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Four constructions of two mean proportionals between two given lines in the Book of Perfection *Istikmal* of Al-Muʾtaman Ibn Hūd

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1. *Introduction*

The construction of two mean proportionals between two given lines is one of the classical problems of Greek geometry. Two straight line segments p and q are given . The problem is to construct two other straight segments (" mean proportionals ") x and y in such a way that $p : x = x : y = y : q$. In modern algebraical notation the problem is equivalent to the cubic equation $x^3 = p^2q$, and therefore two mean proportionals cannot in general be constructed by ruler and compass . The Greek geometers found solutions by means of more complicated instruments or using conic sections or other curves.[1]

Several constructions of two mean proportionals were transmitted from Greek into Arabic . The geometers of medieval Islam continued to write on the subject , but in most cases they rendered the solutions that had been found by their Greek predecessors. This paper is about a construction of two mean proportionals which is not found in the Greek literature, and which is probably of Arabic origin . It is a construction by means of a circle and a parabola found in the Book *Istikmāl* (Perfection) of the Andalusian geometer Yūsuf al-Muʾtaman ibn Hūd[2] , who was the ruler of Saragossa from 1081 – 1085 A. D. I will argue below that al-Muʾtaman was the author of the construction.

In proposition 1 of the " first section of the first species of the fifth species[3] " of the ***Istikmāl*** **al-Muʾtaman** presents four constructions of two mean proportionals between two given lines. The text and translation of

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1. A survey of the Greek constructions of two mean proportionals can be found in *Heath*, *HGM*, vol. 1 , pp. 244 – 270 .
2. On Al-Mu'taman see *Djebbar*.
3. More information on the division of the *Istikmāl* into species and sections can be found in *Hogendijk* 2 , 4.

this proposition are given below. In the following introductory analysis the constructions will be labelled (*a*), (*b*), (*c*) and (*d*), being the order in which they occur in the *Istikmāl*.

A comparison between the *Istikmāl* and other Greek and Arabic mathematical works shows that al-Muʾtaman took many theorems and constructions in the *Istikmāl* from other sources[4]. These sources may have been mentioned in a preface, which is now lost, but the extant parts of the *Istikmāl* contain only straight mathematics in the style of Euclid's *Elements*, and thus the text gives no explicit information on the origin of (*a*), (*b*), (*c*) and (*d*). However, constructions (*a*) (by means of two parabolas) and (*b*) (by means of a parabola and a hyperbola) are the same as two constructions in the commentary by Eutocius (ca. 500 A. D.) on proposition 1 of Book II of *On the Sphere and Cylinder* of Archimedes[5]. In construction (*a*), Al-Muʾtaman draws the parameters of the two parabolas in a very peculiar way, as segments of the axes (AB and AG in Figure 3) outside the parabola, in the same way as Eutocius did in his Commentary on Book II of Archimedes' *On the Sphere and Cylinder*. In the classical Apollonian theory, the parameter of a parabola is always represented as a segment perpendicular to the axis[6]. Therefore it is clear that Al-Muʾtaman took (*a*) from the Arabic translation of Eutocius' commentary. In the text of Eutocius (*a*) follows (*b*), and thus it is plausible that Al-Muʾtaman also took (*b*) from Eutocius. Al-Muʾtaman did not copy (*a*) and (*b*) literally, but he made a number of editorial changes. He handled most other material in the *Istikmāl* in a similar way.

Construction (*d*) (by means of a circle and a hyperbola) appears in many slightly different forms in the Arabic tradition. This construction is probably of Greek origin, and its author may have been Apollonius[7]. Because construction (d) is found in several extant Arabic sources antedating Al-Muʾtaman, it is likely that (*d*) was not his own discovery. I have not been able to identify the source from which he took (*d*).

I now present a paraphrase of (*c*), referring to Figure 1. I render the arguments in the same order as al-Muʾtaman, so that the paraphrase can function as a commentary on the text below.

One is asked to find two mean proportionals between two given segments AB and BG. Suppose that the angle at B is a right angle and that $BG > AB$. Join AG and circumscribe a circle around triangle ABG (because the angle at B is a right angle, AG is a diameter of the circle). Draw a parabola with axis GB, vertex G and parameter GB. Let the parabola intersect the circle at D. Draw the ordinate DE (that is to say, the perpendicular to the axis). Then ED and EG are the required mean proportionals, that is to say $BG : ED = ED : EG = EG : AB$.

Proof :

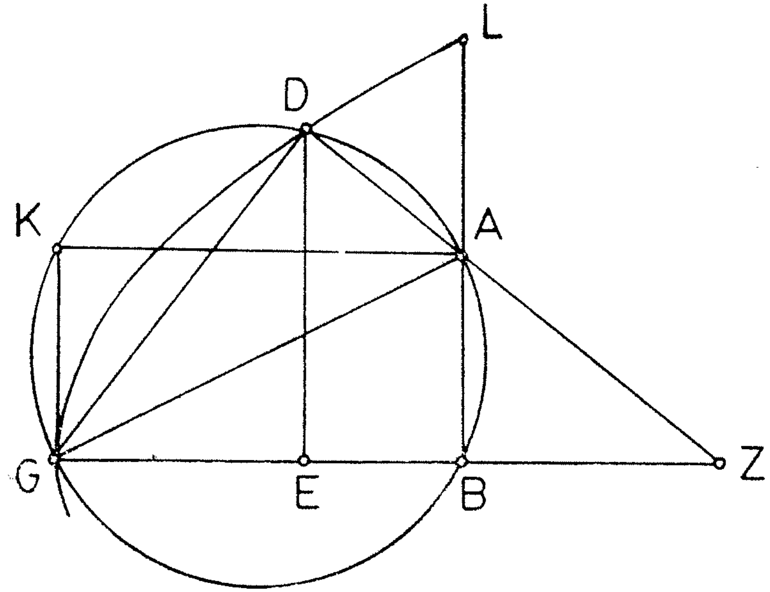


Figure 1

Extend DA and GB to meet at Z, join DG.

(Al-Muʾtaman does not prove that DA and GB meet and that the point of intersection Z is beyond A and B. This can be proved as follows: Complete the rectangle $ABGK$ and let BA meet the parabola at L. Then K is on the circle, GK is tangent to the parabola, and by Apollonius, *Conics* 1 : 11 we have $LB^2 = BG \cdot BG$, so $LB = GB > AB$. Hence L is outside the circle. Therefore the point of intersection D is between K and A, so $DE > AB$. Thus DA and GB meet beyond A and B.)

Since point D is on the circle and AG is a diameter, angle ZDG is a right angle (Euclid, *Elements*, III : 31), so $ZE : ED = ED : EG$ by similar triangles (*Elements* VI : 8). Since D is on the parabola, $ED^2 = BG \cdot EG$ (*Conics* 1 : 11), so $BG : ED = ED : EG$, which is the first equality to be proved. Thus $ZE : ED = BG : ED$, so $ZE = BG$. BY subtraction of BE we obtain $ZB = EG$.

4. See *Hogendijk 4*.
5. For the Greek text of (*a*) and (*b*) see *Heiberg, Archimedes*, vol. 3, pp. 78 – 84; an English translation of (*b*) is in *Bulmer-Thomas* vol. 1, pp. 278 – 283.
6. See *Heath, Apollonius*, pp. 8 – 9.
7. See *Knorr* 3, pp. 252 – 265, 305 and *Thaer*.

We have already shown $ZE : ED = ED : EG$. By similar triangles $ZB : BA = ZE : ED$. Thus $ED : EG = ZB : BA = EG : BA$, q. e. d.

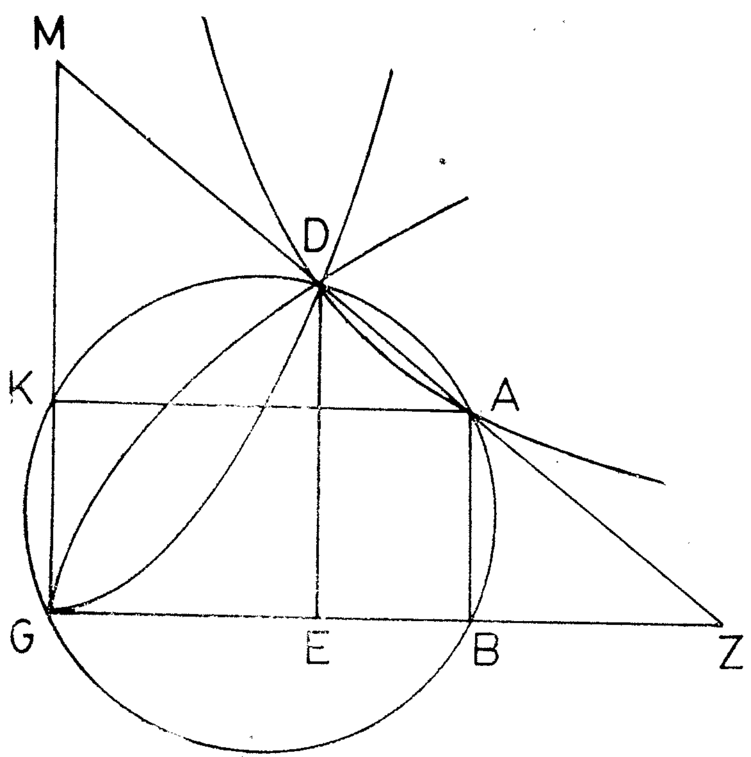


Figure 2

From a modern mathematical point of view construction (*c*) is closely related to the three other constructions (*a*), (*b*) and (*d*). This is illustrated by the following analysis, referring to Figure 2, in which the same notation has been used as in Figure 1. The analysis dœs not take account of the notation used in (*a*), (*b*) and (*d*) in the *Istikmāl*. We suppose that $p = AB = KG$ and $q = AK = BG$ are the two given segments, and that $ABGK$ is a rectangle. Suppose that the two mean proportionals are $Y = DE$ and $X = EG$, and let DE be perpendicular to BG. For later use we extend AD in both directions to meet GB extended at Z and GK extended at M.

We have assumed

$$p : x = x : y = y : q \qquad (1)$$

Hence

$y^2 = qx$, so D is on the parabola (P_1) with axis BG, vertex G and parameter q (cf. *Conics* I : 11);

$x^2 = py$, so D is on the parabola (P_2) with axis KG, vertex G and parameter p (cf. *Conics* I : 11);

$xy = pq$, so D is on the hyperbola (H) with asymptotes KG and GB and passing through A (cf. *Conics* II : 12).

We can easily show that D is also on the circumscribed circle C of rectangle $ABGK$, if we know that D is on P_1 and P_2. Adding the equations of the parabolas we obtain $y^2 + x^2 = qx + py$. This expression can be rewritten as $(x - q/2)^2 + (y - p/2)^2 = (p^2 + q^2)/4$, which is the equation of C.

Now the constructions (*a*) (*b*) (*c*) and (*d*) can be sketched easily: (a) uses P_1, P_2; (*b*) uses P_1, H; (*c*) uses P_1, C and (*d*) uses H, C.

The preceding analysis[8] uses the modern algebraic notation which was introduced by René Descartes in his *Géométrie* (1637). My mathematical analysis gives a misleading impression of the history of (*d*), for it seems that (*d*) was discovered independently of (*a*). This can be inferred from the proofs of (*d*) in the extant sources (including the *Istikmāl*). These proofs are based on the following identities, which do not occur in the preceding analysis:

1. $ZA = DM$ because D and A are points on the hyperbola with asymptotes KG and GB (*Conics* II : 8).

2. $MK \cdot MG = MD \cdot MA$, because K, G, D and A are points on the circle (*Elements* III : 36).

3. $ZB \cdot ZG = ZA \cdot ZD$ because B, G, A and D are points on the circle (*Elements* III : 36).

Nevertheless, my analysis suggests that once (*b*) and (*d*) are known, (*c*) can easily be discovered by someone who realizes that the hyperbolas in (*b*) and (*d*) are the same. It is likely that this is what actually happened. The identities $ZE = BG$ and $ZB = EG$, which occur in the proof of (*c*) in the *Istikmāl*, are closely related to $ZA = DM$, which is used in the proof of (*d*). Hence the author of (*c*) probably knew that D is also on the hyperbola in (*d*).

It is true that the discovery of (*c*) from (*b*) and (*d*) presupposes that sources containing (*b*) and (*d*) are available to the same person. However, it

8. In principle one can rephrase the whole argument in the language of squares and rectangles, used in ancient and medieval mathematics. In this way the equation of C is not easily discovered from the equations of P_1 and P_2.

can be proved that there were at least two capable geometers before al-Mu'taman who knew (*b*) and (*d*), namely Abū Ja'far al-Khāzin (early 10th century) and Al-Sijzī (middle of 10th century). Abū Ja'far al-Khāzin must have known (*b*), because he refers to Eutocius' collection of constructions of two mean proportionals; Abū Ja'far also discussed (*d*) in detail[9]. In a manuscript in Paris (Bibliothèque Nationale, Fonds Arabe 2457, 191b) Al-Sijzī summarized the commentary of Eutocius, so he knew (*b*). In the same manuscript, al-Sijzī copied (*d*) in the version of his contemporary al-Harawī; al-Sijzī also wrote an edited version of (*d*)[10]. Finally it can be noted that Abū Sahl al-Kūhī also knew (*d*)[11], and that (*b*) was known to the 10th-century geometer Abū 'Abdallāh Al-Shannī, who mentions in his *Disclosure of the fallacy of Abū'l-Jūd* (Kashf tamwīh Abī'l-Jūd): "the book of Eutocius who collected in it the statements of the ancients on the construction of two mean proportionals between two given lines, and he rendered in it two methods by Mānāchmus, in one of which he used a hyperbola and a parabola and in the second of which he used two parabolas[12]." Here Al-Shannī refers to (*b*) and (*a*).

To sum up: in the 10th century there were at least four geometers who knew (*d*) and three geometers who knew (*b*), and two geometers who knew (*b*) and (*d*)[13]. However, (*c*) does not appear in any known geometrical work that was written in the entire Eastern Islamic world. We note that especially the 10th-century Eastern Islamic geometers were very fond of finding new solutions to old problems, such as the trisection of the angle and the construction of a regular heptagon, by means of conic sections. Hence there would have been much interest in the 10th century in a solution (*c*) of the prestigious problem of two mean proportionals by means of a parabola and a circle. It is unlikely that such a solution would have vanished without leaving a trace in the quite considerable literature on conic sections that has survived from this period[14]. Hence it seems that (*c*) was unknown to the 10th century geometers in the Eastern Islamic world.

The fact that there were competent geometers who knew (*b*) and (*d*) but who did not know (*c*) shows that the discovery of (*c*) is not a trivial accomplishment in the context of medieval geometry. Such a discovery presupposes perfect insight into the ideas behind the constructions (*b*) and (*d*), independent of the notations that were used (compare Figures 4 and 6), and good a knowledge of conic sections. Thus the author of (*c*) must have been a capable mathematician.

Al-Mu'taman's mathematical abilities are shown by the remarkable simplification in the *Istikmāl* of the difficult solution of the "problem of Alhazen" found in Ibn al-Haytham's *Optics*[15]. Because Ibn al-Haytham died around 1041, less than 45 years before Al-Mu'taman, and because no other traces of the *Optics* are known in the entire Arabic tradition before ca. 1300, it is practically certain that Al-Mu'taman was the author of these simplifications. As we have seen, it is unlikely that (*c*) was known to the geometers of the 10th-century Eastern Islamic world. We can therefore assume that al-Mu'taman was also the author of (*c*).

Text and translation of proposition 1 *of section* 1 *of species* 1 *of species* 5 *of the Istikmāl.*

The following edition of the Arabic text is based on the manuscript Copenhagen, Royal Library, Or. 82, f. 104a – 105a[16]. My own explanatory additions are in parentheses. The abbreviation ms. in the translation indicates erroneous words or letters in the manuscript. Words and passages in pointed brackets < > have been added by me in order to restore the text. In the Arabic manuscript the scribe writes the full names of letters denoting points in geometrical figures (for point *A* the name *alif* instead of the letter *alif*). In my edited text I write the letters and not the names.

Arabic text

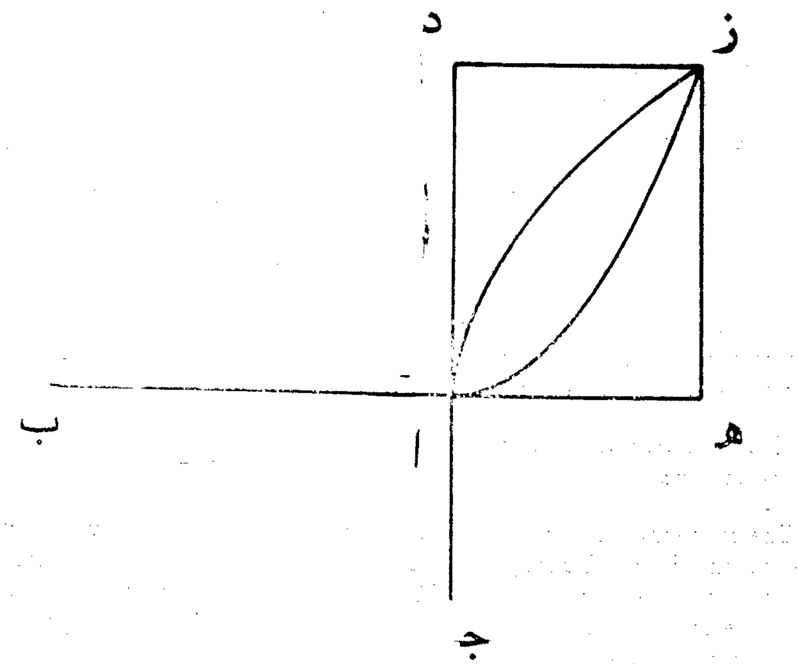
بسم الله الرحمان الرحيم . صلى الله على محمد .

النوع الخامس من جنسي التعاليم الرياضية في إضافة المجسمات وسطوحها بعضها إلى بعض وهو نوعان . النوع الأول في إضافة المجسمات المستقيمة السطوح وسطوحها

9. Cf. *Knorr* 3, pp. 311, 254 – 255.
10. Cf. *GAS* V, p. 130, a; *GAS* VII, p. 409 no. 7.
11. Cf. *Knorr* 3, pp. 252 – 265.
12. Al-Shannī then renders the synthesis of (*a*) beginning with "Mānāchmus said" (ms. Kairo, Dār al-Kutub Muṣṭafā Fāḍil Riyāḍa 41m, 132b, see *Hogendijk* 1, p. 277, M8). Al-Shanni says that (*a*) was plagiarized by Abū'l-Jūd in his work *al-Handasiyyāt* ("Geometrica"), which is now lost. The attribution of (*a*) to Menaechmus is of some interest. Construction (*a*) is presented in the Greek text and in the Arabic translation of Eutocius' commentary as "another way" (Arabic: 'alā wajh ākhar in ms. Escurial 960/2), without reference to any author. The authorship of (*a*) has been repeatedly discussed in the recent literature (cf. *Toomer*, *Knorr* 1, 2, 3). Before these publications, modern historians assumed that (*a*) was by Menaechmus because (*a*) follows (*b*), which is attributed explicitly to Menaechmus. It is interesting to note that Al-Shannī drew the same (natural but not logically necessary) conclusion.
13. Naṣīr al-Dīn al-Ṭūsī (A. D. 1201 – 1274) also knew (*b*) and (*d*). In his edition of Archimedes *On the Sphere and Cylinder*, al-Ṭūsī mentions the commentary of Eutocius several times, so he must have known (*b*) (cf. *Ṭūsī* vol. 2, no. 5, p. 1 line 17, p. 89 line 14). Al-Ṭūsī presents (*d*) in the same text (*Ṭūsī* vol. 2, no. 5, p. 80 – 81, *Knorr* 3, p. 264). It seems therefore that he found (*d*) simpler than the constructions (*a*) or (*b*) of Eutocius' commentary.
14. See for example *Knorr* 3 and *Hogendijk* 1.
15. See *Hogendijk* 3, 5.
16. See *COBRH* 64 – 67 and *Hogendijk* 2 or 4.

بعضها إلى بعض(١) . النوع الأول ينقسم إلى فصلين . الفصل الأول في مقدمات تتصرف فيما بعد من العلوم . الفصل الثاني في إضافة المجسمات المستقيمة السطوح وسطوحها بعضها إلى بعض .

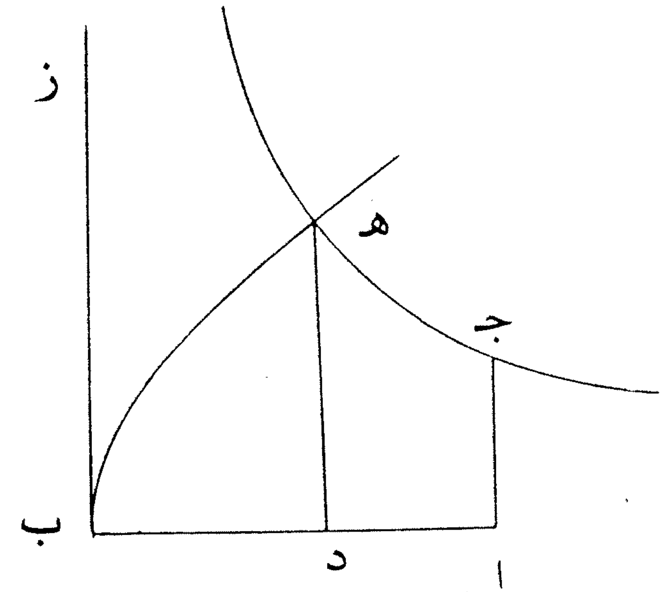
الفصل الأول . نريد أن نبيّن كيف نجد خطّين بين خطّين تتوالى متناسبة .
فليكن الخطان خطي اب اج وليحيطا بزاوية قائمة ولنخرج خط



اب على استقامة إلى هـ و اج على استقامة إلى د . فعلى التحليل إذا فرضنا خطي اد اهـ هما الخطان الوسطان وأن نسبة اب إلى اد كنسبة اد إلى اهـ وكنسبة اهـ إلى اج ، كان مسطح اب في اهـ مساوياً لمربع اد ومسطح اد في اج مساوياً لمربع اهـ . فإذا نحن أخرجنا من نقطتي د هـ خطين موازيين لخطي اهـ اد والتاقيا على نقطة ز كان مسطح اب في اهـ مساويا لمربع هـ ز . فالقطع المكافي الذي سهمه اهـ وضلعه القائم اب يمر بنقطة ز . وكذلك لأن مسطح اج في اد مساوٍ لمربع ز د يكون القطع المكافي الذي سهمه اد وضلعه القائم خط اج يمر بنقطة ز . والقطعـان معلوما الوضع فنقطة ز معلومة الوضع .

(١) حاشية : النوع الثاني في إضافة المجسمات المستديرة وسطوحها بعضها إلى بعض .

فعلى التركيب : إذا جعلنا خط(٢) اب ضلعاً قائماً لقطع يكون رأسه نقطة ا وسهمه خط اهـ وهو قطع از ، ورسمنا قطعاً آخر يكون ضلعه القائم خط اج وسهمه خط اد فلقي قطع(٣) از على نقطة ز ، وأخرجنا من نقطة ز المشتركة خطين على الترتيب وهما خطا ز هـ ز د كانا الوسطين في النسبة . برهان ذلك : لأن قطع از قطع مكافٍ وضلعه القائم اب يكون مسطح اب في اهـ مساوياً لمربع اد . فنسبة اب إلى اد كنسبة اد إلى اهـ . وكذلك أيضاً لأن سطح اج في اد مساوٍ لمربع اهـ تكون نسبة اد إلى اهـ كنسبـة اهـ إلى اج . ونسبة اد إلى اهـ كنسبة اب إلى اد . فنسبة اب إلى اد كنسبة اد إلى اهـ وكنسبة اهـ إلى اج .

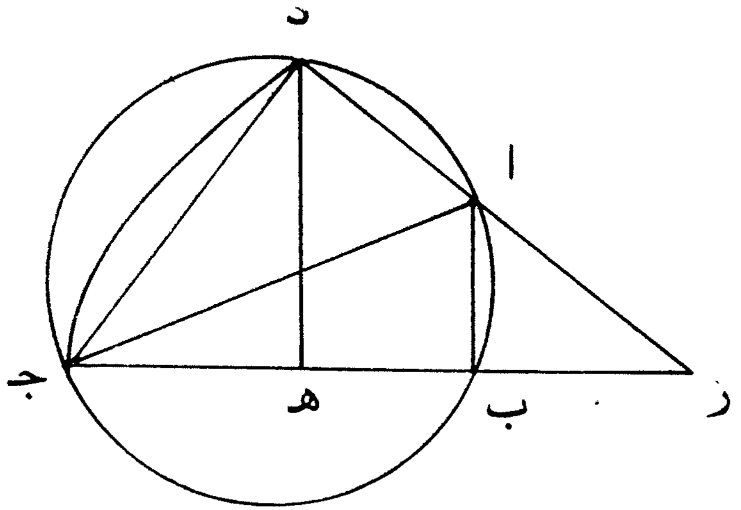


وقد نجد ذلك بقطعٍ مكافٍ وزائد . فلنفرض < أن > ذلك قد كان على التحليل ولتكن نسبة اب إلى هـ د كنسبة هـ د إلى د ب وكنسبة د ب إلى اج . فيكون مسطح اب في ب د مساوياً لمربع د هـ . فإذا جعلنا خط د هـ موازياً لخط اج كان القطع المكافي الذي ضلعه القائم خط اب يمرّ بنقطـة هـ . ولأن مسطح اج في اب مساوٍ(٤) لمسطح هـ د في

(٢) خط : خطي .
(٣) قطع : قطر .
(٤) حاشية : تبين ذلك في الشكل التاسع عشر من الفصل الأول من النوع الثالث من < النوع الرابع > .

دب(٥) ، إذا أخرجنا من نقطة ب خط ب ز موازياً لخط ده كان القطع الزائد الذي يرسم على نقطة ج والخطّان اللذان لايقعان عليه خطا اب ب ز يمرّ بنقطة ه ، فهي مفروضة.

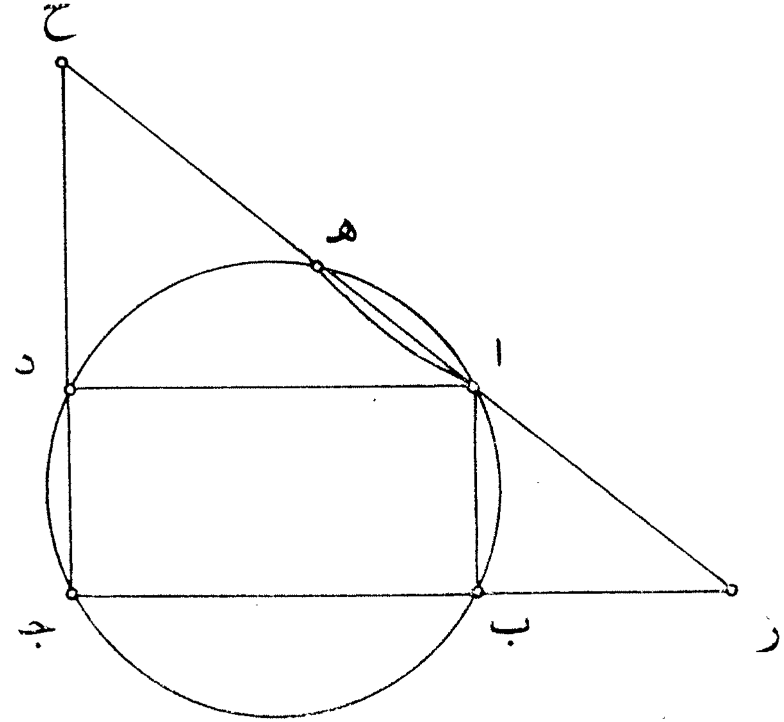
فعلى التركيب نفرض خطّي اب < ا > ج يحيطان بزاوية . ونعمل على خطّ اب قطعاً مكافياً رأسه نقطة ب وضلعه القائم خط اب . ونخرج من نقطة ب خط ب ز موازياً لخط اج ، ونرسم على نقطة ج قطعاً زائداً يكون الخطان اللذان لايقعان عليه خطي اب ب ز وهو قطع جه ، وليلق القطع المكافي على نقطة ه . ولنخرج خط ده على الترتيب . فلأن قطع جه زائد يكون مسطح اج في اب مساوياً لمسطح ه د في د ب . فتكون نسبة اب إلى ده كنسبة د ب إلى اج . ولأن قطع ب ه مكاف وضلعه القائم اب يكون مسطح اب في ب د مساوياً لمربع ده . فنسبة اب إلى ده كنسبة ده إلى د ب .



وقد نجد ذلك بقطع مكاف ودائرة(٦) . فليكن الخطّان خطّي < اب > ب ج ويحيطان بزاوية قائمة . ولنرسم على نقط ا ب ج دائرة . وليكن ب ج أعظم الخطّين ولنجعله(٧) ضلعاً قائماً وسهماً لقطع مكافٍ رأسه نقطة ج . وليلق الدائرة على نقطة

(٥) دال با : دال فا .
(٦) دائرة : وهي (نص) ، دائرة (حاشية) .
(٧) ولنجعله : ولنجعل .

د . ونخرج من نقطة د خطاً على الترتيب وهو ده . فأقول إن خطّي ده ه ج هما الوسطان في النسبة . برهان ذلك : إنا نصل ا د ونخرجه على استقامة إلى أن يلقى خط ب ج على نقطة(٨) ز . ونصل د ج . فلأن الزاوية التي عند نقطة د قائمة تكون نسبة ز ه إلى ه د كنسبة ه د إلى ه ج . ولأن ب ج ضلعٌ قائمٌ لقطع ج د يكون مسطح ب ج في ج ه مساوياً لمربع د ه فنسبة ب ج إلى د ه كنسبة د ه إلى ه ج . فخطّ ب ج مساوٍ لخطّ ز ه . وإذا أسقطنا ب ه المشترك كان ز ب مساوياً لخطّ ه ج . فنسبة ز ه إلى ه د كنسبة خط ه د إلى خط ه ج ، ونسبة خط ه ز إلى خط ه د كنسبة خط ب ز إلى خط ب ا . فنسبة خط ب ج إلى خط ه د كنسبة خط ه د إلى خط ه ج وكنسبة خط ه ج إلى خط ا ب .



(٨) نقطة : استقامة (نص) ، نقطة (حاشية) .

وقد يستبين ذلك أيضاً بقطع زائد ودائرة. وذلك أنا إذا[٩] جعلنا خطي ا ب ب ج[١٠] يحيطان بزاوية قائمة ، وتممنا سطح ا ب ج د المتوازي الأضلاع ، وعملنا عليه دائرة وعملنا على نقطة ا قطعاً زائداً يكون الخطّان اللذان لا يقعان عليه خطي ب ج ج د وليلق الدائرة على نقطة ه . ونصل ا ه ونخرجه حتى يلقى خطي جب ج د على نقطتي ز ح . فأقول : إن خطي ب ز د ح كما أردنا . برهان ذلك : لأن خط ا ز مساو لخط ه ح يكون[١١] < مسطح ا ز في ز ه مساوياً لمسطح ا ح في ح ه فيكون > مسطح ج ز في ز ب مساوياً[١٢] لمسطح ج ح في ح د . فنسبة ج ح إلى ج ز ، التي هي كنسبة ا ب إلى ب ز وكنسبة ح د إلى د ا ، كنسبة ب ز إلى د ح[١٣]. فنسبة ا ب إلى ب ز كنسبة ب ز إلى د ح وكنسبة د ح إلى د ا . وذلك ماأردنا أن نبين .

Translation

In the name of God, the Merciful, the Compassionate. God bless Muḥammad.

The fifth species from the two genera on the mathematical sciences, on the combination of solids and their surfaces. It is in two species.

The first species on the combination of solids with plane surfaces and their surfaces[17]. The first species is divided into two sections: the first section, on the preliminaries, which play a role in the theories that follow, and the second section on the combination of solids with plane surfaces and their surfaces.

The first section. We want to demonstrate how we find two lines between two lines in continued proportion (Figure 3). Thus let the two lines be lines AB, AG and let them contain a right angle. Let us produce line AB rectilinearly to E, and AG rectilinearly to D. Then, in the way of analysis if we assume that lines AD, AE are the two means and that the ratio of AB to AD is equal to the ratio of AD to AE and equal to the ratio of AE to AG, then the rectangle AB, AE is equal to the square of AD, and the rectangle AD, AG is equal to the square of AE. Thus, if we draw from points

(٩) إذا : (في الحاشية فقط) .
(١٠) با جيم : الف جيم .
(١١) مساو لخط ها حا يكون : (في الحاشية فقط) .
(١٢) حاشية : تبين ذلك في الشكل الثامن عشر من الفصل الأول من النوع الثالث من النوع الرابع .
(١٣) دال حا : زاي حا .

17. A marginal note in a different hand adds: "The second species, on the combination of round solids and their surfaces."

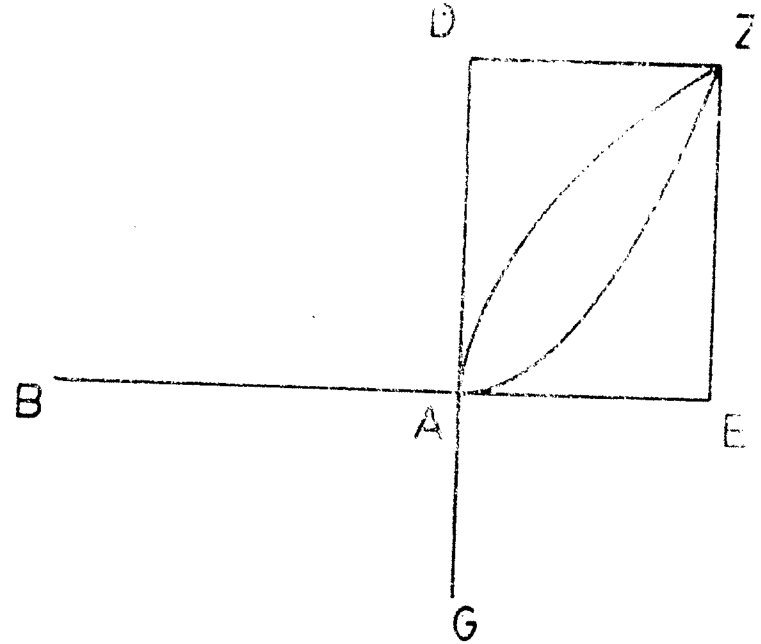


Figure 3

D, E two lines parallel to lines AE, AD meeting at point Z, then the rectangle AB, AE is equal to the square of EZ. Therefore the parabola with axis AE and parameter AB passes through point Z. Similarly, because the rectangle AG, AD is equal to the square of ZD, the parabola with axis AD and parameter line AG passes through point Z. The two (conic) sections are known in position, so point Z is known in position.

Synthesis: If we make line AB the parameter of a (conic) section with vertex point A and axis line AE, namely (conic) section AZ, and if we draw another (conic) section with parameter line AG and axis line AD, then it will meet (conic) section (ms. diameter) AZ at point Z, and (if) we draw from the common point Z two ordinates, namely lines ZE, ZD, then they are the two mean proportionals. Proof of this: Since (conic) section AZ is a parabola with parameter AB, the rectangle AB, AE is equal to the square of AD. Thus the ratio of AB to AD is equal to the ratio of AD to AE. Similarly also, since the rectangle AG, AD is equal to the square of AE, the ratio of AD to AE is equal to the ratio of AE to AG. But the ratio of AD to AE is equal to the ratio of AB to AD. Thus the ratio of AB to AD is equal to the ratio of AD to AE and equal to the ratio of AE to AG.

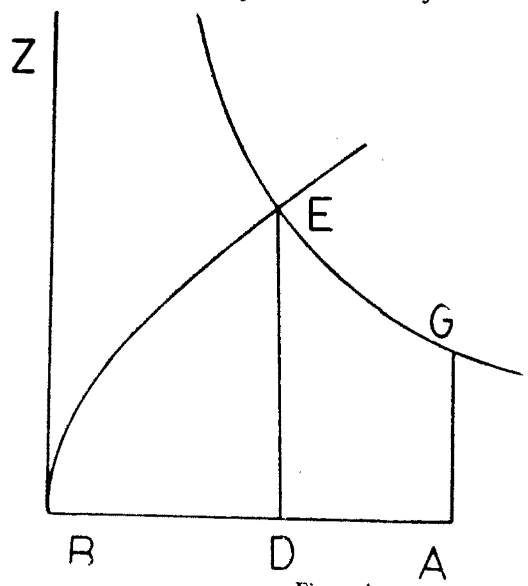


Figure 4

(Figure 4) We can also find this by means of a parabola and a hyperbola. Let us assume that this exists, by way of analysis, and let the ratio of AB to ED be equal to the ratio of ED to DB and equal to the ratio of DB to AG. Then the rectangle AB, BD is equal to the square of DE. Thus, if we make line DE parallel to line AG, the parabola with parameter line AB[18] passes through point E. Since the rectangle AG , AB is equal[19] to the rectangle ED, DB , therefore if we draw through point B line BZ parallel to line DE, then the hyperbola drawn through point G with asymptotes lines AB , BZ passes through point E. It (point E) is therefore assumed (i. e. given).

Synthesis : We assume two lines AB , AG containing an angle. We construct on line AB a parabola with vertex point B and parameter line AB[18] . We draw from point B line BZ parallel to line AG, and we construct on point G a hyperbola with asymptotes lines AB , BZ, namely (conic) section GE, and let it meet the parabola at point E. Let us draw the ordinate DE. Then, since (conic) section GE is a hyperbola, the rectangle AG , AB is equal to the rectangle ED , DB. Thus the ratio of AB to DE is equal to the ratio of DB to AG. Since (conic) section BE is a parabola with para - meter AB, the rectangle AB, BD is equal to the square of DE . Thus the ratio of AB to DE is equal to the ratio of DE to DB .

18. The ordinates of the parabola are assumed to be parallel to AG . The angle is not necessarily a right angle.

19. A marginal remark states : " this has been proven in the 19th proposition of the first section of the third species of < the fourth species >. This (extant) theorem of the *Istikmāl* includes the equivalent of *Conics* II:12. Point E is on the hyperbola by the converse of *Conics* II:12 .

(Figure 5) We can also find this by means of a parabola and a circle. Let the two lines be lines <AB>, BG, and they contain a right angle. Let us draw through points A, B, G a circle. Let BG be the greater of the two lines, and let us make <it> the parameter and the axis of a parabola with vertex at point G. Let it meet the circle at point D . We draw from point D an ordinate DE . I say that lines DE, EG are the two mean proportionals. Proof of this: We join AD, and we extend it rectilinearly to meet line BG at point Z. We join DG. Then, since the angle at point D is (a) right (angle), the ratio of ZE to ED is equal to the ratio of ED to EG . Since BG is a parameter of (conic) section GD, the rectangle BG, GE is equal to the square of DE, so the ratio of BG to DE is equal to the ratio of DE to EG. Thus line BG is equal to line ZE. If we subtract the common (part) BE, ZB is equal to line EG . Thus the ratio of ZE to ED is equal to the ratio of line ED to line EG, and the ratio of line EZ to line ED is equal to the ratio of line BZ to line BA . Thus the ratio of line BG to line ED is equal to the ratio of line ED to line EG, and equal to the ratio of line EG to line AB.

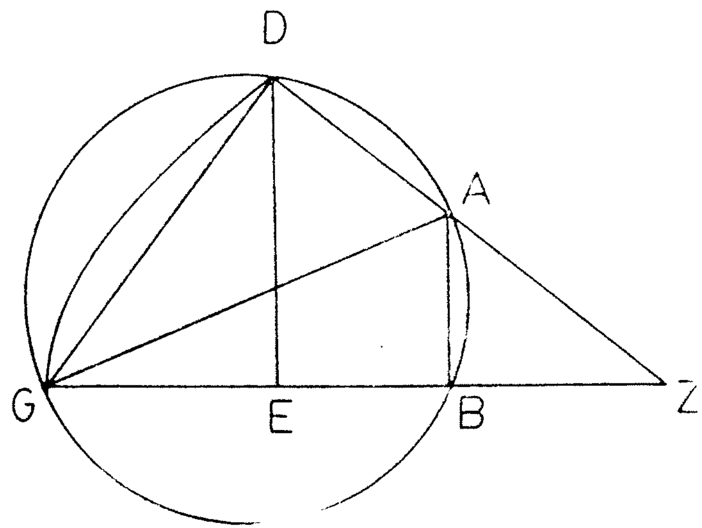


Figure 5

(Figure 6) This can also be demonstrated by means of a hyperbola and a circle. That is: if we let lines AB, BG (ms. AG) contain a right angle, and (if) we complete parallelogram $ABGD$, and construct on it a circle, and construct on point A a hyperbola with asymptotes lines BG , GD , and let it meet the circle at point E . We join AE and extend it to meet lines GB , GD at points Z , H. Then I say that lines BZ, DH are as we wished.

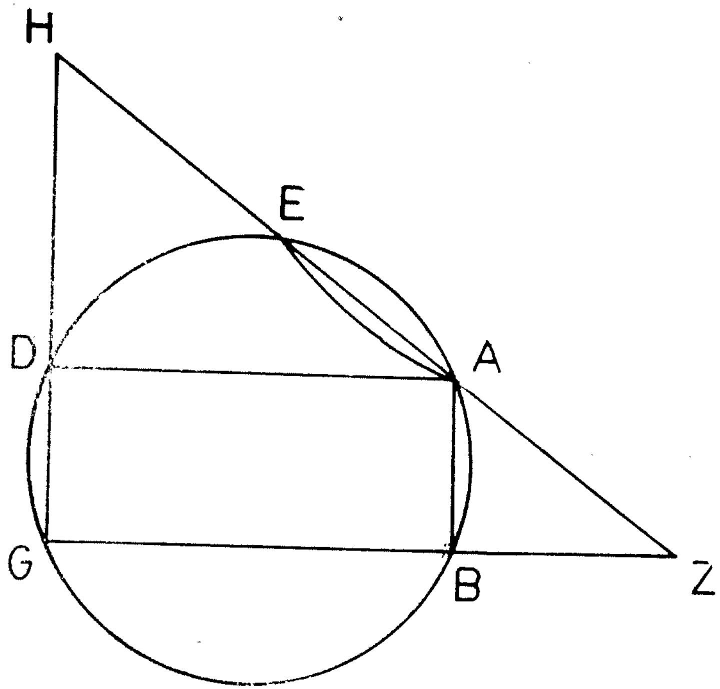


Figure 6

Proof of this: Since line AZ is equal to line $EH <$ the rectangle AZ, ZE is equal to the rectangle AH, HE, so >[20] the rectangle GZ , ZB is equal to the rectangle GH, HD . Thus the ratio of GH to GZ, which is equal to the ratio of AB to BZ , and equal to the ratio of HD to DA , is equal to the ratio of BZ to DH . Thus the ratio of AB to BZ is equal to the ratio of BZ to DH , and equal to the ratio of DH to DA . That is what we wanted to demonstrate.

20. I have tentatively reconstructed the incomplete text. A marginal remark states that " this has been proved in the 18th proposition of the first section of the third species of the fourth species". The theorem in question includes *Conics* II:8, which is to the effect that $AZ = EH$ because points A and E are on the hyperbola and Z and H on its asymptotes . There are more marginal remarks at the top of f. 105a, but I was not able to read them because most of the text has been destroyed by worms . Because A , E , G and B are on a circle we have $AZ . ZE = GZ . ZB$ by Euclid , *Elements*, III : 36 , and similarly $AH . HE = GH . HD$ because A , E , G and D are on a circle .

References

Apollonius, Conics (Books I - IV) . Greek text in *Heiberg, Apollonius*. French translation in *Ver Eecke*. English paraphrase in *Heath, Apollonius*.

Bulmer - Thomas I. Bulmer-Thomas, *Greek mathematical works*. Cambridge : Harvard University Press, London: William Heinemann Ltd. 2 vols. , 1967. Reprint of the 1939 edition.

COBRH Codices Orientalis Bibliothecae Regiae Hafnensis. Pars altera, codices hebraicos et arabicos continens. Copenhagen 1851.

Djebbar A. Djebbar, *Deux mathématiciens peu connus de l'Espagne du XI^e siècle: Al-Mu'taman et Ibn Sayyid*. Université de Paris-Sud, Département de Mathématique , 91405 Orsay Cedex, France 1984. Also in : M. Folkerts, J. P. Hogendijk (eds.), *Vestigia Mathematica*, Studies in medieval and early modern mathematics in honour of H. L. L. Busard, Amsterdam 1993, pp. 79 - 92.

Euclid, see *Heath, Euclid*.

GAS F. Sezgin, *Geschichte des arabischen Schrifttums*. Vol. 5, *Mathematik* bis ca. 430 *H*. Vol. VII, *Astrologie, Meteorologie und Verwandtes*. Leiden: Brill, 1974, 1978.

Heath, Apollonius T. L . Heath, *Apollonius of Perga. Treatise on conic sections* . Cambridge : Cambridge University Press, 1896.

Heath, HGM T. L. Heath, *A history of Greek mathematics*. Oxford: Clarendon Press 1921. Reprint: New York (Dover) 1981.

Heath, Euclid T. L. Heath, *Euclid. The thirteen Books of the Elements*. Cambridge: Cambridge University Press. Second edition.

Heiberg, Apollonius J. L. Heiberg, *Apollonii pergaei quae Graece exstant cum commentariis antiquis*. Leipzig : Teubner, 1893.

Heiberg, Archimedes J. L. Heiberg, *Archimedis Opera Omnia cum commentariis Eutocii*. Stuttgart: Teubner, 3 vols., 1972 (reprint of the second edition, 1915).

Hogendijk 1 J. P. Hogendijk, " Greek and Arabic constructions of the regular heptagon ", *Archive for History of Exact Sciences* 30 (1984), 197 - 330 .

Hogendijk 2 J. P. Hogendijk , Discovery of an 11 - th century geometrical compilation: the *Istikmāl* of Yusuf al-Mu'taman ibn Hud, king of Saragossa". *Historia Mathematica* 13 (1986), 43 - 52.

Hogendijk 3 J. P. Hogendijk, " Le roi-géomètre Al-Mu'taman ibn Hud et son livre de la perfection (Kitab al-Istikmāl)". In : *Actes du premier colloque international sur l'histoire des mathématiques arabes, Alger*, 1 , 2 , 3, *décembre* 1986, pp. 53 - 66. Algiers: Maison des livres, 1988.

Hogendijk 4 J. P. Hogendijk, " The geometrical parts of the *Istikmāl* of Yusuf al-Mu'taman ibn Hud. An analytical table of contents " , *Archives Internationales d'Histoire des Sciences* 41 (1992), 207 - 281.

Hogendijk 5 J. P. Hogendijk, " Al-Mu'taman's simplified lemmas for solving " Alhazen's problem". To appear.

Knorr 1 W. R. Knorr " Observations on the early history of conics", *Centaurus* 26 (1982), 1 - 24.

Knorr 2 W. R. Knorr, *The ancient tradition of geometric problems*. Boston: Birkhäuser, 1986.

Knorr 3. W. R. Knorr, *Textual studies in ancient and medieval geometry*. Boston: Birkhäuser, 1988.

Thaer C. Thaer, " Die Würfelverdoppelung des Apollonios " , *Deutsche Mathematik* 5 (1940) , 241 - 243.

Tūsī Naṣir al-Din al-Tūsī. *Majmū^c al-Rasā'il* , 2 vols. Hyderabad: Osmania oriental publications Bureau, 1358 A. H. / 1952 A. D.

Toomer G. J. Toomer, *Diocles on Burning Mirrors*. New York: Springer, 1976.

Ver Eecke P. Ver Eecke, *Les Coniques d'Apollonius de Perge*. Paris: Blanchard, 1959. Reprint of the 1922 edition.