problem for $D$ inside the circle, by finding the points at which the other branch of the hyperbola intersects the circle with center $A$ and radius $I$, and by exactly the same reasoning (broken line in Figure 5). Al-Mu'taman gives a general proof for the two different cases. Ibn al-Haytham has a new and essentially different solution for $D$ inside the circle.

(3) Al-Mu'taman observed that $GABQ$ can be an arbitrary parallelogram instead of a rectangle. Thus, it is not necessary to assume that $BG$ is a diameter;

[5] Details: as $A, B, G, H$ are on the same circle, $DA{\cdot}DH = DB{\cdot}DG$, so $DH : DG = DB : DA =$ (by similar triangles) $BG : AZ$. Hence $BG : DH = AZ : DG =$ (by similar triangles) $JF : BG$. $JF = AC$ is proved just as $JF = MC$ in Figure 3. The text will appear in [6].

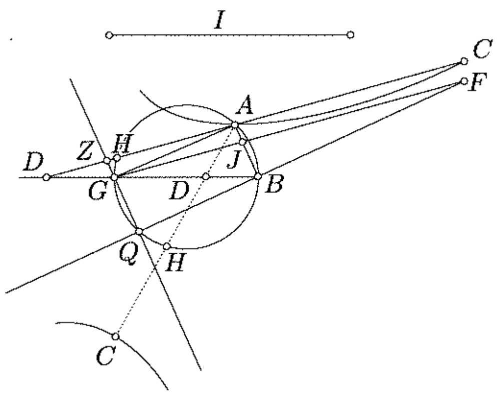


Figure 5

$A$, $B$, and $G$ can be three arbitrary different points on a circle. In general, point $Q$ is not on the circle.

Summarizing: al-Mu'taman solved a more general problem than Ibn al-Haytham in a much shorter way. Ibn al-Haytham's solution of the two cases is ten times as long as al-Mu'taman's general solution. This shows that al-Mu'taman was able to improve on Ibn al-Haytham, who was one of the most important geometers of the Islamic tradition.

Al-Mu'taman must be the author of the general solution (Figure 5) for the following reasons. The solution occurs in a series of geometrical constructions that are clearly adapted from the *Optics*. Ibn al-Haytham wrote this work around the year 1030 in Egypt at an advanced age, less than 50 years before al-Mu'taman wrote his *Book of Perfection*. In the Eastern part of the Islamic world, nobody seems to have seriously studied Ibn al-Haytham's *Optics* in the eleventh century. We have a description of Islamic Spanish science made around 1065 by the biographer Ṣāᶜid al-Andalusī, in which we can read that there are only three people interested in "natural philosophy", namely Al-Mu'taman and two others (whose names are mentioned). The two other people are not known to have written any mathematical works, and their mathematical reputations among their contemporaries are nowhere near that of al-Mu'taman. This leaves al-Mu'taman as the only plausible author.

His authorship is confirmed by the fact that there are many more solutions of a similar nature in the *Book of Perfection*. An example is the famous problem of Apollonius (to construct by ruler and compass a circle tangent to three given circles). Al-Mu'taman presents a solution that was also inspired by the same Ibn al-Haytham, but that is also much shorter and much clearer than Ibn al-Haytham's confused original. Another example is al-Mu'taman's simplification of a quadrature of the parabola by Ibrāhīm ibn Sinān, a geometer from tenth century Baghdad. Al-Mu'taman was not the only eleventh-century Islamic Spanish geometer working on complicated subjects. Between 1087 and 1096, the geometer Ibn Sayyid of Valencia



developed a theory of higher-order curves, and he used these to divide an angle into an arbitrary number of parts and to construct an arbitrary number of geometric means between two given lines.[6] His work is lost; all we have is rather a vague description of it by the philosopher Ibn Bājja. [1].

These examples show that geometry was studied on a much higher level in Islamic Spain than had been thought 15 years ago.

## 6. Ibn Muᶜādh and the astrological "aspects"

One cannot get an adequate view of mathematics in Islamic Spain without looking at the applications in astronomy and astrology. To give you some of the flavor, I have chosen a nontrivial problem from astrology, which was solved by the eleventh-century mathematician Ibn Muᶜādh. Astrology is nowadays considered to be a pseudo-science, but it was very important in the history of mathematics in the Middle Ages, because it was one of the main fields where mathematics was applied in a nontrivial way. I first recall a few astronomical preliminaries.

In ancient and medieval planetary theory, the positions of the planets were represented on the celestial sphere, in coordinates called "celestial longitude" and "celestial latitude". The basic circle of reference is the ecliptic; that is, the apparent orbit of the sun around the earth against the background of the fixed stars. The zero point on the ecliptic is the vernal point, that is the position of the sun at the beginning of spring, and the celestial longitude of a planet is the arc between the vernal point and the perpendicular projection of the planet on the ecliptic. This arc is measured in the direction of the motion of the sun. I will ignore the celestial latitude, that is the arc measuring the deviation of the planet from the ecliptic. In the Islamic Middle Ages, there was a satisfactory theory for the prediction of the celestial longitudes of the sun, moon, and planets on the ecliptic at any place and at any moment of time, and there were many handbooks explaining the necessary computations and containing the necessary tables. The fact that astronomy was geocentric, not heliocentric, did not affect the exactness of these predictions.

I will now explain the astrological concept of "aspect". The basic assumption of medieval astrology is that the sun, moon, and planets have an influence on events on earth, and that the positions of these celestial bodies can be used to predict the future with some probability. Such predictions were very complicated, and one of the many things that the astrologer had to take into account was certain special configurations between two planets, the so-called aspects. The idea is as follows. The astrologers believed that from its position on the ecliptic each planet emitted seven "visual rays" to other points of the ecliptic. (According to the ancient theory of vision, a human being sees by emitting visual rays from the eye, and not because light enters his eye.) If a second planet is sufficiently close to the endpoint of such a visual ray, it is "seen" by the first planet ("looked at", in Latin: adspectus), and then the planets are said to have an aspect.

There were two theories for the computation of these aspects. According to the simplest theory, the seven rays are emitted to points in the ecliptic at angular

[6] $n$ geometric means between two given line segments $a$ and $b$ are $n$ straight segments $x_1 \dots x_n$ such that $a : x_1 = x_1 : x_2 = \cdots = x_n : b$.

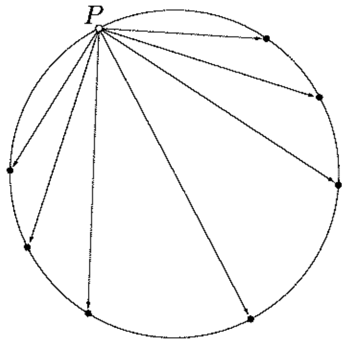


Figure 6

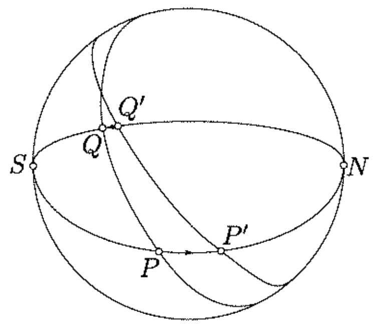


Figure 7

distances of 60°, 90°, 120°, and 180° in both directions (Figure 6). Thus, two planets have an aspect if the difference in their celestial longitude is 60°, etc. This does not lead to interesting mathematical problems.

For reasons unknown to me, several medieval Islamic astrologers believed in a more complicated theory, which I will call the "equatorial" theory of aspects. The idea is as follows. We first project the planet $P$, which is on the ecliptic, on the intersection $P'$ of the celestial equator[7] with the great semicircle through $P$ and the north point $N$ and the south point $S$ of the horizon. From $P'$ we measure the angles of 60°, 90°, 120°, and 180° on the celestial equator in both directions (in Figure 7 we have $P'Q'$=60°). For each of the seven points $Q'$ that we find this way, we apply the inverse projection: we draw the great semicircle $NQ'S$ and we call $Q$ its intersection with the ecliptic. Then according to the "equatorial" theory the planet at $P$ emits its seven visual rays to the seven points $Q$.

If $P'Q' = 60°, 90°$, or 120°, arc $PQ$ has a variable magnitude, which depends not only on the celestial longitude of $P$, but also, oddly enough, on the geographical latitude of the astrologer and on the (local) time of day. (This leads to the rather surprising conclusion that the planet emits its visual rays differently for different observers on earth. This does not seem to have worried the astrologers.)

The "equatorial" theory of aspects was quite popular, in spite of (or maybe because of) its difficulty. In the ninth century, al-Khwārizmī computed tables for the computation of the aspects according to the equatorial theory for Baghdad, using a rather crude approximation. This al-Khwārizmī is the famous mathematician who wrote on algebra and whose name has been corrupted in the word *algorithm*.

These tables were recomputed for Córdoba, around the year 1000, by the leading Islamic Spanish mathematician of that time, Maslama ibn Aḥmad al-Majrīṭī. (Al-Majrīṭī is Arabic for: of Madrid.) Maslama used a different geographical latitude and a more sophisticated (but still approximate) mathematical method.

I will now discuss the exact computation of the aspects according to the "equatorial" theory by Ibn Muᶜādh al-Jayyānī [3]. He was one of the best math-

[7] This is the intersection of the celestial sphere and the plane through the observer perpendicular to the world axis, i.e. the line through the celestial north and south pole.



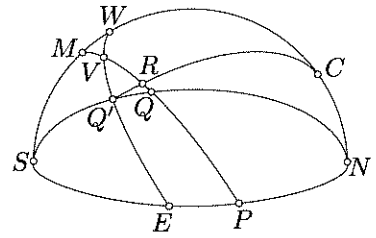


Figure 8

ematicians of the eleventh century, and also a qāḍī, that is an Islamic judge. He lived in Jaén, in southern Spain.

For simplicity of explanation, I will assume that the planet $P$ is on the eastern horizon and that the vernal point is close to the meridian. The general case is not essentially more difficult. I will skip some computations that were standard in Ibn Muᶜādh's time, and I will use modern algebraical notation for the sake of clarity. My source is an unpublished Arabic text by Ibn Muᶜādh on the theory of "aspects", which survives in a unique Arabic manuscript, written in thirteenth century (Christian) Spain.

Figure 8 displays the quadrant of the celestial sphere above the horizon and East of the meridian. All arcs in this figure are arcs of great circles, i.e. intersections of the sphere with planes through its center. Points $E$, $S$, and $N$ are the East, South, and North points of the horizon. Because point $P$ is on the eastern horizon, its "projection" $P'$ coincides with the East point $E$. Point $C$ is the celestial north pole, $NCWMS$ is the meridian, which intersects the ecliptic at $M$ and the celestial equator $WVE$ at $W$. Point $V$ is the vernal point. Then $\angle PVE = \epsilon$, the obliquity of the ecliptic, a known angle for which the value 23°35′ was often used in Arabic astronomy; $\angle VES = 90° - \phi$; here $\phi$ is the geographical latitude, which is also assumed to be known (we assume that the astrologer is in the Northern Hemisphere, between the equator and the Arctic Circle). Note that arc $CN = \phi$. We also assume that we know arc $VP$, that is the celestial longitude of $P$.

To compute the so-called right sextile aspect of $P$, we mark off $EQ'$=60°on the celestial equator, and we draw semicircle $SQ'N$ to intersect the ecliptic at $Q$. We want to compute arc $VQ$.

Ibn Muᶜādh draws arc $CQ'$ to meet the ecliptic at point $R$. Because $C$ is the celestial north pole and $Q'$ is on the equator, $\angle CQ'V$ is a right angle.

Arc $VE$, the so-called "rising time of point $P$" can be found by a standard computation, or can be looked up in a "table for rising times" for the particular geographical latitude.

On the celestial equator, we know arc $EQ'$=60°, $EW$=90°, so from $EV$ we find $Q'V$ and $VW$. Using these arcs we can compute or look up the following arcs: $VR$, $VM$ (from a right ascension table; these are the longitudes that belong to right ascensions $VQ', VW$) and $Q'R$, $MW$ (from a declination table). We have arc $WS = 90° - \phi$, hence we find arc $MS$ = arc $WS -$ arc $WM$. The following quantities are now known: $a$ = arc $MS$, $\Lambda$ = arc $RM$, $\delta$ = arc $Q'R$; we also know arc $CS = 180° - \phi$.

The next step in the argument is crucial.

I set $x$ = arc $QR$. Ibn Muᶜādh uses the spherical theorem of Menelaus, to the effect that

$$\frac{\sin MQ}{\sin QR} = \frac{\sin MS}{\sin SC} \cdot \frac{\sin CQ'}{\sin Q'R}.$$

Menelaus was a Greek astronomer who lived in Rome around a.d. 70, and whose work on geometry of the sphere is lost in Greek but extant in Arabic. Menelaus used a different trigonometric function, the "chord", which the Arabs replaced by the sine, a function of Indian origin. The medieval sine is a constant factor times the modern sine. Because we are dealing with proportions between sines, the constant factor can be ignored.

Thus, we get

$$\frac{\sin(\Lambda + x)}{\sin x} = \frac{\sin a}{\sin \phi} \cdot \frac{\sin 90°}{\sin \delta}.$$

The right-hand side of this equation is a known quantity $c$. Thus, Ibn Muᶜādh has to solve $x$ from

$$\frac{\sin(\Lambda + x)}{\sin x} = c, \tag{1}$$

for known $\Lambda, c$.

In an earlier work [7], Ibn Muᶜādh had shown how equation (1) can be reduced to

$$\tan(\frac{\Lambda}{2} + x) = \frac{c+1}{c-1} \cdot \tan\frac{\Lambda}{2}. \tag{2}$$

There Ibn Muᶜādh had also tabulated the tangent[8] function for every degree up to 89°.[9] By means of this table, $x$ can easily be computed. We have arc $QV$ = arc $RV + x$.

Ibn Muᶜādh reduced (1) to (2) by means of the following geometrical reasoning (Figure 9, [7, Fig. 4]). On a circle of reference we represent the known arc $\Lambda$ as arc $AG$, bisected at $B$, and the unknown arc $x$ as arc $GL$. Let $D$ be the center, extend lines $DL$ and $AG$ to meet at point $K$, and drop perpendiculars $GN$, $BZ$, and $AM$ onto $DK$. Let $BD$ intersect $AG$ at $O$.

Now $\tan(\frac{\Lambda}{2} + x) = BZ : ZD$. By similar triangles, $BZ : ZD = KO : OD = (KO : OG) \cdot (OG : OD)$. We have $AK : KG = AM : GN = \sin(\Lambda + x) : \sin x = c$. Thus, $KO : OG = (KG + KA) : AG = (c+1) : (c-1)$. Finally, $OG : OD = \tan\frac{\Lambda}{2}$.

I have discussed the solution of the problem for a particular configuration. The procedure is general; in some cases one gets a minus sign instead of a plus sign in equation (1). Ibn Muᶜādh also discusses this situation in [7].

It is not certain whether Ibn Muᶜādh ever computed tables for the aspects, to replace the tables of his famous predecessors al-Khwārizmī and al-Majrīṭī. We know, however, that Ibn Muᶜādh computed tables of the so-called astrological houses on the basis of the same computation. The astrologers divided the celestial sphere into 12 "houses" somewhat like an orange into 12 parts, and according to Ibn Muᶜādh this should be done in the same "equatorial" way as in the theory of aspects. The arcs of the equator between the meridian and the horizon had to be divided into three equal parts of 30°, and through the division points $Q'$ one had to draw semicircles $NQ'S$, etc., which were the boundaries of the houses. Ibn Muᶜādh computed tables for the celestial longitude of the arcs $VQ$ as a function of the celestial longitude of point $P$ on the Eastern horizon, so that the astrologers could use his system of houses.

[8]Ibn Muᶜādh did not have a special name for the tangent; he called it the quotient of the sine and the cosine.

[9]Also for every quarter of a degree between 89°and 89°45′.



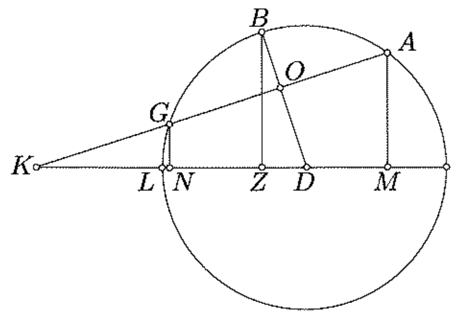


Figure 9

It is clear that astrology created nontrivial work for the mathematicians. Not all astrologers needed to understand the computations, but there was always a certain demand for people who understood the technicalities. The applications in astrology and similar ones in astronomy were much more advanced than the trivial applications of mathematics in land measurement, administration, and commerce.

The purpose of my two examples (of al-Mu'taman and Ibn Muᶜādh) has been to convey some idea of the work of the eleventh-century Islamic Spanish mathematicians in geometry and trigonometry. There were no revolutions in the mathematics of this period, comparable to those in antiquity and the seventeenth century. However, the geometrical methods of antiquity and the Islamic East were handled independently and creatively. In these areas the Western European mathematicians did not catch up with their Islamic Spanish predecessors until the Renaissance.[10] Therefore Islamic Spain has a unique position in the history of mathematics in medieval Europe.

*Note added in proof:* A complete manuscript of a 13-th century recension of the Istikmāl has turned up in 1995.

[10]The Renaissance scholar Regiomontanus (1436–1476) was influenced by Ibn Muᶜādh; they both used the same system of astrological "houses".

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