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## Four constructions of two mean proportionals between two given lines in the Book of Perfection *Istikmāl* of Al-Mu'taman Ibn Hūd

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### 1. Introduction

The construction of two mean proportionals between two given lines is one of the classical problems of Greek geometry. Two straight line segments  $p$  and  $q$  are given. The problem is to construct two other straight segments ("mean proportionals")  $x$  and  $y$  in such a way that  $p : x = x : y = y : q$ . In modern algebraical notation the problem is equivalent to the cubic equation  $x^3 = p^2q$ , and therefore two mean proportionals cannot in general be constructed by ruler and compass. The Greek geometers found solutions by means of more complicated instruments or using conic sections or other curves.<sup>1</sup>

Several constructions of two mean proportionals were transmitted from Greek into Arabic. The geometers of medieval Islam continued to write on the subject, but in most cases they rendered the solutions that had been found by their Greek predecessors. This paper is about a construction of two mean proportionals which is not found in the Greek literature, and which is probably of Arabic origin. It is a construction by means of a circle and a parabola found in the Book *Istikmāl* (Perfection) of the Andalusian geometer Yūsuf al-Mu'taman ibn Hūd<sup>2</sup>, who was the ruler of Saragossa from 1081–1085 A. D. I will argue below that al-Mu'taman was the author of the construction.

In proposition 1 of the "first section of the first species of the fifth species"<sup>3</sup> of the *Istikmāl* al-Mu'taman presents four constructions of two mean proportionals between two given lines. The text and translation of

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1. A survey of the Greek constructions of two mean proportionals can be found in Heath, *HGM*, vol. 1, pp. 244–270.

2. On Al-Mu'taman see *Djebbar*.

3. More information on the division of the *Istikmāl* into species and sections can be found in *Hogendijk* 2, 4.

this proposition are given below. In the following introductory analysis the constructions will be labelled (a), (b), (c) and (d), being the order in which they occur in the *Isikimāl*.

A comparison between the *Isikimāl* and other Greek and Arabic mathematical works shows that al-Mu<sup>2</sup>taman took many theorems and constructions in the *Isikimāl* from other sources<sup>4</sup>. These sources may have been mentioned in a preface, which is now lost, but the extant parts of the *Isikimāl* contain only straight mathematics in the style of Euclid's *Elements*, and thus the text gives no explicit information on the origin of (a), (b), (c) and (d). However, constructions (a) (by means of two parabolas) and (b) (by means of a parabola and a hyperbola) are the same as two constructions in the commentary by Eutocius (ca. 500 A. D.) on proposition I of Book II of *On the Sphere and Cylinder* of Archimedes<sup>5</sup>. In construction (a), Al-Mu<sup>2</sup>taman draws the parameters of the two parabolas in a very peculiar way, as segments of the axes ( $AB$  and  $AG$  in Figure 3) outside the parabola, in the same way as Eutocius did in his Commentary on Book II of Archimedes' *On the Sphere and Cylinder*. In the classical Apollonian theory, the parameter of a parabola is always represented as a segment perpendicular to the axis<sup>6</sup>. Therefore it is clear that Al-Mu<sup>2</sup>taman took (a) from the Arabic translation of Eutocius' commentary. In the text of Eutocius (a) follows (b), and thus it is plausible that Al-Mu<sup>2</sup>taman also took (b) from Eutocius. Al-Mu<sup>2</sup>taman did not copy (a) and (b) literally, but he made a number of editorial changes. He handled most other material in the *Isikimāl* in a similar way.

Construction (d) (by means of a circle and a hyperbola) appears in many slightly different forms in the Arabic tradition. This construction is probably of Greek origin, and its author may have been Apollonius<sup>7</sup>. Because construction (d) is found in several extant Arabic sources antedating Al-Mu<sup>2</sup>taman, it is likely that (d) was not his own discovery. I have not been able to identify the source from which he took (d).

I now present a paraphrase of (c), referring to Figure 1. I render the arguments in the same order as al-Mu<sup>2</sup>taman, so that the paraphrase can function as a commentary on the text below.

One is asked to find two mean proportionals between two given segments  $AB$  and  $BC$ . Suppose that the angle at  $B$  is a right angle and that  $BC > AB$ . Join  $AC$  and circumscribe a circle around triangle  $ABC$  (because

the angle at  $B$  is a right angle,  $AC$  is a diameter of the circle). Draw a parabola with axis  $CB$ , vertex  $C$  and parameter  $CB$ . Let the parabola intersect the circle at  $D$ . Draw the ordinate  $DE$  (that is to say, the perpendicular to the axis). Then  $ED$  and  $EG$  are the required mean proportionals, that is to say  $BC : ED = ED : EG = EG : AB$ .

Proof :

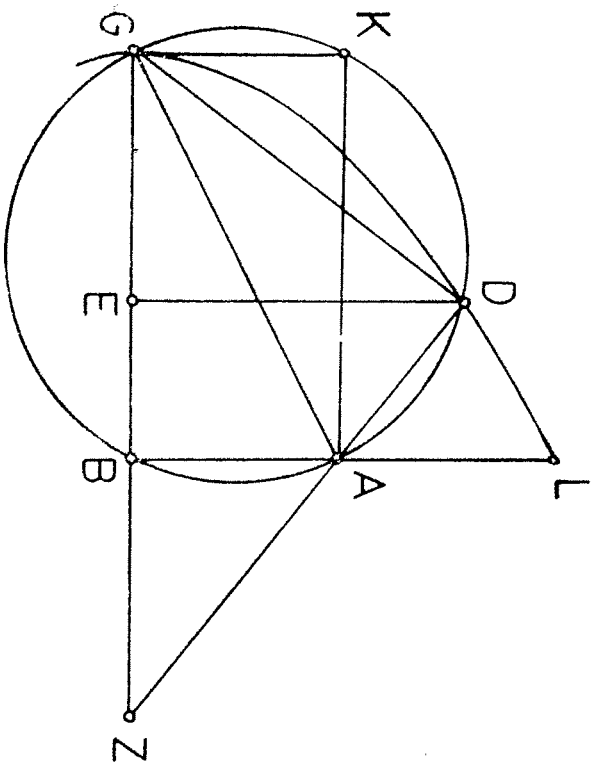


Figure 1

Extend  $DA$  and  $GB$  to meet at  $Z$ , join  $DG$ .

(Al-Mu<sup>2</sup>taman does not prove that  $DA$  and  $GB$  meet and that the point of intersection  $Z$  is beyond  $A$  and  $B$ . This can be proved as follows : Complete the rectangle  $ABCK$  and let  $BA$  meet the parabola at  $L$ . Then  $K$  is on the circle,  $CK$  is tangent to the parabola, and by Apollonius, *Conics* I : 11 we have  $LB^2 = BC \cdot BG$ , so  $LB = GB > AB$ . Hence  $L$  is outside the circle. Therefore the point of intersection  $D$  is between  $K$  and  $A$ , so  $DE > AB$ . Thus  $DA$  and  $GB$  meet beyond  $A$  and  $B$ .)

Since point  $D$  is on the circle and  $AG$  is a diameter, angle  $ZDC$  is a right angle (Euclid, *Elements*, III : 31), so  $ZE : ED = ED : EG$  by similar triangles (*Elements* VI : 8). Since  $D$  is on the parabola,  $ED^2 = BC \cdot EG$  (*Conics* I : 11), so  $BC : ED = ED : EG$ , which is the first equality to be proved. Thus  $ZE : ED = BC : ED$ , so  $ZE = BC$ . By subtraction of  $BE$  we obtain  $ZB = EG$ .

4. See Hogendijk 4.

5. For the Greek text of (a) and (b) see Heiberg, *Archimedes*, vol. 3, pp. 78–84; an English translation of (b) is in Bulmer-Thomas vol. 1, pp. 278–283.

6. See Heath, *Apollonius*, pp. 8–9.

7. See Knorr 3, pp. 252–265, 305 and Thaeer.



can be proved that there were at least two capable geometers before al-Mu<sup>2</sup>-taman who knew (b) and (d), namely Abū Ja'far al-Khāzin (early 10th century) and Al-Sijzi (middle of 10th century). Abū Ja'far al-Khāzin must have known (b), because he refers to Eutocius' collection of constructions of two mean proportionals: Abū Ja'far also discussed (d) in detail<sup>9</sup>. In a manuscript in Paris (Bibliothèque Nationale, Fonds Arabe 2457, 191b) Al-Sijzi summarized the commentary of Eutocius, so he knew (b). In the same manuscript, al-Sijzi copied (d) in the version of his contemporary al-Harawī; al-Sijzi also wrote an edited version of (d)<sup>10</sup>. Finally it can be noted that Abū Sahl al-Kūhī also knew (d)<sup>11</sup>, and that (b) was known to the 10th-century geometer Abū 'Abdallāh Al-Shannī, who mentions in his *Disclasure of the fallacy of Abū'l-Jūd* (Kashf tamwīh Abū'l-Jūd): "the book of Eutocius who collected in it the statements of the ancients on the construction of two mean proportionals between two given lines, and he rendered in it two methods by Mānāchmus, in one of which he used a hyperbola and a parabola and in the second of which he used two parabolae<sup>12</sup>." Here Al-Shannī refers to (b) and (a).

To sum up: in the 10th century there were at least four geometers who knew (d) and three geometers who knew (b), and two geometers who knew (a) and (d)<sup>13</sup>. However, (c) does not appear in any known geometrical work that was written in the entire Eastern Islamic world. We note that especially the 10th-century Eastern Islamic geometers were very fond of finding new solutions to old problems, such as the trisection of the angle and the construction of a regular heptagon, by means of conic sections. Hence there would have been much interest in the 10th century in a solution (c) of the prestigious problem of two mean proportionals by means of

9. Cf. Knorr 3, pp. 311, 254 - 255.

10. Cf. GAS V, p. 130, a; GAS VII, p. 409 no. 7.

11. Cf. Knorr 3, pp. 252 - 265.

12. Al-Shannī then renders the synthesis of (a) beginning with "Mānāchmus said" (ms. Karo. Dar al-Kutub Mustafā Fādl Riyāda 41m. 132b, see Hogendijk 1, p. 277, M6). Al-Shannī says that (a) was plagiarized by Abū'l-Jūd in his work *al-Handasiyyāt* ("Geometria"), which is now lost. The attribution of (a) to Menaechnus is of some interest. Construction (a) is presented in the Greek text and in the Arabic translation of Eutocius' commentary as "another way" (Arabic: 'alā waḥd akhar in ms. Escorial 960/2), without reference to any author. The authorship of (a) has been repeatedly discussed in the recent literature (cf. Toomer, Knorr 1, 2, 3). Before these publications, modern historians assumed that (a) was by Menaechnus because (a) follows (b), which is attributed explicitly to Menaechnus. It is interesting to note that Al-Shannī drew the same (natural but not logically necessary) conclusion.

13. Naṣīr al-Dīn al-Tūsī (A. D. 1201 - 1274) also knew (b) and (d). In his edition of Archimedes *On the Sphere and Cylinder*, al-Tūsī mentions the commentary of Eutocius several times, so he must have known (b) (cf. *Tūsī* vol. 2, no. 5, p. 1 line 17, p. 89 line 14). Al-Tūsī presents (d) in the same text (*Tūsī* vol. 2, no. 5, p. 80 - 81, Knorr 3, p. 264). It seems therefore that he found (d) simpler than the constructions (a) or (b) of Eutocius' commentary.

a parabola and a circle. It is unlikely that such a solution would have vanished without leaving a trace in the quite considerable literature on conic sections that has survived from this period<sup>14</sup>. Hence it seems that (c) was unknown to the 10th century geometers in the Eastern Islamic world.

The fact that there were competent geometers who knew (b) and (d) but who did not know (c) shows that the discovery of (c) is not a trivial accomplishment in the context of medieval geometry. Such a discovery presupposes perfect insight into the ideas behind the constructions (b) and (d), independent of the notations that were used (compare Figures 4 and 6), and good a knowledge of conic sections. Thus the author of (c) must have been a capable mathematician.

Al-Mu<sup>2</sup>taman's mathematical abilities are shown by the remarkable simplification in the *Istikmāl* of the difficult solution of the "problem of Alhazen" found in Ibn al-Haytham's *Optics*<sup>15</sup>. Because Ibn al-Haytham died around 1041, less than 45 years before Al-Mu<sup>2</sup>taman, and because no other traces of the *Optics* are known in the entire Arabic tradition before ca. 1300, it is practically certain that Al-Mu<sup>2</sup>taman was the author of these simplifications. As we have seen, it is unlikely that (c) was known to the geometers of the 10th-century Eastern Islamic world. We can therefore assume that al-Mu<sup>2</sup>taman was also the author of (c).

#### *Text and translation of proposition 1 of section 1 of species 1 of species 5 of the Istikmāl.*

The following edition of the Arabic text is based on the manuscript Copenhagen, Royal Library, Or. 82, f. 104a - 105a<sup>16</sup>. My own explanatory additions are in parentheses. The abbreviation ms. in the translation indicates erroneous words or letters in the manuscript. Words and passages in pointed brackets <> have been added by me in order to restore the text. In the Arabic manuscript the scribe writes the full names of letters denoting points in geometrical figures (for point A the name *alif* instead of the letter *alif*). In my edited text I write the letters and not the names.

#### Arabic text

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ . صَلَّيْ اللَّهَ عَلَى مُحَمَّدٍ .

النوع الخامس من جنسي العالم الرياضية في إضافة الجسام وسطوحها بعضها إلى بعض وهو نوعان . النوع الأول في إضافة الجسام المستقيمة السطوح وسطوحها

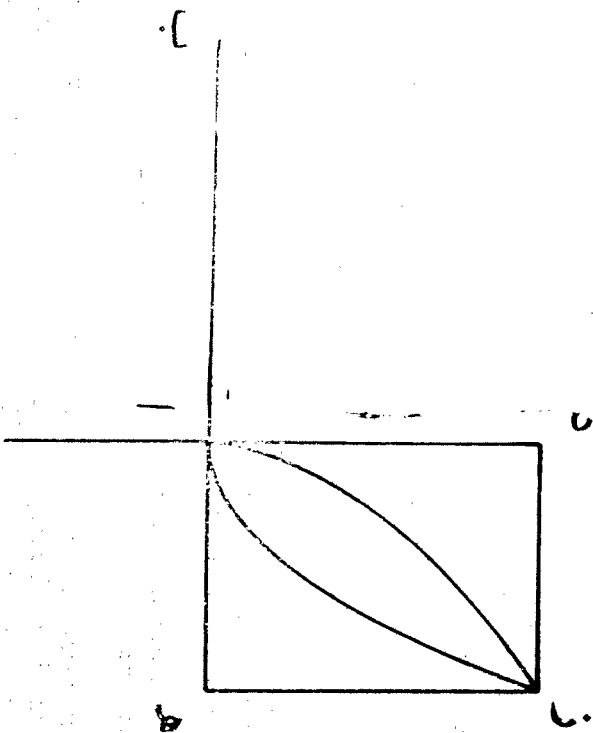
14. See for example Knorr 3 and Hogendijk 1.

15. See Hogendijk 3, 5.

16. See COBRH 64 - 67 and Hogendijk 2 or 4.

بعضها إلى بعض (1). النوع الأول ينقسم إلى فصلين . الفصل الأول في مقدمات تتصرف فيما بعد من المعلوم . الفصل الثاني في إضافة الجسومات المستقيمة المسطوح وسطوحها بعضها إلى بعض .

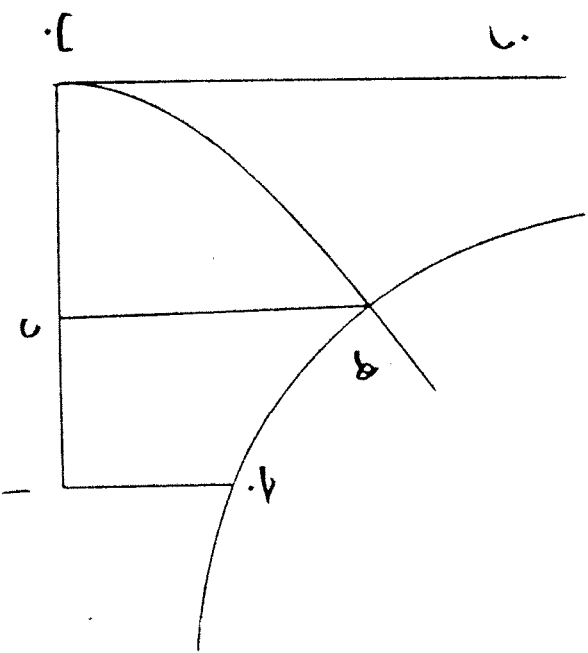
الفصل الأول . نريد أن نبين كيف نجد خطين بين خطين تتوال متناسبة . فإمكن الخطان خطي  $آب$  و  $آج$  وليجربا بزواوية قائمة ولخرج خط



$آب$  على استقامة إلى  $هـ$  و  $آج$  على استقامة إلى  $د$  . فعل التحليل إذا فرضنا خطي  $آد$   $آهـ$  هما الخطان الراسطان وأن نسبة  $آب$  إلى  $آد$  كنسبة  $آد$  إلى  $آهـ$  وكنسبة  $آهـ$  إلى  $آج$  ، كان مسطح  $آب$  في  $آهـ$  مساوياً لربع  $آد$  ومسطح  $آد$  في  $آهـ$  مساوياً لربع  $آهـ$  . فإذا نحن أخرجنا من تقطعي  $د هـ$  خطين موازيين لخطي  $آهـ$   $آد$  والتأقيا على نقطة  $ز$  كان مسطح  $آب$  في  $آهـ$  مساوياً لربع  $هـ ز$  . فالقطع المكافئ الذي سهمه  $آهـ$  وضامه القائم  $آب$  يمر بنقطة  $ز$  . وكذلك لأن مسطح  $آج$  في  $آد$  مساوياً لربع  $ز د$  يكون القطع المكافئ الذي سهمه  $آد$  وضامه القائم خطي  $آج$  يمر بنقطة  $ز$  . والقطعة معلوما الوضع فنقطه  $ز$  معلومة الوضع .

(1) حاشية : النوع الثاني في إضافة الجسومات المستوية وسطوحها بعضها إلى بعض .

فعل التركيب : إذا جمعنا خط (1)  $آب$  ضاماً قائماً لقطع يكون رأسه نقطة  $آ$  وسهمه خط  $آهـ$  وهو قطع  $آز$  ، ورسمنا قطعاً آخر ، يكون ضامه القائم خط  $آج$  وسهمه خط  $آد$  فلقى قطع (2)  $آز$  على نقطة  $ز$  ، وأخرجنا من نقطة  $ز$  المبتدئة خطين على الترتيب وهما خطا  $ز د$  و  $ز هـ$  كانا الواسطين في النسبة . برهان ذلك : لأن قطع  $آز$  قطع مكافئ وضامه القائم  $آب$  يكون مسطح  $آب$  في  $آهـ$  مساوياً لربع  $آد$  . فنسبة  $آب$  إلى  $آد$  كنسبة  $آد$  إلى  $آهـ$  . وكذلك أيضاً لأن مسطح  $آج$  في  $آد$  مساوياً لربع  $آهـ$  تكون نسبة  $آد$  إلى  $آهـ$  كنسبة  $آهـ$  إلى  $آج$  . ونسبة  $آد$  إلى  $آهـ$  كنسبة  $آب$  إلى  $آد$  . فنسبة  $آب$  إلى  $آد$  كنسبة  $آد$  إلى  $آهـ$  وكنسبة  $آهـ$  إلى  $آج$  .



وقد نجد ذلك بقطع مكافئ وزاوية . فلنفترض  $> أن$  ذلك قد كان على التحليل ولكن نسبة  $آب$  إلى  $هـ د$  كنسبة  $هـ د$  إلى  $د ب$  وكنسبة  $د ب$  إلى  $آج$  . فبكون مسطح  $آب$  في  $ب د$  مساوياً لربع  $د هـ$  . فإذا جمعنا خط  $د هـ$  موازياً لخط  $آج$  كان القطع المكافئ الذي ضامه القائم خط  $آب$  يمر بنقطة  $هـ$  . ولأن مسطح  $آج$  في  $آب$  مساوياً لقطع (1)  $آب$  في  $هـ د$  في

(1) خط : خطي .

(2) قطع : قطع .

(4) حاشية : تبين ذلك في الشكل الخامس عشر من الفصل الأول من النوع الثالث من > النوع الرابع <



وقد يستبين ذلك أيضاً بقطع زائد ودائرة. وذلك أنا إذا<sup>(١٠)</sup> جعلنا خطي  $\overline{AB}$  و  $\overline{B\Gamma}$  يحيطان بزاوية قائمة ، وتمسنا سطح  $\overline{AB\Gamma}$  المتوازي الأضلاع ، وعمنا عليه دائرة وعمنا على نقطة  $\overline{A}$  قطعاً زائلاً يكون القطان اللذان لا يتعان عليه خطي  $\overline{B\Gamma}$  و  $\overline{AD}$  وليق اللائرة على نقطة  $\overline{H}$  . ونصل  $\overline{AH}$  ونخرجه حتى يلقى خطي  $\overline{B\Gamma}$  و  $\overline{AD}$  على نقطة  $\overline{Z}$  . فأقول : إن خطي  $\overline{B\Gamma}$  و  $\overline{AD}$  كما أردنا . برهان ذلك : لأن خط  $\overline{AZ}$  مساوٍ لخط  $\overline{AZ}$  يكون<sup>(١١)</sup>  $\overline{AZ} > \overline{AZ}$  مساوياً لسطح  $\overline{AZ}$  في  $\overline{Z}$  مساوياً لسطح  $\overline{AZ}$  في  $\overline{Z}$  ، التي هي كسبة  $\overline{AB}$  إلى  $\overline{B\Gamma}$  و كسبة  $\overline{AD}$  إلى  $\overline{AD}$  ، كسبة  $\overline{B\Gamma}$  إلى  $\overline{AD}$  . فنسبة  $\overline{AB}$  إلى  $\overline{B\Gamma}$  كسبة  $\overline{B\Gamma}$  إلى  $\overline{AD}$  . وذلك ما أردنا أن نبيِّن .

Translation

In the name of God , the Merciful , the Compassionate. God bless Muhammad.

The fifth species from the two genera on the mathematical sciences, on the combination of solids and their surfaces. It is in two species .

The first species on the combination of solids with plane surfaces and their surfaces<sup>17</sup> . The first species is divided into two sections : the first and the second section on the preliminaries, which play a role in the theories that follow, their surfaces .

The first section . We want to demonstrate how we find two lines between two lines in continued proportion (Figure 3). Thus let the two lines be lines  $AB$  ,  $AG$  and let them contain a right angle. Let us produce line  $AB$  rectilinearly to  $E$ , and  $AG$  rectilinearly to  $D$ . Then, in the way of analysis if we assume that lines  $AD$ ,  $AE$  are the two means and that the ratio of  $AB$  to  $AD$  is equal to the ratio of  $AD$  to  $AE$  and equal to the ratio of  $AE$  to  $AG$ , then the rectangle  $AB$ ,  $AE$  is equal to the square of  $AD$ , and the rectangle  $AD$ ,  $AG$  is equal to the square of  $AE$ . Thus, if we draw from points

(٩) إذا : (في المائية فقط) .

(١٠) با جمع : الف جمع .

(١١) مساوٍ لخط  $\overline{HA}$  يكون : (في المائية فقط) .

(١٢) مائية : تبين ذلك في الشكل الثاني عشر من الفصل الأول من النوع الثالث من النوع الرابع .

(١٣) دال حا : زاي حا .

17 . A marginal note in a different hand adds : "The second species, on the combination of round solids and their surfaces."

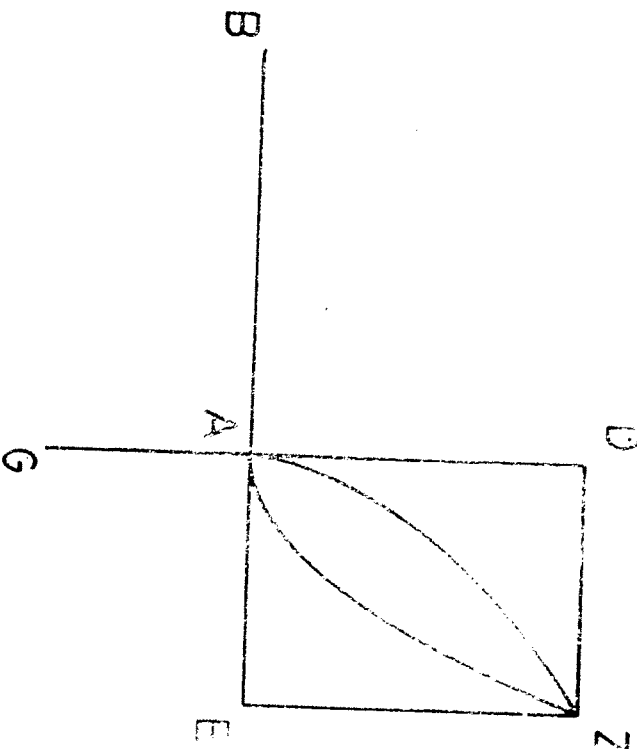


Figure 3

$D$ ,  $E$  two lines parallel to lines  $AE$ ,  $AD$  meeting at point  $Z$ , then the rectangle  $AB$ ,  $AE$  is equal to the square of  $EZ$ . Therefore the parabola with axis  $AE$  and parameter  $AB$  passes through point  $Z$ . Similarly , because the rectangle  $AG$ ,  $AD$  is equal to the square of  $ZD$ , the parabola with axis  $AD$  and parameter line  $AG$  passes through point  $Z$ . The two (conic) sections are known in position, so point  $Z$  is known in position.

Synthesis : If we make line  $AB$  the parameter of a (conic) section with vertex point  $A$  and axis line  $AE$ , namely (conic) section  $AZ$ , and if we draw another (conic) section with parameter line  $AG$  and axis line  $AD$ , then it will meet (conic) section (ms. diameter)  $AZ$  at point  $Z$ , and (if) we draw from the common point  $Z$  two ordinates, namely lines  $ZE$ ,  $ZD$ , then they are the two mean proportionals. Proof of this : Since (conic) section  $AZ$  is a parabola with parameter  $AB$ , the rectangle  $AB$ ,  $AE$  is equal to the square of  $AD$ . Thus the ratio of  $AB$  to  $AD$  is equal to the ratio of  $AD$  to  $AE$ . Similarly also , since the rectangle  $AG$ ,  $AD$  is equal to the square of  $AE$ , the ratio of  $AD$  to  $AE$  is equal to the ratio of  $AE$  to  $AG$ . But the ratio of  $AD$  to  $AE$  is equal to the ratio of  $AB$  to  $AD$  . Thus the ratio of  $AB$  to  $AD$  is equal to the ratio of  $AD$  to  $AE$  and equal to the ratio of  $AE$  to  $AG$  .

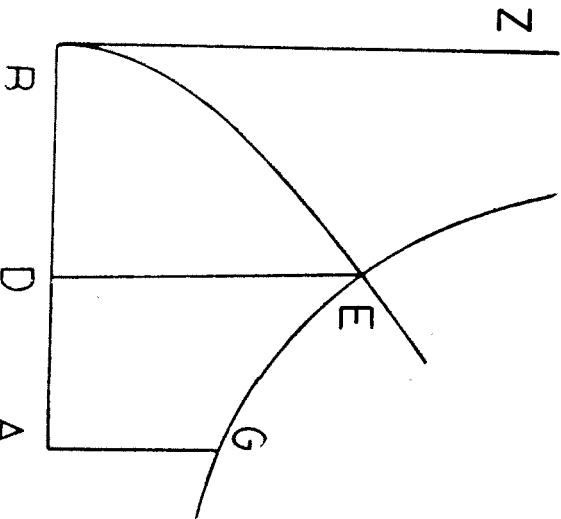


Figure 4

(Figure 4) We can also find this by means of a parabola and a hyperbola. Let us assume that this exists, by way of analysis, and let the ratio of  $AB$  to  $ED$  be equal to the ratio of  $ED$  to  $DB$  and equal to the ratio of  $DB$  to  $AG$ . Then the rectangle  $AB, BD$  is equal to the square of  $DE$ . Thus, if we make line  $DE$  parallel to line  $AG$ , the parabola with parameter line  $AB$ 's passes through point  $E$ . Since the rectangle  $AG, AB$  is equal<sup>18</sup> to the rectangle  $ED, DB$ , therefore if we draw through point  $B$  line  $BZ$  parallel to line  $DE$ , then the hyperbola drawn through point  $E$  with asymptotes lines  $AB, BZ$  passes through point  $E$ . It (point  $E$ ) is therefore assumed (i. e. given).

Synthesis : We assume two lines  $AB, AG$  containing an angle. We construct on line  $AB$  a parabola with vertex point  $B$  and parameter line  $AB$ 's. We draw from point  $B$  line  $BZ$  parallel to line  $AG$ , and we construct on point  $G$  a hyperbola with asymptotes lines  $AB, BZ$ , namely (conic) section  $GE$ , and let it meet the parabola at point  $E$ . Let us draw the ordinate  $DE$ . Then, since (conic) section  $GE$  is a hyperbola, the rectangle  $AG, AB$  is equal to the rectangle  $ED, DB$ . Thus the ratio of  $AB$  to  $DE$  is equal to the ratio of  $DB$  to  $AG$ . Since (conic) section  $BE$  is a parabola with parameter  $AB$ , the rectangle  $AB, BD$  is equal to the square of  $DE$ . Thus the ratio of  $AB$  to  $DE$  is equal to the ratio of  $DE$  to  $DB$ .

18. The ordinates of the parabola are assumed to be parallel to  $AG$ . The angle is not necessarily a right angle.

19. A marginal remark states : "this has been proven in the 19th proposition of the first section of the third species of < the fourth species >. This (extant) theorem of the *Istihendi* includes the equivalent of *Conics* II :12. Point  $E$  is on the hyperbola by the converse of *Conics* II:12.

(Figure 5) We can also find this by means of a parabola and a circle. Let the two lines be lines  $\angle AB \angle$ ,  $BG$ , and they contain a right angle. Let us draw through points  $A, B, G$  a circle. Let  $BG$  be the greater of the two lines, and let us make  $\angle it \angle$  the parameter and the axis of a parabola with vertex at point  $G$ . Let it meet the circle at point  $D$ . We draw from point  $D$  an ordinate  $DE$ . I say that lines  $DE, EG$  are the two mean proportionals. Proof of this: We join  $AD$ , and we extend it rectilinearly to meet line  $BG$  at point  $Z$ . We join  $DG$ . Then, since the angle at point  $D$  is (a) right (angle), the ratio of  $ZE$  to  $ED$  is equal to the ratio of  $ED$  to  $EG$ . Since  $BG$  is a parameter of (conic) section  $GD$ , the rectangle  $BG, GE$  is equal to the square of  $DE$ , so the ratio of  $BG$  to  $DE$  is equal to the ratio of  $DE$  to  $EG$ . Thus line  $BG$  is equal to line  $ZE$ . If we subtract the common (part)  $BE, ZB$  is equal to line  $EG$ . Thus the ratio of  $ZE$  to  $ED$  is equal to the ratio of line  $ED$  to line  $EG$ , and the ratio of line  $EZ$  to line  $ED$  is equal to the ratio of line  $BZ$  to line  $BA$ . Thus the ratio of line  $BG$  to line  $ED$  is equal to the ratio of line  $ED$  to line  $EG$ , and equal to the ratio of line  $EG$  to line  $AB$ .

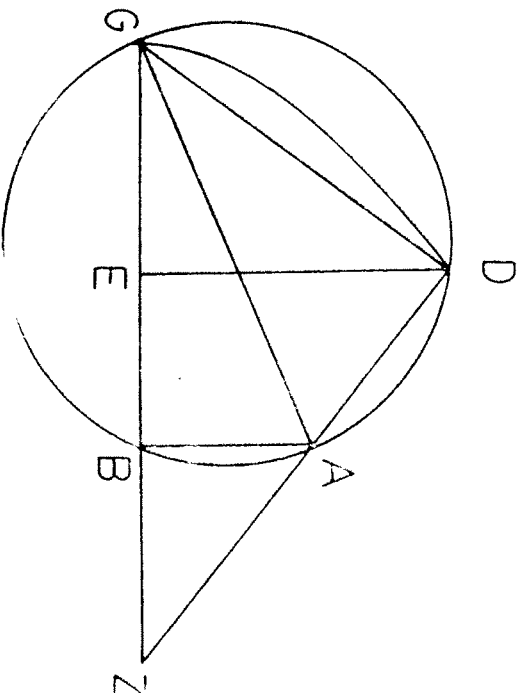


Figure 5

(Figure 6) This can also be demonstrated by means of a hyperbola and a circle. That is: if we let lines  $AB, BG$  (ms.  $AG$ ) contain a right angle, and (if) we complete parallelogram  $ABGD$ , and construct on it a circle, and construct on point  $A$  a hyperbola with asymptotes lines  $BG, GD$ , and let it meet the circle at point  $E$ . We join  $AE$  and extend it to meet lines  $CB, CD$  at points  $Z, H$ . Then I say that lines  $BZ, DH$  are as we wished.

