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TRACES OF THE LOST *GEOMETRICAL ELEMENTS* OF MENELAUS IN TWO TEXTS OF AL-SIJZĪ

JAN P. HOGENDIJK*

1. *Introduction and summary*

Menelaus was a mathematician and astronomer who lived around A.D. 100 in Rome.[1] His most important work was the *Spherics*, a treatise on the geometry of the sphere. The original Greek text of the *Spherics* is lost but the work has come down to us in various Arabic, Hebrew and medieval Latin translations. In the Greek and Arabic literature, reference is made to other works by Menelaus which have not come down to us. This paper is about one of these works, of which the Greek title is unknown, but which is entitled *Geometrical Elements* (*Uṣūl Handasiyya*) in Arabic.

Many conjectures have been made about the contents of this work,[2] but most of these are not supported by explicit references in ancient or medieval sources. To show how little is known about the *Geometrical Elements*, I have listed the published references to the work in Section 2 of this paper. From these references we can deduce the following information: The *Geometrical Elements* was a work in three books, which was known in the ninth century A.D. to Thābit ibn Qurra,[3] who wrote a commentary to it, and in the tenth century to Abū Ja'far al-Khāzin (ca. A.D. 950).[4] Al-Bīrūnī[5] describes a problem which Menelaus solved

* Mathematical Institute, Utrecht University, P.O. Box 80010, NL-3508 TA Utrecht.

[1] On Menelaus see Bulmer-Thomas and GAS V, 158–164.

[2] See, for example, Björnbo pp. 4–8; Bulmer-Thomas, pp. 299–300; Knorr, pp. 101, 115, 219; it has been conjectured that the *Geometrical Elements* contained an alternative proof of Euclid's *Elements* I:25 ascribed to Menelaus by Proclus; a proof of the plane version of the "theorem of Menelaus" (on a triangle and a transversal); a construction of two mean proportionals by Archytas; a version of this construction by means of a certain "paradoxical curve" mentioned by Pappus; some Archimedean theorems on chords; a trisection of the angle by means of a circle and a hyperbola, see Section 5.

[3] On the mathematical works of Thābit ibn Qurra see GAS V, 264–272, Matvievskaya and Rozenfeld vol. 2, pp. 85–103.

[4] On Abū Ja'far Muḥammad ibn al-Ḥusayn al-Khāzin see GAS VI, 189–190, GAS V, 298–300, 305–307.

[5] On al-Bīrūnī see e.g. GAS V, 375–383, Matvievskaya and Rozenfeld vol. 2, pp. 264–295.

in proposition 2 of book III of the *Geometrical Elements*, but he does not give the solution by Menelaus. Al-Bīrūnī says that Abū'l-Jūd (ca. A.D. 970)[6] wrote a treatise on this problem, which was criticized by al-Sijzī, who solved the problem in an easy way. This is all information on the *Geometrical Elements* that can be extracted from the published references.

In this paper I present some new information on the *Geometrical Elements* of Menelaus and its influence in the medieval Islamic tradition. My sources are two unpublished texts by al-Sijzī. Abū Saᶜīd Aḥmad ibn Muḥammad ibn ᶜAbdaljalīl al-Sijzī[7] was a very productive geometer and astrologer, who originated from Sijistān in South-Eastern Iran, and who was active in the second half of the tenth century A.D.

Section 3 is about a new trace of the *Geometrical Elements* of Menelaus, in al-Sijzī's hitherto unpublished *Treatise on the Properties of the Perpendiculars Drawn From a Given Point To a Given Equilateral Triangle, By Way of Determination.* This text shows that in the beginning of the *Geometrical Elements*, Menelaus studied the sum (or difference) of perpendiculars drawn from a given point to the sides of a given equilateral triangle.

In Section 4, I discuss the relation between *Geometrical Elements* III:2 and al-Sijzī's hitherto unpublished *Letter to Abū ᶜAlī Naẓīf ibn Yumn the Physician on the Construction of an Acute-Angled Triangle From Two Different Straight Lines.* In this letter al-Sijzī presents a ruler-and-compass construction of certain triangles, as an improvement over the construction by his contemporary Abū Saᶜd al-ᶜAlāʾ ibn Sahl, who had used an ellipse. This construction is to be found in Text 3. Thus *Geometrical Elements* III:2 was related to a series of studies by geometers in the Arabic-Islamic tradition.

In Section 5, I will use the new evidence to disprove a suggestion by the late Wilbur Knorr that the *Geometrical Elements* contained a trisection of the angle by means of conic sections. The *Geometrical Elements* seems to have been a collection of elementary theorems and problems on straight lines, circles and triangles in the plane. The Arabic-Islamic geometers were very interested in such theorems and problems, and the influence of the *Geometrical Elements* of Menelaus in the Arabic-Islamic tradition was probably much more extensive than the few extant references suggest. Further study and analysis of unpublished Arabic texts may well reveal more details about the contents of this work, and one cannot exclude the possibility that Arabic fragments of the *Geometrical Elements* will be discovered in the future.

[6] On Abū'l-Jūd see GAS V, 353–355, Matvievskaya and Rozenfeld vol. 2, pp. 260–262, on his date see Hogendijk 1, p. 243.

[7] On al-Sijzī see GAS V, 329–334, GAS VI, 224–226, GAS VII, 177–182, 409–410, al-Sijzī pp. vii–ix.

The Appendix to this paper contains the Arabic texts with English translations of the following treatises:

Text 1: Al-Sijzī's *Treatise on the Properties of the Perpendiculars Drawn From a Given Point To a Given Equilateral Triangle, By Way of Determination.* This text is special since it contains a very large geometrical figure (Figure 10 below), in which no less than 46 points are labeled by letters.

Text 2: Al-Sijzī's *Letter to Abū ᶜAlī Naẓīf ibn Yumn the Physician on the Construction of an Acute-Angled Triangle From Two Different Straight Lines.*

Text 3: A fragment of al-Sijzī's treatise *On the Selected Problems Which Were Discussed by Him and the Geometers of Shīrāz and Khorāsān, and His Annotations*, containing a construction by Abū Saᶜd al-ᶜAlāʾ ibn Sahl by means of an ellipse.

2. *References to the* Geometrical Elements *in published sources*

No explicit references to the *Geometrical Elements* have been found in ancient Greek texts. The work is mentioned in three medieval Arabic sources which have been published thus far. I now render these three references in Arabic with English translation. I will not discuss references in the Greek and Arabic literature to unspecified geometrical works by Menelaus, because it is uncertain that any of these references concern the *Geometrical Elements*.[8]

[8] For example, in the *Book on the Measurement of Plane and Spherical Figures* (GAS V, 251 no. 1) the Banū Mūsā present a construction of two mean proportionals. The Banū Mūsā say:

و هذا العمل لرجل من القدماء اسمه مانالاوس أورده في كتاب له في الهندسة.

"This construction is by a man among the ancients, whose name is Menelaus, and who presented it in a book by him on geometry." See al-Ṭūsī no. 1, p. 20, line 2–3. It does not follow that this book was the *Geometrical Elements*. The construction is essentially the same as that by Archytas (fourth century B.C.) which has been transmitted in a Greek version in the commentary by Eutocius on Archimedes' *On the Sphere and Cylinder* II:1.

1. The bio-bibliographer Ibn al-Nadīm (ca. 970) says in the *Fihrist* that Menelaus wrote:

كتاب أصول الهندسة عمله ثابت بن قرة ثلاث مقالات، كتاب المثلّثات وخرج منه إلى العربي شيء يسير.

"The book on the Elements of Geometry, which Thābit ibn Qurra made (i.e. edited) in three treatises (i.e. Books); the Book of the Triangles, of which a small part came out in Arabic."[9] In a passage quoted below, al-Bīrūnī says that Thābit ibn Qurra commented on the *Geometrical Elements*. Thus it is likely that Thābit revised the existing translation of the three Books of the *Geometrical Elements* and added his own commentary. The reference to the *Book on Triangles* shows that Menelaus wrote other works on elementary plane geometry as well.

2. Around A.D. 1000 Abū Naṣr ibn ʿIrāq wrote a letter to al-Bīrūnī in which he corrected some errors in the *Astronomical Handbook of the Plates* (Zīj al-Ṣafā'iḥ) by Abū Jaʿfar al-Khāzin (ca. A.D. 950).[10] Abū Naṣr says that even excellent mathematicians can make mistakes, and he gives the following example:

وأبو جعفر نفسه استدرك على مانالاناوس في كتابه المرسوم بالأصول الهندسيّة غلطًا أو سهوًا وقع له.

"And Abū Jaʿfar al-Khāzin himself corrected a mistake or omission which Mānālānāwus (i.e. Menelaus) happened to make in his book called the *Geometrical Elements*."[11]

Unfortunately Abū Naṣr does not give us more information on Abū Jaʿfar's criticisms.[12]

3. In the *Extraction of Chords* al-Bīrūnī mentions the *Geometrical Elements* of Menelaus in the following passage:

إخراج خطين من نقطتين مفروضتين يحيطان بزاوية مفروضة يساوي مجموعهما خطًا مفروضًا. إن مانالاوس رام في الشكل الثاني من المقالة الثالثة من كتابه في الأصول الهندسيّة أن يبيّن كيف نعطف في نصف دائرة مفروضة خطًّا منعطفًا مساويًا لخطّ مفروض فسلك إليه مسلكًا طويلًا جدًّا ثمّ عمله ثابت بن قرّة حين فسر ذلك الكتاب بعمل في طول عمل مانالاوس. فأمّا بعد تقديم ما تقدّم من خاصية الخط المنحني في تقعير كل قوس فقد يسهل عمل ما رامه مانالاوس ويكون عامًا في جميع قسي الدائرة المفروضة دون نصفها فقط.

وإن أبا الجود أفرد لهذا المعنى مقالة واستخرجه بطريق تجاوز كل طوالة وصعوبة. فلما وقف عليها أبو سعيد السجزي استخرجه بطريق هو في نهاية السهولة ولن نقصر عنه فيها.

"To construct two lines from two assumed points such that they (the lines) contain an assumed angle and their sum is equal to an assumed line. Menelaus wanted to show in the second proposition of the third book of his *Geometrical Elements*, how we inscribe in a given semicircle a broken line which is equal to a given line, but he used a very long method. Then Thābit ibn Qurra constructed it when he commented on this book, by means of a construction of the same length as the construction of Menelaus. By means of the above-mentioned property of the broken line in the concavity of any arc, the construction of what Menelaus wanted is easy, and it is general, for all arcs of the circle, not only for the semicircle. Abū'l-Jūd devoted a treatise to this problem and he solved it in an outrageously long and difficult way. When Abū Saʿīd al-Sijzī saw it (the treatise), he solved it (the problem) in an extremely easy way. Our solution will not be less (easy) than his."[13]

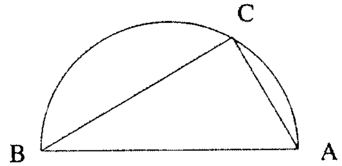

Figure 1

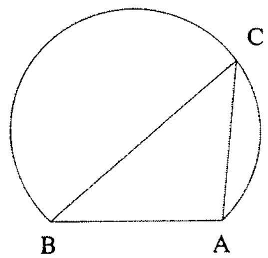

Figure 2

The problems mentioned by al-Bīrūnī can be illustrated by Figures 1 and 2. In Figure 1, AB is the diameter of a semicircle. Menelaus constructed a "broken line" ACB with C on the semicircle such that $AC + CB$ is equal to a given length.[14] Since any point C is on the

[9] See al-Nadīm p. 327.

[10] On Abū Naṣr ibn ʿIrāq see GAS V, 338–341, GAS VI, 242–245, the letter in question is mentioned in GAS VI, 243 no. 3. A manuscript of the *Zīj al-Ṣafā'iḥ* has been found recently, see Samsó, pp. 594–601.

[11] Abū Naṣr no. 3, p. 3, lines 13–14.

[12] The other references to Menelaus in this letter by Abū Naṣr concern the *Spherics* (al-Kuriyyāt) (see Abū Naṣr no. 3, p. 5, line 10).

[13] Al-Bīrūnī no. 1, pp. 49–50; Suter, pp. 31–32.

[14] In his translation of al-Bīrūnī's *Extraction of Chords*, Suter makes the following curious remark (p. 69): "Die Aufgabe, die Menelaus im 2. Satze des 3. Buches seiner

semicircle if and only if $\angle ACB = 90°$,[15] the problem can also be stated thus: to construct two lines AC, BC from two assumed points (A, B) such that they contain a right angle ($\angle ACB$) and their sum is equal to a given line. In the beginning of the quoted passage al-Bīrūnī mentions a more general problem, which can also be stated in two equivalent forms (Figure 2): (a) Let A and B be two given points. To construct point C such that $AC + CB$ is a given length and $\angle ACB$ is a given angle (which need not be a right angle); (b) Let A and B be two given points on a given circle. To construct point C on the circle such that $AC + CB$ is equal to a given length. The two forms of the problem are equivalent as a consequence of Euclid's *Elements* III:26–27.[16] I will discuss al-Sijzī's solution of this problem in Section 4 of this paper. The treatise by Abū'l-Jūd is lost, but the story that al-Sijzī criticized him is plausible because he and al-Sijzī were enemies and al-Sijzī also criticized him elsewhere.[17]

3. *A new trace of the* Geometrical Elements *of Menelaus*

A new trace of the *Geometrical Elements* of Menelaus is to be found in al-Sijzī's *Treatise on the Properties of the Perpendiculars Drawn From a Given Point To a Given Equilateral Triangle, By Way of Determination*, that is Text 1 in the Appendix to this paper. Al-Sijzī begins Text 1 by saying that "Mayālāwūs mentioned in the beginning of his book on the *Geometrical Elements* the property of equality (resulting) from drawing the perpendiculars in an equilateral triangle to its perimeter. He followed in this the method of division (of the problem into special cases) but the necessary division was not exhaustive. So I intended to follow in this the method of division (into cases) according to the (different) position of the points."

Mayālāwus ميالاوس is evidently an error for Manālāwus منالاوس, caused by misplacement of one diacritical mark, but we do not know

Elemente gelöst hat, muß aber doch etwas anders gelautet haben, denn sonst könnte er kaum 'einen sehr langen Weg' für die Lösung gebraucht haben." Perhaps Suter thought that a Greek geometer could not give a long solution of an easy problem. See, however, Section 4 for a conjecture about a problem which may have been treated in the *Geometrical Elements* and for which proposition 2 of Book III may have been a preliminary.

15 Euclid, *Elements* III:31, Heath vol. 2, pp. 61–63.

16 Heath vol. 2, pp. 56–59.

17 On the relations between Abū'l-Jūd and al-Sijzī see Hogendijk 1, Chapter 5.

when and by whom the error was made, and al-Sijzī may have believed that Mayālāwus was the right spelling.

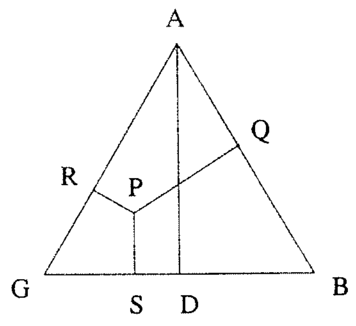

Figure 3

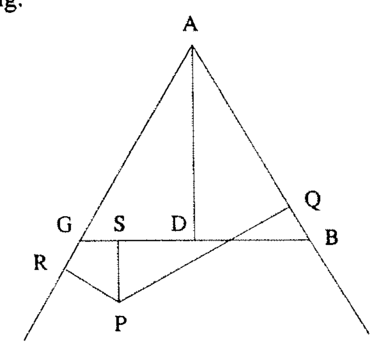

Figure 4

In the rest of Text 1, al-Sijzī discusses the three perpendicular distances from any point P to the sides of a fixed equilateral triangle. If P is inside the triangle, al-Sijzī proves that the sum of these three distances is constant and equal to the altitude of the triangle, that is to say, $PQ + PR + PS = AD$ in Figure 3. He divides the proof into four cases: (1) P is an angular point of the triangle, (2) P is on a side of the triangle, (3) P is on an altitude of the triangle, (4) P is inside the triangle but not on an altitude. Al-Sijzī also discusses the case where P is outside the triangle, as in Figure 4; in that case the sum of two perpendicular distances *minus* the third distance is equal to the altitude of the triangle, that is to say $PQ + PR - PS = AD$. Here al-Sijzī distinguishes 7 cases. Al-Sijzī tacitly assumes that point P is inside one of the angles of the triangle; Figure 4 is drawn for the case where P is inside angle BAG, with AB and AG extended indefinitely. We now raise the question, which cases were treated by Menelaus in his lost *Geometrical Elements*. Al-Sijzī's preface shows that Menelaus proved the theorem at least for points P inside the triangle as in Figure 3. For an ancient or medieval mathematician, who did not work with negative quantities, the generalization of the theorem to points P outside the triangle as in Figure 4 is not at all straightforward. If al-Sijzī had discovered this generalization, he would probably have said so.[18] al-Sijzī's statement that

18 In other works, al-Sijzī claims authorship of constructions which he adapted with very minor modifications from other geometers. For example, he adapted the trisection of the angle by means of a hyperbola from Abū Sahl al-Kūhī, see Woepcke p. 118, Knorr pp. 293–309; I disagree with Knorr's suggestion on pp. 308–309 that al-Kūhī's

"the necessary division (of cases) was not exhaustive" suggests that his own contribution consisted of the elaboration of cases which Menelaus had not treated explicitly. I conclude that Menelaus stated and proved the theorem in some form for points P outside the triangle, but that he did not make the explicit distinction into 7 cases which al-Sijzī made in Text 1.

If point P is inside an angle opposite to one of the angles of the triangle, as in Figure 5, we have $PQ - PR - PS = AD$. Al-Sijzī missed the theorem for this case. It is likely that Menelaus did not treat the theorem explicitly for this case; he may have missed it as well.

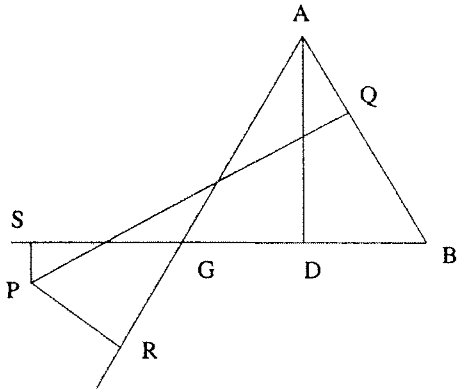

Figure 5

For points P inside the triangle as in Figure 3, the theorem is also stated and proved in an ancient Greek treatise which survives in an Arabic translation under two different titles, namely the *Book of the Geometrical Elements* of (Pseudo-)Archimedes, and *Book of Assumptions* of Aqāṭun.[19] My conclusion that the *Geometrical Elements* of Menelaus contained the theorem for Figure 4, agrees with the conclusion of Yvonne Dold-Samplonius[20] that Menelaus is not the same author as Aqāṭun or Pseudo-Archimedes, who did not give the theorem for P outside the triangle.

Ibn al-Haytham (ca. 960–ca. 1040) proved the theorem for points P inside the equilateral triangle and he generalized the theorem for points P inside an isosceles triangle.[21] Ibn al-Haytham did not discuss the case where P is outside these triangles, so it is likely that he did not know the *Geometrical Elements* of Menelaus.

trisection is an adaptation of the trisection in Pappus of Alexandria's *Mathematical Collection* IV:43.

19 See Hermelink and Dold-Samplonius, propositions 8–10, pp. 21–23.

20 See Dold-Samplonius p. 12.

21 See Hermelink.

4. *Al-Sijzī's* Letter to Abū ʿAlī Naẓīf ibn Yumn *and the* Geometrical Elements *of Menelaus*

Al-Bīrūnī's discussion of *Geometrical Elements* III:2 has been quoted in Section 2 of this paper. There al-Bīrūnī discussed a problem which is as follows in modern notation (Figure 6): Let A and B be two given points, c a given length (i.e. line segment) and α a given angle. To construct point C such that $AC + CB = c$ and $\angle ACB = \alpha$. Menelaus solved this problem for $\alpha = 90°$, that is to say that C is on the semicircle with diameter AB.

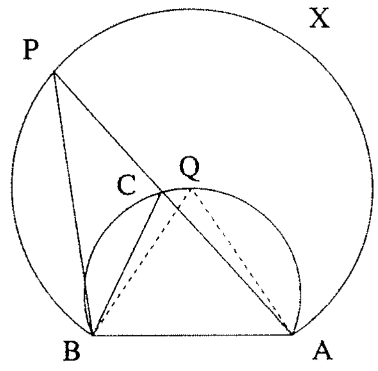

Figure 6

Al-Bīrūnī says that al-Sijzī solved the problem (for arbitrary α) in an easy way. It is not difficult to reconstruct al-Sijzī's solution, either from the indications given by al-Bīrūnī himself, or from various texts by al-Sijzī, such as Text 2 below.[22] It is as follows (cf. Figure 6):

Construct the circular arc through A and B which "contains" an angle equal to $\frac{\alpha}{2}$. This arc is, in modern terms, the collection of all points X on one side of line AB such that $\angle AXB = \frac{\alpha}{2}$. The construction of this circular arc is explained in Euclid's *Elements* III:33.[23] Then draw the circle with centre A and radius c, let it intersect the arc at P. To avoid confusion, the circle with centre A and radius c is not drawn in Figure 6, but its point of intersection P with circle AXB is displayed. Draw PA. Draw the circular arc through A and B and "containing" an angle α. Let AP intersect this arc at C, and draw CB and PB. Since $\angle ACB = \angle CPB + \angle CBP$, $\angle ACB = \alpha$, and $\angle APB = \frac{\alpha}{2}$, it follows that $\angle CBP = \frac{\alpha}{2} = \angle CPB$, so $CP = CB$. Thus $AC + CB = AC + CP = AP = c$, as required. The construction is

22 The construction is also found in al-Sijzī's treatise *On the Selected Problems Which Were Discussed by Him and the Geometers of Shīrāz and Khorāsān, and His Annotations*, GAS V, p. 333 no. 23, see Text 3 below. There, al-Sijzī tried to use this construction in order to inscribe a triangle of given perimeter in a given circle. His solution of this problem was incomplete. See MS. Dublin, Chester Beatty 3652, ff. 39a:30–39b:4, MS. Istanbul, Reşit 1191, ff. 39a:20–39b:12.

23 Heath vol. 2, pp. 67–70.

possible if $c \leq 2AQ$, where AQB is the isosceles triangle with AB as base and $\angle AQB = \alpha$ (dotted lines in Figure 6). For $c < 2AQ$ the two solutions C are located symmetrically with respect to Q.

Text 2 in the Appendix is al-Sijzī's hitherto unpublished *Letter to Abū ʿAlī Naẓīf ibn Yumn the Physician on the Construction of an Acute-Angled Triangle From Two Different Straight Lines* (GAS V, 332 no. 13). Naẓīf ibn Yumn was a Christian theologian, physician and mathematician who lived in Baghdād (GAS V, 313–314). Although Text 2 does not contain an explicit reference to the *Geometrical Elements* or to Menelaus, the text is of interest here for two reasons. First, its mathematical contents are closely related to al-Sijzī's solution outlined above, and hence to the theorem which Menelaus solved in *Geometrical Elements* III:2. Thus Text 2 may shed light on the possible context of this problem. Secondly, since Text 2 was written A.H. 359 / A.D. 970, it is likely that al-Sijzī had read the *Geometrical Elements* by that time. This information will be useful in Section 5 of this paper.

In Text 2, al-Sijzī discusses the construction of an acute-angled triangle from "two different straight lines" AB and AG, with $AG > AB$. This is a triangle ABZ with acute angles such that $AZ + ZB = AG$ (see Figure 7).

Al-Sijzī draws the segment AG perpendicular to AB. He constructs the circular arc through points A, B, "containing" an angle of 45 degrees. He draws perpendiculars BD and AH to intersect this arc at D and H and he notes that AD is a diameter of the circle to which this arc belongs. (Note that $ABDH$ is a square, $\angle ADB = 45°$, and $AG > AH$.) He then draws a second circle with centre A and radius AG. In Figure 7 the two circles intersect at two points K and T, but it is also possible that the two circles are tangent at D or do not intersect. Al-Sijzī gives figures for these three cases, see the Appendix, Figures 11–13.

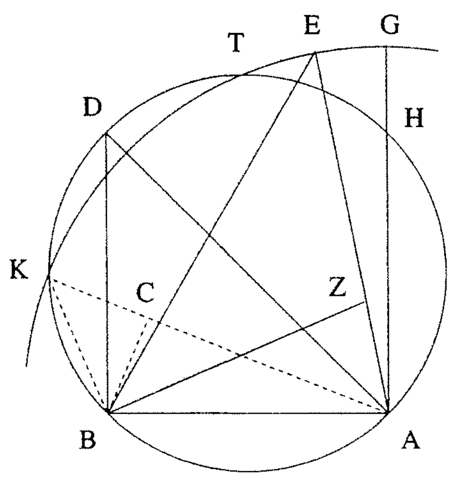

Figure 7

For any point E on the second circle but outside the first circle, we have $\angle AEB < 45°$. Al-Sijzī constructs Z on AE such that $EZ = ZB$ and he shows that $\angle AZB = \angle ZEB + \angle ZBE = 2\angle AEB < 2 \cdot 45° = 90°$. If point E is located in the strip between AG extended and BD extended, the angle ZAB is also acute. Al-Sijzī claims that one obtains all possible acute-angled triangles ABZ such that $AZ + ZB = AG$ by this method. This claim is incorrect, but it is true that his method provides all acute-angled triangles ABZ such that $AZ + ZB = AG$ and $AZ < ZB$. A triangle AZB such that $AZ = ZB = \frac{1}{2}AG$ is very easy to construct, and triangles AZB such that $AZ > ZB$ and $AZ + ZB = AG$ can be constructed in the way of al-Sijzī by interchanging B and A. Thus Text 2 is really about the construction of *all* acute-angled triangles ABZ such that $AZ + ZB = AB$. The problem to construct *one* acute-angled triangle AZB such that $AZ + ZB = AG$ is much easier, because one can take the isosceles triangle with $AZ = ZB = \frac{1}{2}AG$ and one does not need al-Sijzī's solution.

Text 2 is in the following way related to al-Sijzī's solution of the problem in *Geometrical Elements* III:2 of Menelaus: If in Figure 7 we choose $E = T$ or $E = K$, a point of intersection of the two circles, we have $\angle ATB = \angle AKB = 45°$. Thus we obtain triangle ACB with $\angle C = 90°$, $AC + CB = AG$, hence a solution of the problem of Menelaus for $c = AG$. Figure 7 displays in dotted lines the case $E = K$. Figure 7 can also be used to construct triangle ABZ with $\angle AZB = \alpha < 90°$, $AZ + ZB = c = AG$. All we have to do is to draw the cirular arc through A and B "containing" an angle $\frac{\alpha}{2}$ and taking E as a point of intersection of this arc and the circle with centre A and radius c, that is circle GTK in Figure 7. Points E and Z in Figure 7 then correspond to points P and C in Figure 6. Thus Text 2 is closely related to al-Sijzī's solution of the problem of Menelaus.

We now discuss the historical background of this problem. In Text 2 al-Sijzī says that he will be dealing with the construction of "an acute-angled triangle from two different straight lines" (i.e. AB and $c = AG$), and that he had also solved this problem in his work *On Triangles*, now lost, but known to Naẓīf ibn Yumn.[24] Al-Sijzī says that the problem had also been solved by Abū Saʿd al-ʿAlāʾ ibn Sahl by means of an ellipse. Al-Sijzī discusses the solution by al-ʿAlāʾ not in Text 2 but in another unpublished treatise, and the relevant part has been included in this paper as Text 3.

[24] Note that Menelaus also wrote a work called *On Triangles*, which is mentioned by Ibn al-Nadīm in the *Fihrist* in the passage quoted in Section 2.

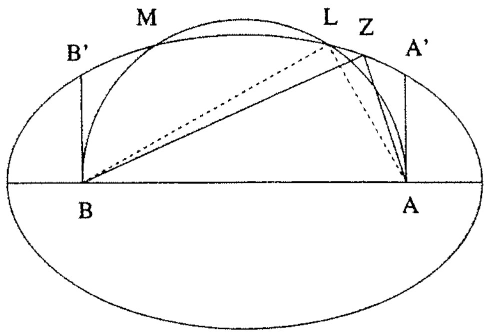

Figure 8

In the notations of Figure 7, the idea is as follows (see Figure 8). Consider the ellipse with foci A and B and major axis $c > AB$, draw the semicircle with diameter AB, and draw perpendiculars AA' and BB' to meet the ellipse at points A' and B'. If point Z is on arc $A'B'$ of the ellipse, but outside the circle, triangle AZB will be acute-angled and $AZ + ZB = c$, as required. The intersections L, M of the ellipse and the semicircle provide the solutions to the problem of Menelaus to construct a "broken line" ALB equal to c in the semicircle. Again, we see that al-ʿAlāʾ dealt with the construction of *all* triangles AZB such that $AZ + ZB = c$.

The relations between Text 2, al-Sijzī's solution of the problem of Menelaus, and the text of al-ʿAlāʾ suggest that the problem "to construct the (i.e. every) acute-angled triangle from two different straight lines" may have occurred in the *Geometrical Elements* as well, and that *Geometrical Elements* III:2 was a preliminary to the solution. Al-Bīrūnī informs us that the solution by Menelaus (probably by ruler and compass) was complicated, and therefore al-ʿAlāʾ must have preferred his own elegant solution by means of the ellipse. Al-Sijzī then showed that a simple ruler-and-compass construction was possible, although he did not work out all details in a satisfactory way. Thus the *Geometrical Elements* of Menelaus may have stimulated an interesting series of investigations by Arabic-Islamic geometers.

5. *The* Geometrical Elements *did not contain a trisection of the angle by means of conic sections*

In *Textual studies in Ancient and Medieval Geometry*, pp. 219, 288–289, the late Wilbur Knorr suggests that the *Geometrical Elements* of Menelaus contained a trisection of the angle by means of a circle and a hyperbola. Knorr argued that this trisection was adapted from this work by Thābit ibn Qurra in his own treatise on the trisection of the angle (GAS V, 271 no. 16). I will now argue that this suggestion is incorrect by means of Texts 1 and 2 and al-Sijzī's treatise on the trisection of the angle (GAS V, 331 no. 7) which was summarized by Franz Woepcke in his edition of the *Algebra* of ʿUmar al-Khayyāmī. My argument is as follows.

Text 1 shows that al-Sijzī knew the *Geometrical Elements* of Menelaus at some stage in his career, and Text 2 strongly suggests that he knew this work in A.D. 970 / 359 H. In the treatise on the trisection, al-Sijzī cites the work of al-Bīrūnī, who was born in A.D. 972 / 362 H. The treatise on the trisection is therefore much later than Text 2 (and probably also much later than Text 1), and we can therefore assume that al-Sijzī knew the *Geometrical Elements* of Menelaus when he wrote his treatise on the trisection of the angle.

In the treatise on the trisection of the angle, al-Sijzī says that the ancient geometers were unable to trisect the angle, and that the only modern geometers who could solve this problem were Thābit ibn Qurra and Abū Sahl al-Kūhī. In the same text, al-Sijzī renders one "proposition by Thābit ibn Qurra." This proposition is evidently taken from Thābit's text on the trisection of the angle.[25] It follows that al-Sijzī knew Thābit's trisection by means of a circle and a hyperbola and considered it to be a correct solution. If Menelaus' *Geometrical Elements* contained the same trisection as Thābit's text, as suggested by Knorr, or any other trisection of the angle by means of conic sections, al-Sijzī would not have said in his own treatise on the trisection of the angle that the ancients were unable to trisect the angle. I conclude that the *Geometrical Elements* of Menelaus did not contain a trisection of the angle by means of conic sections.

Thus there is no evidence that the *Geometrical Elements* contained anything else than ruler-and-compass constructions and theorems on straight lines, triangles and circles in plane geometry.

[25] See Woepcke, pp. 117–118.

Appendix: Arabic texts and translations

Text I

Treatise of Aḥmad ibn Muḥammad ibn ʿAbdaljalīl al-Sijzī
on the Properties of the Perpendiculars Drawn From a Given Point
To a Given Equilateral Triangle, By Way of Determination.

The following edition and translation is based on the manuscripts Dublin, Chester Beatty Library 3562, ff. 66b–67b (Arberry vol. 3, p. 60), and Istanbul, Süleymaniye Library, Reşit 1191, ff. 124b–125b (GAS V, 333 no. 19). The Dublin manuscript is dated Friday 7 Ramaḍān 611 H. = January 9, 1215 A.D.

The manuscripts contain numerous scribal errors, and some passages had to be restored to make mathematical sense. Such passages appear in angular brackets < ... > in the text and the translation. The text is followed by a critical apparatus, in which I have indicated the manuscripts by the symbols د (for Dublin) and ر (for Reşit). I have made some changes in the orthography, but I have not corrected grammatical errors in the text. Only the Dublin manuscript indicates by slashes the letters in the text referring to points in geometrical figures.

My own explanatory additions to the translation are in parentheses.

The manuscript contains 11 figures (my Figures 9–I to 9–XI), displaying special cases of the theorem in the text. The Roman numbers I, II, ..., XI which appear in these figures and in the text are my transcriptions of the Arabic alphabetical numbers in the figures in the Dublin manuscript. These 11 figures are derived from an enormous composite figure at the end of the text (Figure 10). This figure is special because no less than 46 points are labelled by a letter. Al-Sijzī first used single letters of the Arabic alphabet to label these points. He used these letters roughly in order of increasing numerical values, with $W = 6$ and $Y = 10$ as exceptions.[26] I now list the letters of the Arabic alphabet and their numerical values, together with the transcriptions I have used, following the system of E.S. Kennedy. ا $= 1 = A$, ب $= 2 = B$, جـ $= 3 = G$, د $= 4 = D$, هـ $= 5 = E$, و $= 6 = W$, ز $= 7 = Z$, ح $= 8 = H$, ط $= 9 = T$, ي $= 10 = Y$, ك $= 20 = K$, ل $= 30 = L$, م $= 40 = M$, ن $= 50 = N$, س $= 60 = S$, ع $= 70 = O$, ف $= 80 = F$, ص $= 90 = C$, ق $= 100 = Q$, ر $= 200 = R$, ش $= 300 = X$, ت $= 400 = U$, ث $= 500 = \Theta$, خ $= 600 = J$, ذ $= 700 = \Phi$, ض $= 800 = \Sigma$, ظ $= 900 = V$, غ $= 1000 = I$.

After he had exhausted the Arabic alphabet, al-Sijzī continued with two-letter combinations يا $= YA$, يب $= YB$, يجـ $= YG$ and يد $= YD$. These combinations have numerical values 11, 12, 13 and 14. For sake of clarity, I have used the symbols α, β, γ and δ instead of YA, YB, YG, YD. Al-Sijzī then abandoned the principle of numerical values and used the symbols EE, KK, LL, MM, FF, II for the remaining points which he needed. Perhaps he chose these double letters because these combinations do not look the same as the two single letters in Arabic. For example, هه is not the same as ه plus ه, كك is not the same as ك plus ك, etc. I have transcribed the double letters by a lower-case letter, thus: k for KK, ℓ for LL, i for II, e for EE, m for MM, f for FF.

[26] The letters و and ي were infrequently used in diagrams in Arabic texts, perhaps because the corresponding Greek letters digamma (= 6) and iota (= 10) occur only rarely in Greek mathematical diagrams. The impopularity of و=6 in diagrams may also be due to the possible confusion with the Arabic word و meaning "and". For details on the correspondence between Arabic and Greek letters in geometrical figures, see Toomer vol. 1, pp. xci–xciii.

بسم الله الرحمن الرحيم رب تمّم[1]

قول أحمد بن محمد بن عبد الجليل السجزي في خواص الأعمدة الواقعة من النقطة المعطاة إلى المثلث المتساوي الأضلاع المعطى بطريق التحديد.

إن ميلاوس ذكر في أول كتابه في الأصول الهندسية خاصة مساواة من إخراج الأعمدة في المثلث المتساوي الأضلاع إلى محيطه وسلك فيه طريق القسمة إلا أنه لم يستوفي في ذلك أمرًا ما وجب من القسمة. فقصدت أن أسلك فيه طريق[2] القسمة بوقوع النقط وأبرهن لذلك بطريق قريب في شكل واحد ليقع وقوع النقط تحت حس البصر ويسهل على مَن نظر في ذلك ما يحدث من خواص الأعمدة باختلاف وقوع النقط. وبالله التوفيق وهو حسبني ونعم الوكيل.

فليكن المثلث المتساوي الأضلاع المعطى ابج و العمود المخرج من زاوية ا إلى قاعدة بج عمود اد. المطلوب أن كل نقطة معطاة في داخل المثلث أو على زاوية أو على أحد أضلاعه إذا أخرج منها أعمدة إلى أضلاع المثلث الثلاثة فهي مساوية لعمود اد. وإذا فرضت النقطة خارج المثلث وأخرجت الأعمدة الثلاثة

إلى أضلاع المثلث أو إلى استقامته فإن العمودين الواقعين على ضلعي المثلث المتساوي الأضلاع أو على استقامته زائدان على عمود اد بالعمود[3] الواقع على قاعدة المثلث أو على استقامته.

فلنعدد وقوع النقط فنقول إن النقطة تقع من المثلث إما على الزاوية منه وإما على الضلع منه وإما داخل المثلث وإما خارجًا عنه. أما الذي يقع داخل المثلث فيقع إما على عمود اد وإما على غير عمود اد . فأما الذي يقع خارج المثلث فيقع إما على أضلاعه الثلاثة وإما على استقامة أضلاعه الثلاثة وإما بعضها على أضلاعه وبعضها على استقامة أضلاعه وذلك إما أن يقع العمود على قاعدة المثلث أعني خط بج والعمودين الباقيين على استقامة ضلعي اب اج وإما أن يقع العمود على أحد ضلعي اب اج والباقيين على استقامة قاعدة بج وعلى استقامة أحد ضلعي اب اج وإما أن يقع العمودان على قاعدة بج وعلى أحد ضلعي اب اج والعمود الباقي على استقامة أحد ضلعي اب اج .

آ[4] . فإن وقعت النقطة على الزاوية كنقطة ب فلنخرج بل على ضلع[5] من أضلاع مثلث ابج فخط بل مساو لعمود اد .

ب[6] . وإن وقعت النقطة على ضلع من أضلاع المثلث كنقطة ه فلنخرج عمودي هم هن على ضلعي اب اج وليس يخرج عمود ثالث سواهما. فلأن هم عمود مثلث متساوي الأضلاع يكون ضلع المثلث به و هن عمود مثلث متساوي الأضلاع يكون ضلع المثلث هج فنسبة هم[7] إلى به كنسبة هن إلى هج وكنسبة اد إلى بج . ففي التركيب تكون نسبة هم هن إلى به هج كنسبة اد إلى بج فخطا هم هن مساويان لخط اد .

ج[8] . وإن وقعت النقطة على خط اد كنقطة ز فلنخرج عمودي زس زع[9] على ضلعي اب اج فلنخرج خط ذزض[10] موازيًا لخط بج فخطا زس زع يعدلان خط از فخطوط زس زع زد تعدل از زد أعني خط اد .

د[11] . وإن وقعت النقطة في سطح ابج كنقطة ح فلنخرج خط[12] ذحض موازيًا لخط بج ولنخرج أعمدة حف حص[13] حق على أضلاع اب اج بج فعلى ما بينا خطا حف حص[14] يعدلان عمود از وخط حق مواز لخط زد ومساو له فخطوط حف حص[15] حق تعدل خطي از زد أعني خط اد .

ه[16] . وإن وقعت النقطة خارج المثلث فلنخرج من نقطتي ب ج عمودي بط جط[17] على ضلعي اب اج ولنخرج اب اج إلى غير نهاية[18] ونخرج بط جط حتى يلقى اظ اغ على نقطتي ظ غ ولنخرج عمودي ب غغ ج هه .

و < إن > وقعت < النقطة >[19] على نقطة ط أقول إن عمودي طب طج زائدان على عمود اد بعمود طد. برهانه إن نسبة اج إلى جد كنسبة اط إلى طج و اج ضعف جد ف اط ضعف طج ف بط طج يعدلان اط فيزيدان على اد خط دط.

ز[20] . وإن[21] وقعت النقطة على عمود دط كنقطة ك فنخرج عمودي كر كش فخطا كر كش يقعان في المثلث المتساوي الأضلاع الذي عموده اك < فهما إذًا مساويان لخط اك فهما إذًا زائدان على عمود اد بعمود كد . وإن وقعت النقطة على خط طج كنقطة ي فنخرج عمودي يت يث فخطا يت يث يقعان في المثلث المتساوي الأضلاع الذي عموده اك >[22] فهما إذًا مساويان لخط[23] اك فهما إذًا زائدان على عمود اد بعمود كد وخط يث[24] مساو لخط كد فهما زائدان على خط اد بخط يث[25] .

ز[26] . وإن وقعت النقطة داخل مثلث بحط كنقطة و فلنخرج أعمدة وت وش وق ونخرج وخ موازيًا ﻟ بج فلأن عمودي وت وش يقعان في المثلث المتساوي الأضلاع الذي عموده اخ فهما مساويان لعمود اخ لكن وق يعدل دخ فهما إذًا زائدان على عمود اد بعمود وق .

ح[27] . وإن وقعت النقطة فيما بين خطوط غغ م ط فف هه تحدث أعمدة على قاعدة بج وعلى استقامة ضلعي اب اج أعني بظ جغ كنقطة يا وأعمدة يايد يايج يايب.

ط[28] . وإن وقعت النقطة فيما بين مثلث ج فف غ فتحدث أعمدة على استقامة قاعدة بج أعني ج يج وعلى استقامة ضلع اج اعني جغ وعلى ضلع اب كنقطة يا وأعمدة يايد يايج يا يب .

ي[29] < . وإن وقعت النقطة فيما بين خطوط هه فف غ كك تحدث أعمدة على استقامة قاعدة بج أعني ج يج وعلى استقامة ضلعي اب اج أعني بظ جغ كنقطة يا وأعمدة يايد يايج يا يب . >[30]

يا[31] . وإن وقعت النقطة فيما بين مثلث طج فف فتحدث أعمدة على

قاعدة بجـ وعلى ضلع اب وعلى استقامة خط اجـ أعني جغ كنقطة يا وأعمدة ياد ياجـ يا يب.

والبرهان على ذلك إن نخرج من نقطة يا خطًا موازيًا لقاعدة بجـ يلقى ضلعي اكك الل فيكون عمودا ياد يايب يعدلان عمود المثلث فهما زائدان على عمود اد بعمود ياجـ على ما بينا متقدمًا. فأما في المواضع الخارجة من هذا التحديد فليس يقع هذه الخاصة البتة وذلك ما أردنا أن نبين. تمت الرسالة[٣٢] .

Apparatus for Text 1

١ بسم الله الرحمن الرحيم رب تمّم: ناقص في مخطوط ر ‖ ٢ طريق: في مخطوط د: طريقه ‖ ٣ بالعمود: في المخطوطين: فالعمود ‖ ٤ آ: في مخطوط ر: ١ ‖ ٥ ضلع: في المخطوطين: ضلعي ‖ ٦ ب: في مخطوط ر: ٢ ‖ ٧ فنسبة هم: ناقص في مخطوط ر، في حاشية مخطوط د فقط ‖ ٨ جـ: في مخطوط ر: ٣ ‖ ٩ زع: في مخطوط ر: رخ ‖ ١٠ ذزض: في مخطوط ر: در ‖ ١١ د: في مخطوط ر: ٤ ‖ ١٢ خط: ناقص في مخطوط ر ‖ ١٣ حص: في المخطوطين: حض ‖ ١٤ حص: في المخطوطين: حض ‖ ١٥ حص: في المخطوطين: حض ‖ ١٦ ه: في مخطوط ر: ٥ ‖ ١٧ بط جط: في المخطوطين بد جد ‖ ١٨ نهاية: في مخطوط ر: النهاية ‖ ١٩ وإن وقعت النقطة: في المخطوطين: ووقع ‖ ٢٠ و: ناقص في مخطوط ر ‖ ٢١ وإن: تكرر في المخطوطين ‖ ٢٢ فهما ... اك: ناقص في المخطوطين يقتضيه السياق ‖ ٢٣ لخط: في المخطوطين: لخطي ‖ ٢٤ يث: في المخطوطين يب ‖ ٢٥ يث: في المخطوطين يت ‖ ٢٦ ز: ناقص في مخطوط ر ‖ ٢٧ ح: ناقص في مخطوط ر ‖ ٢٨ ط: ناقص في مخطوط ر ‖ ٢٩ ي: ناقص في مخطوط ر ‖ ٣٠ وإن ... يا يب: ناقص في الاصل يقتضيه السياق ‖ ٣١ يا: ناقص في مخطوط ر ‖ ٣٢ تمت الرسالة: في مخطوط د: تم المجلد الأول من الهندسيات للسجزي ضاحي نهار الجميع السابع من رمضان سنة ٦١١.

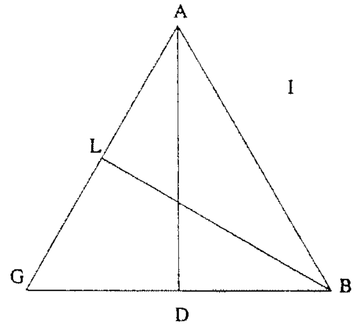

Figure 9-I

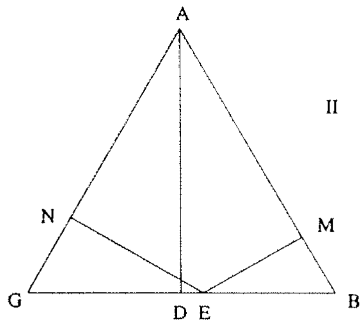

Figure 9-II

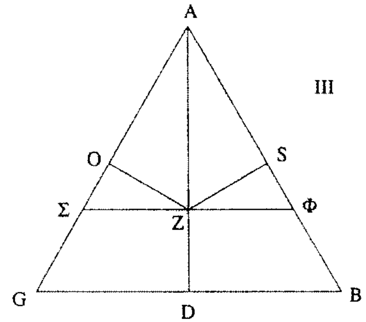

Figure 9-III

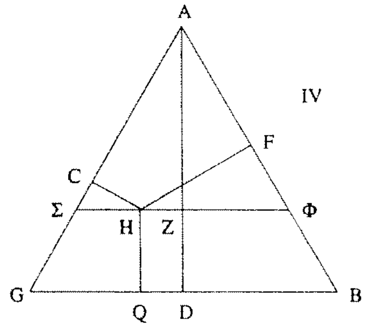

Figure 9-IV

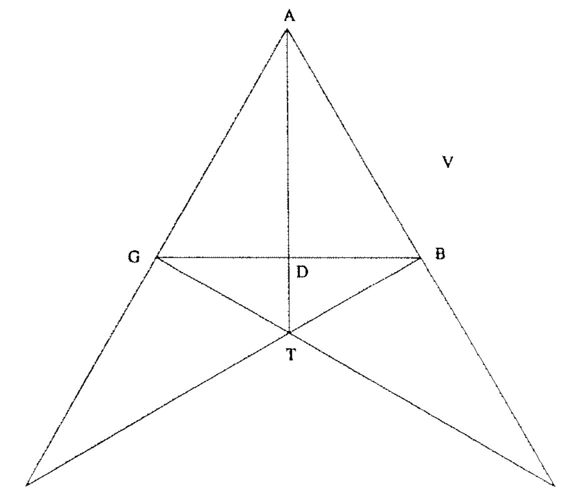

Figure 9-V

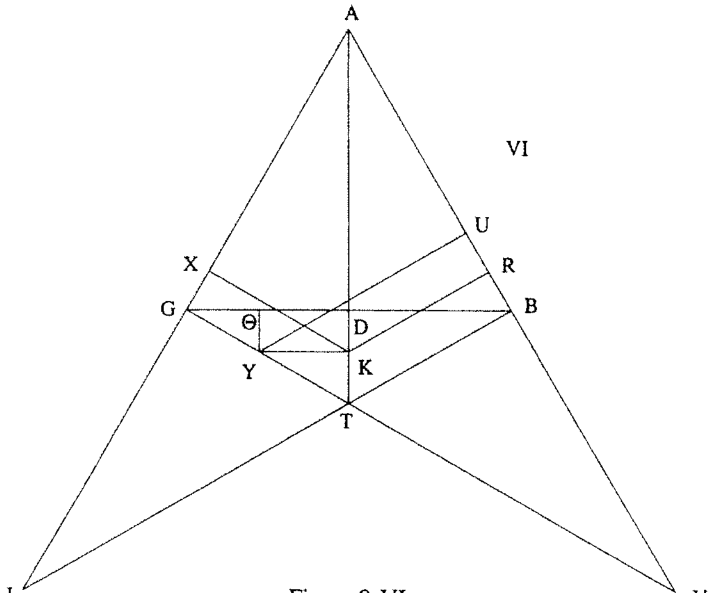

Figure 9-VI

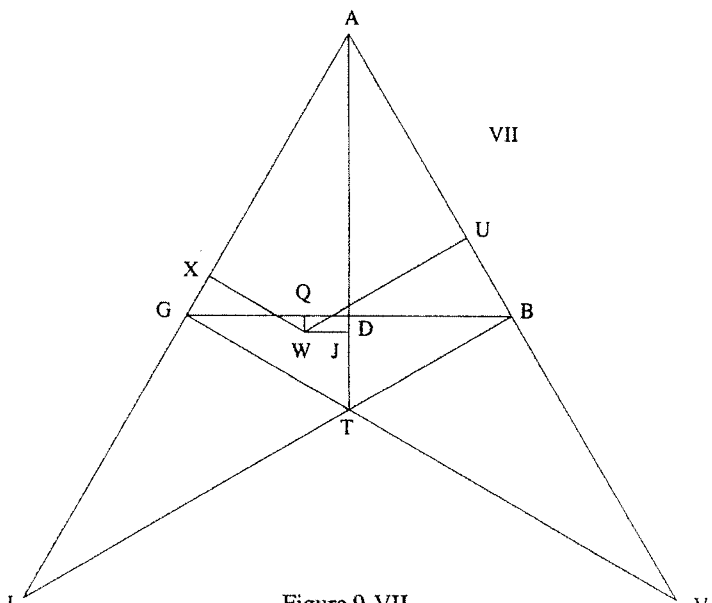

Figure 9-VII

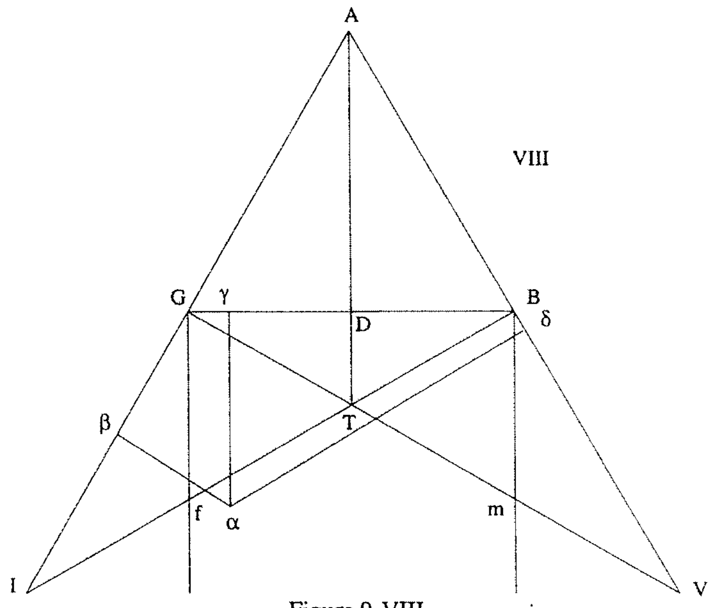

Figure 9-VIII

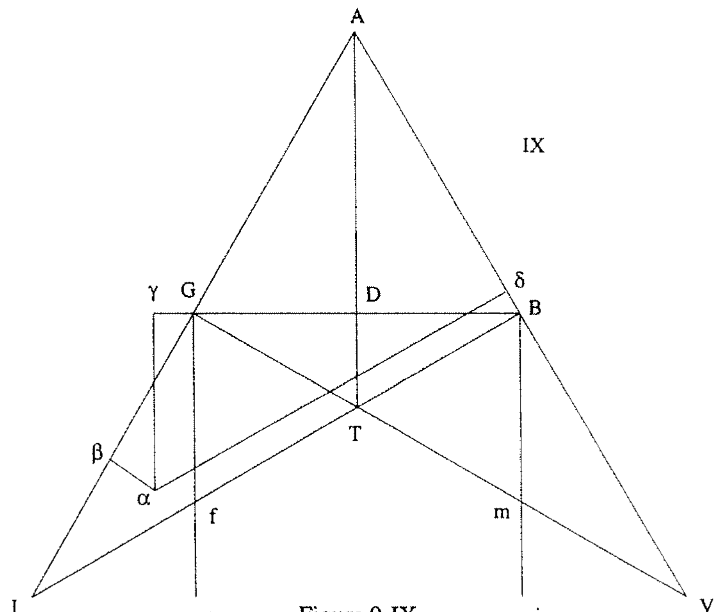

Figure 9-IX

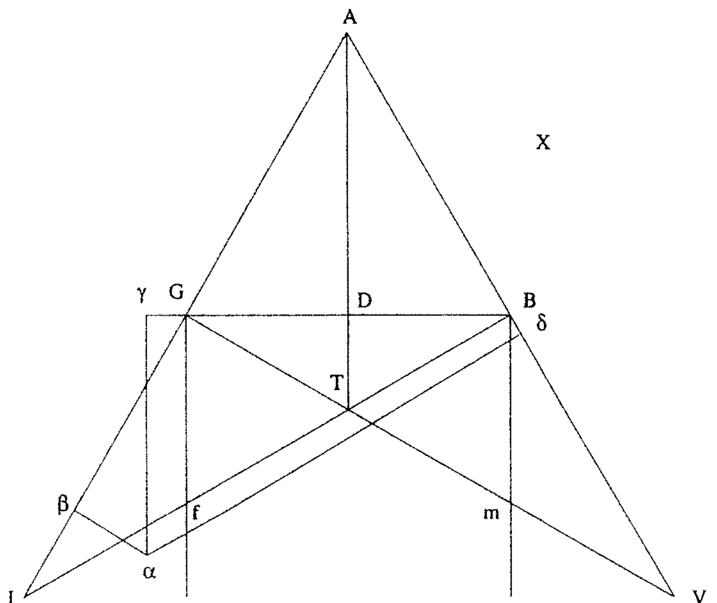

Figure 9-X

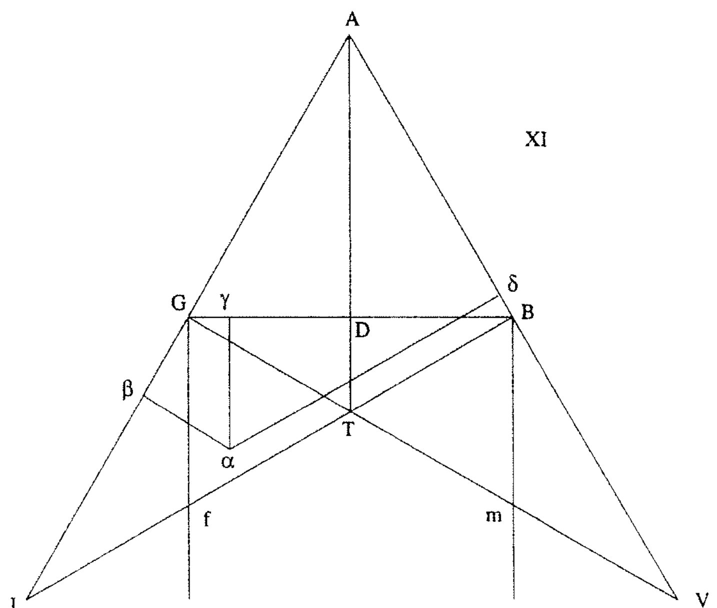

Figure 9-XI

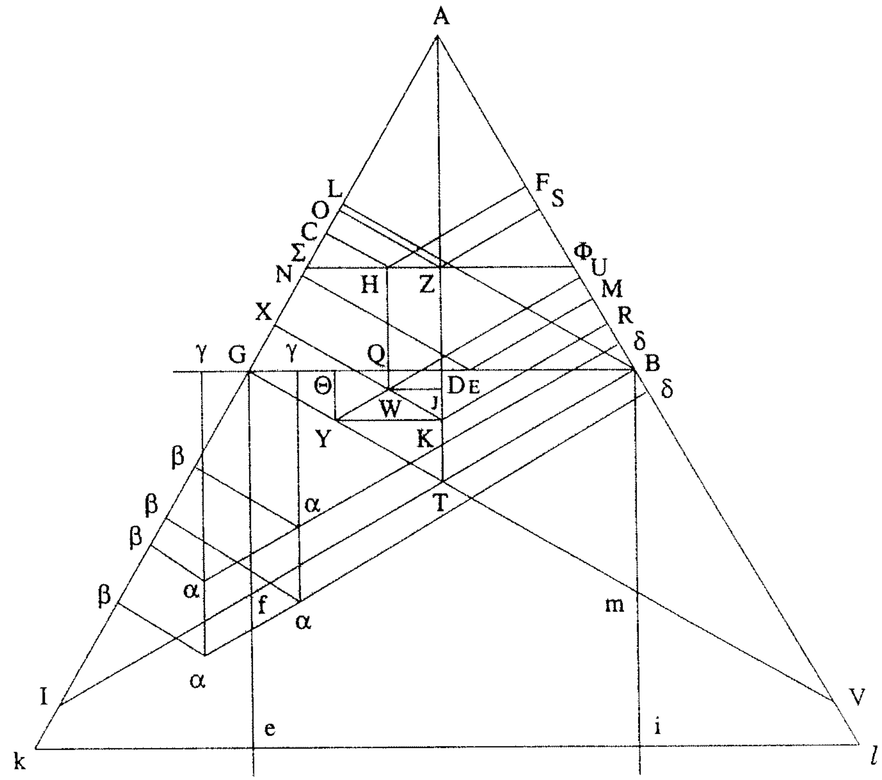

Figure 10

Translation of Text 1

In the name of God, the Merciful, the Compassionate. Lord, make (our work) complete.

Treatise of Aḥmad ibn Muḥammad ibn ʿAbdaljalīl al-Sijzī on the properties of the perpendiculars drawn from a given point to a given equilateral triangle, by way of determination (of all possible cases).

Mayālāwūs mentioned in the beginning of his book on the *Geometrical Elements* the property of equality (resulting) from drawing the perpendiculars in an equilateral triangle to its perimeter. He followed in this the method of division (of the problem into special cases) but the necessary division was not exhaustive. So I intended to follow in this the method of division (into cases) according to the (different) position of the points, and to prove that in an easy way, in one figure (Figure 10), so that the position of the points can be seen by the eyes, and the properties of the perpendiculars which occur according to the (different) positions of the points become easy for someone who studies this (matter). Success is with God. On Him I count, verily, He is the Trustworthy.

Thus let the given equilateral triangle be ABG and let the perpendicular[27] drawn from angle A to the base BG be perpendicular AD. Required (to prove): if from any given point inside the triangle, or at an angle (i.e. angular point) of it, or on one of its sides, perpendiculars are drawn to the three sides of the triangle, they (i.e. their sum) are equal to perpendicular AD. If the point is assumed outside the triangle and the three perpendiculars are drawn to the sides of the triangle or their rectilinear extensions, then the two perpendiculars drawn to the two sides of the equilateral triangle or their extensions exceed perpendicular AD by the perpendicular drawn to the base of the triangle or its rectilinear extension.

Thus let us enumerate the positions of the points. We say: the point can fall with respect to the triangle at an angle (i.e. an angular point, Figure 9–I) of it, or on a side (Figure 9–II) of it, or inside the triangle (Figures 9–III, 9–IV), or outside it (Figures 9–V to 9–XI).

If it (the point) falls inside the triangle, it falls on perpendicular AD (Figure 9–III) or not on perpendicular AD (Figure 9–IV).

If it (the point) falls outside the triangle, it falls (i.e. the feet of the three perpendiculars are) either on its three sides (Figures 9–V, 9–VI, 9–VII), or on the rectilinear extensions of its sides (Figure 9–X), or some of them fall on the sides and the others on the rectilinear extensions of the sides, that is to say, either the (i.e. one) perpendicular falls on the base of the triangle, that is line BG, and the two remaining perpendiculars fall on the rectilinear extensions of sides AB, AG (Figure 9–VIII), or the (i.e. one) perpendicular falls on one of the sides AB, AG and the two remaining perpendiculars fall on the rectilinear extension of the base

[27] The Arabic uses the same word (ʿamūd) for altitude of a triangle and perpendicular to a straight line.

BG and the rectilinear extension of one of the sides AB, AG (Figure 9–IX), or the two (i.e. two) perpendiculars fall on base BG and one of the sides AB, AG and the remaining perpendicular falls on the rectilinear extension of one of the sides AB, AG (Figure 9–XI).

I. (Figure 9–I) If the point falls at an angle, as point B, then let us draw BL to one of the sides of triangle ABG. Then line BL is equal to perpendicular (i.e. altitude) AD.

II. (Figure 9–II) If the point falls at one of the sides of the triangle, as point E, then let us draw perpendiculars EM, EN to sides AB, AG. No third perpendicular can be drawn. Then, since EM is a perpendicular (i.e. altitude) of an equilateral triangle with side BE and EN is a perpendicular (i.e. altitude) of an equilateral triangle with side EG, the ratio of EM to BE is as the ratio of EN to EG, and as the ratio of AD to BG. By addition,[28] the ratio of EM (plus) EN to BE (plus) EG is as the ratio of AD to BG. So lines EM (plus) EN are equal to line AD.

III. (Figure 9–III) If the point falls on line AD, as point Z, then let us draw perpendiculars ZS, ZO to sides AB, AG. Then let us draw line $\Phi Z \Sigma$ parallel to line BG. Then lines ZS (plus) ZO are equal to line AZ. Thus lines ZS (plus) ZO (plus) ZD are equal to lines AZ (plus) ZD, that is line AD.

IV. (Figure 9–IV) If the point falls in the area ABG (i.e. inside the triangle), as point H, then let us draw line $\Phi H \Sigma$ parallel to line BG, and let us draw the perpendiculars HF, HC, HQ to sides AB, AG, BG. Then, as we have proved, lines HF (plus) HC are equal to perpendicular (i.e. altitude) AZ and line HQ is parallel to line ZD and equal to it. Thus lines HF (plus) HC (plus) HQ are equal to lines AZ (plus) ZD, that is line AD.

V. (Figure 9–V) If the point falls outside the triangle, let us draw from points B, G perpendiculars BT, GT to sides AB, AG, let us extend AB, AG indefinitely and let us extend BT, GT to meet AV, AI at points V, I.[29] Let us draw perpendiculars Bi, Ge.[30] And < if the point> falls at point T, I say that the two perpendiculars TB, TG exceed perpendicular AD by perpendicular TD. Proof: The ratio of AG to GD is as the ratio of AT to TG, but AG is twice GD, so AT is twice TG,

[28] Here al-Sijzī uses Euclid's *Elements* V:12, to the effect that (in modern notation) if $a : b = c : d = e : f$, then $(a+b) : (c+d) = e : f$, see Heath, vol. 2, pp. 159–160.

[29] The text means: Let us extend BT, GT to meet AG extended and AB extended at points I, V respectively.

[30] Points i and e occur in Figures 9-VIII to 9-XI and 10.

so BT (plus) TG are equal to AT, so they exceed AD by line DT.

VI. (Figure 9–VI) If the point falls at perpendicular DT, such as point K, we draw perpendiculars KR, KX. Then lines KR, KX fall in the equilateral triangle with perpendicular (i.e. altitude) AK, < thus they are equal to line AK, so they exceed perpendicular AD by perpendicular KD. And if the point falls at line GT, such as point Y, we draw perpendiculars YU, $Y\Theta$. Then lines YU, YG fall in the equilateral triangle with perpendicular AK, >[31] thus they are equal to line AK, so they exceed perpendicular AD by perpendicular KD. Line $Y\Theta$ is equal to line KD so they exceed line AD by line $Y\Theta$.

VII. (Figure 9–VII) If the point falls inside triangle BGT, as point W, then let us draw perpendiculars WU, WX, WQ. We draw WJ parallel to BG. Then, since the perpendiculars WU, WX fall in the equilateral triangle with perpendicular (i.e. altitude) AJ, they are equal to perpendicular AJ, but WQ is equal to DJ, so they exceed perpendicular AD by perpendicular WQ.

VIII. (Figure 9–VIII) If the point falls between lines $imTfe$, the resulting perpendiculars meet side BG and the rectilinear extensions of sides AB, AG, that is, BV, GI; for example point α and perpendiculars $\alpha\delta, \alpha\gamma, \alpha\beta$.

IX. (Figure 9–IX) If the point falls between (i.e. inside) triangle GfI, the resulting perpendiculars meet the rectilinear extension of base BG, that is $G\gamma$, and the rectilinear extension of side AG, that is GI, and side AB; for example point α and perpendiculars $\alpha\delta, \alpha\gamma, \alpha\beta$.

X. (Figure 9–X) < If the point falls between lines $efIk$, the resulting perpendiculars meet the rectilinear extension of base BG, that is $G\gamma$, and the rectilinear extensions of sides AB, AG, that is, BV, GI; for example point α and perpendiculars $\alpha\delta, \alpha\gamma, \alpha\beta$. >[32]

XI. (Figure 9–XI) If the point falls between (i.e. inside) triangle TGf, the resulting perpendiculars meet the base BG, side AB and the rectilinear extension of side AG, that is GI; for example point α and perpendiculars $\alpha\delta, \alpha\gamma, \alpha\beta$.

[31] I have restored a lacuna in the manuscript, which must have occurred because the eye of the scribe moved from one passage "in the equilateral triangle with perpendicular AK" to the next. I have also restored perpendicular $Y\Theta$ in Figure 9–VI. I assume that YK and GB are parallel, so that $Y\Theta = KD$.

[32] I have tentatively restored the missing passage describing the case of Figure 9–X. The figure appears in the manuscript, and the case is mentioned explicitly in the beginning of the text.

The proof for this (i.e. Figures 9–VIII to 9–XI): We draw through point α a line[33] parallel to the base BG, to meet sides Ak, $A\ell$.[34] Then perpendiculars $\alpha\delta, \alpha\beta$ are equal to the perpendicular (i.e. altitude) of the triangle, so they exceed perpendicular AD by perpendicular $\alpha\gamma$ as we have shown before.

In cases other than in this determination (of possible cases), the property is not found at all.[35] That is what we wanted to demonstrate. End of the treatise.[36]

Text 2

Letter by Aḥmad ibn Muḥammad ibn ʿAbdaljalīl (al-Sijzī) to Abū ʿAlī Naẓīf ibn Yumn the Physician on the Construction of an Acute-Angled Triangle From Two Different Straight Lines.

The following edition and translation of Text 2 is based on the manuscript: Paris, Bibliothèque Nationale, Fonds Arabe 2457, ff. 136b–137a.[37] On the date in this manuscript different views have been proposed; Kunitzsch and Lorch assume that the manuscript was an autograph of al-Sijzī, but Sezgin argued in GAS VI, 192 that the manuscript is a 13th century copy. I have given further arguments supporting Sezgin's view in al-Sijzī p. viii. The Arabic text is well legible, a few scribal errors have been corrected in the apparatus. My own explanatory additions to the translation are in parentheses.

بسم الله الرحمان الرحيم

رسالة أحمد بن محمد بن عبد الجليل إلى أبي علي نظيف بن يمن المتطبب في عمل مثلث حاد الزوايا من خطين مستقيمين مختلفين.

سألت أدام الله سعادتك عن عمل المثلث الحاد الزوايا من خطين مستقيمين

[33] This line is not drawn in the figures in the manuscript.

[34] Points k and ℓ are not shown in Figures 9–VIII to 9–XI, but these points are found in the composite Figure 10.

[35] This statement is incorrect, see Figure 5 above, in which we have $PQ - PR - PS = AD$. This can be proved by the same reasoning.

[36] The Dublin manuscript ends with the remark: "The first volume of geometrical (works) of al-Sijzī has been completed, in the morning of Friday the seventh of Ramaḍān of the year 611 (corresponding to January 9, A.D. 1215).

[37] De Slane p. 432, GAS V, 332 no. 13.

مختلفين وذكرت أن أبا سعد العلاء بن سهل عمل ذلك من القطع الناقص من الشكل من المقالة الثالثة من كتاب أبللونيوس في المخروط وذكرت أني استخرجته وهو في كتابنا في المثلثات لكن ما أتينا في كتابنا لم يكن على طريق التحديد فاستخرجته على طريق القسمة والتحديد[1] من المقالة الأولى والثالثة من كتاب أقليدس في الأصول ليبين سيدي أيده الله أن الأشكال المستخرجة من الطرق السهلة والمبادئ القريبة من مقالات كتاب أقليدس في الأصول تكون أفضل من سلوك الطرق الصعبة وخاصة من كتاب المخروطات. فأما المسائل الغير الممكنة إخراجها من كتاب الأصول فيجوز أن يتعلق بالطرق الغريبة الغامضة ولسنا نحتاج إلى إظهار الحجة على هذا من جهة ظهوره وبالله التوفيق.

السؤال نريد أن نعمل من خطين مستقيمين مفترضين مثلثًا حاد الزوايا.

الجواب إن وقوع هذا المثلث يكون على ثلاثة أوجه. فليكن الخطين المستقيمين المفترضين خطي اب اج ونريد ما قلنا. فندير دائرة على وتر اب يكون قوس ادب تقبل زاوية نصف قائمة وهي دائرة ادب ونخرج بد عمود على اب إلى محيط الدائرة ونصل اد فبين أن اد هو قطر الدائرة. وندير على مركز ا وببعد اج قوس دائرة جك . فإما أن تماس دائرة ادب على ما في الأولى على نقطة د وإما أن تقطعها في موضعين على ما في الثانية على ط ك وإما أن تقع خارج الدائرة على ما في الثالثة. فإن وقع خارج الدائرة فلنخرج بد إلى ط[2] ولنصل اط وإن قطعها على ما في الثانية فلتقطعها على نقطتين جنبتي قطر اد على نقطتي ط ك فلنصل اط وإن ماسها على ما في الأولى فلا بد من أن تماسها على نقطة د .

أقول إن الزاوية التي وترها خط اب تقع أبدًا فيما بين قطاع جاد في الأولى وفيما بين قطاع جاط في الثانية والثالثة وبالخطين المخرجين من نقطتي اب إلى قوس جهد وبإخراج خط من نقطة ب إلى الخط المخرج من ا إلى قوس جهد المحيط مع الخط المخرج من ب إلى قوس جهد بزاوية مساوية للزاوية التي تحدث على قوس جهد من الخطين المخرجين من نقطتي ا ب يلتئم[3] مثلث حاد الزوايا وبما سوى ذلك فلا عكن من هذين الخطين المفترضين عمل المثلث الحاد الزوايا.

فلنخرج اه إلى قوس جهد ونصل به ونعمل على نقطة ب من خط اب[4] زاوية مساوية لزاوية اهب وهي هبز . أقول إن مثلث ازب حاد الزوايا.

برهانه لأن زاوية اهب أصغر من نصف قائمة وزاوية ازب ضعف زاوية اهب لأنها مساوية لزاويتي اهب هبز المتساويتين تكون زاوية ازب أصغر من قائمة فهي حادة فلأن الخطين المخرجين من نقطتي ا ب إلى قوس جهد يحيطان مع خط اب بزاويتين أصغرين من قائمتين أعني حادتين فإن كل واحدة منهما حادة فمثلث ازب حاد الزوايا. وبين أن الخط المخرج من نقطة ا نحو جهة ح يحيط مع اب بزاوية منفرجة وكذلك الخط المخرج من ب نحو جهة د يحيط مع اب بزاوية منفرجة لحد خروجها داخل خطي اج بد المتوازيين وذلك ما أردنا أن نبين.

فهذا ما أتينا به على جهة التقسيم والتحديد بطريق كلي قريب المأخذ سهل المسلك وإيجاز من القول بحسب ما يليق بذهنك وفهمك فكن به مستفيدًا جعلك الله به سعيدًا. تمت الرسالة بحمد الله ومنه. كتبته يوم الخميس دي روز من آبان ماه سنة سلط يزدخردية .

Apparatus for Text 2

١ على طريق القسمة والتحديد : في الحاشية فقط ‖ ٢ إلى ط : في الاصل اط ‖ ٣ يلتئم : في الاصل يلتام ‖ ٤ اب : في الاصل اد

Translation of Text 2

In the name of God, the Merciful, the Compassionate.

Letter by Aḥmad ibn Muḥammad ibn ʿAbdaljalīl (al-Sijzī) to Abū ʿAlī Naẓīf ibn Yumn the Physician on the Construction of an Acute-Angled Triangle From Two Different Straight Lines.

You asked, may God make your happiness everlasting, about the construction of the acute-angled triangle from two different straight lines, and you mentioned that Abū Saʿd al-ʿAlāʾ ibn Sahl constructed that by means of the ellipse and proposition …[38] of the third book of

[38] The manuscript has an empty space here. In *Conics* III:52 Apollonius proves that the sum of the focal distances of any point of an ellipse is equal to the major axis. See Ver Eecke, pp. 271–272. The last few lines of the text were also available to me in a Lahore manuscript, on which see GAS VII, 409.

the Book of Apollonius on the cone (i.e. his *Conics*). You mentioned that I solved this, and that is in our book *On Triangles*, but what we presented in our book was not in the way of determination (of possible and impossible cases). Therefore I solved it, in the way of division (of the problem into cases) and determination (of possible and impossible cases) by means of the first and third Book of Euclid's *Elements*, in order to make clear to my Lord (i.e. you), may God help him, that propositions by easy ways and simple principles from Euclid's *Elements*, are to be preferred over (solutions by) difficult methods, especially from the Book of *Conics*. Problems which cannot be solved by means of the *Elements* can be related to strange and deep methods, and we do not have to clarify the reason for this because it is clear. Our success is by God.

The question. We want to construct from two assumed straight lines an acute-angled triangle.

The answer. There are three cases for this triangle.

Let the assumed straight lines be lines AB, AG. We want what we have said. We draw a circle on chord AB in such a way that arc ADB contains half a right angle, namely circle ADB. We draw BD perpendicular to AB to (meet) the circumference of the circle. We join AD. Then it is clear that AD is the diameter of the circle. We draw with centre A and distance AG an arc of a circle GK. Either it is tangent to circle ADB at point D as in the first (figure) (Figure 11), or it intersects it at two places, as in the second (figure) (Figure 12), at T, K, or it falls outside the circle as in the third (figure) (Figure 13). If it falls outside the circle, let us extend BD towards T and let us join AT. If it intersects it, as in the second (figure), let it intersect it at two points, on both sides of diameter AD, (namely) points T, K, and let us join AT. If it is tangent to it as in the first (figure) it must be tangent to it at point D.[39]

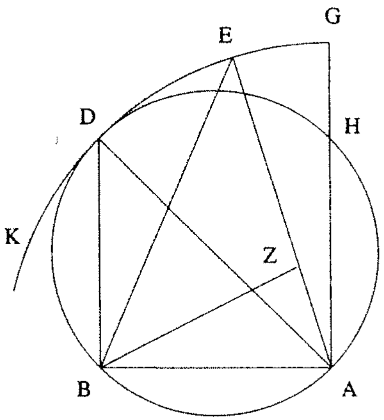

Figure 11

[39] By *Elements* III:16, Heath vol. 2, pp. 37–39, the tangent is the perpendicular to the diameter through the point of contact.

I say that the angle[40] whose chord is line AB always falls between (i.e. inside) sector GAD in the first (figure), and between (i.e. inside) sector GAT in the second and third (figure),[41] and (that) by means of the two lines drawn from points A, B to arc GED and by drawing a (new) line from point B to the line drawn from A to arc GED, (the new line) containing with the line drawn from B to arc GED an angle equal to the angle which is produced at arc GED by the two lines (already) drawn from points A, B, an acute-angled triangle can be put together, and (that) from other (new lines through B) and these two assumed lines no acute-angled triangle can be constructed.

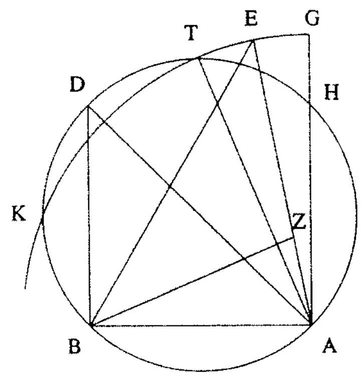

Figure 12

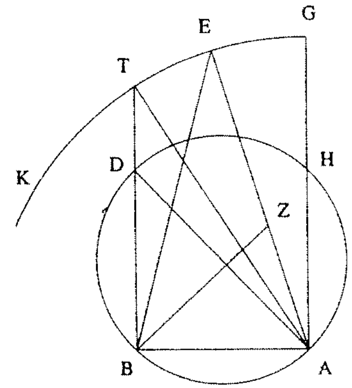

Figure 13

Thus let us draw AE to arc GED.[42] We join BE and we make on point B of line AB an angle equal to angle AEB, namely EBZ. I say that triangle AZB is acute-angled. Proof of this: Since angle AEB is less than half a right angle,[43] and angle AZB is twice angle AEB, since it is equal to the angles AEB, EBZ which are equal to one another, angle AZB is less than a right angle, so it is acute. Since the two lines drawn from points A, B to arc GED contain with line AB two angles less than two right angles, I mean, two acute angles, each of them is acute, so triangle AZB is acute-angled. It is clear that the line

[40] al-Sijzī probably means $\angle AEB$ here, because he has not yet defined point Z.

[41] Line AG is placed perpendicularly to line AB, and al-Sijzī supposes $AG > AB$.

[42] The notation in the following proof is for the first figure (Figure 11). For the second and third figure (12, 13), D should be changed to T.

[43] If BE intersects the circle at X, by Euclid's *Elements* III:27 $\angle AXB = \angle ADB = 45°$. Thus $\angle AEB < 45°$ because point E is outside the circle through A, D, B.

drawn from point A towards the side of H[44] contains with line AB an obtuse angle, and similarly the line drawn from B towards the side of D contains with AB an obtuse angle.[45] Therefore the limit of drawing them is in between the parallel lines AG, BD. That is what we wanted to show.

This is what we have presented by way of division (into cases) and determination (of possible and impossible cases), in a general method, easy to grasp and easy to follow, concisely phrased, according to what suits your mind and understanding. So profit from it, and may God make you happy with it. The letter has ended, with the praise to God and His grace. I wrote this on Thursday, the day Dey of the month Ābān of the year 339 of the Yazdgerd era.[46]

Text 3

A construction by Abū Saʿd al-ʿAlāʾ ibn Sahl, from al-Sijzī's treatise *On the Selected Problems Which Were Discussed by Him and the Geometers of Shīrāz and Khorāsān, and His Annotations*

Al-Sijzī's treatise *On the Selected Problems Which Were Discussed by Him and the Geometers of Shīrāz and Khorāsān, and His Annotations* (fī al-masāʾil al-mukhtāra allatī jarat baynahu wa-bayna muhandisī Shīrāz wa-Khurāsān wa-taʿlīqātuhu) (GAS V, p. 333 no. 23) has come down to us in the two manuscripts Dublin, Chester Beatty 3652, ff. 35a–52b (Arberry vol. 3, p. 59) and Istanbul, Süleymaniye Library, Reşit

[44] The text means: a line AZ drawn on the other side of line AH, where point H is the second point of intersection of AG and circle ABD.

[45] Here the text is not clear. Al-Sijzī seems to be talking about $\angle ABE$, but the required triangle is ABZ, and $\angle ABZ$ may be acute even if $\angle ABE$ is a right or obtuse angle. This part of the reasoning should be changed as follows. Suppose we want to construct all acute-angled triangles AZB such that A and B are given and $AZ + ZB = AG > AB$. It is easy to construct an isosceles triangle AZB such that $AZ = BZ = \frac{1}{2}AG$. To construct a scalene triangle, we can assume without loss of generality $AZ < ZB$, so $\angle ABZ < \angle ZAB$. Then we can use the construction of al-Sijzī, which guarantees that $\angle AZB$ and $\angle ZAB$ are acute. Thus $\angle ABZ$ is acute as well.

[46] The month Ābān of the year 339 of the Yazdgerd era corresponds to October 20 – November 19, A.D. 970. The day called "Dey" can be the 8th, 15th and 23th day of the month; these were called Dey beh Āzar, Dey beh Mehr, Dey beh Dīn. Because the day is a Thursday, two possibilities remain: Thursday 8 Ābān 339 = October 27, 970 A.D. and Thursday 15 Ābān 339 = November 3, 970 A.D. I owe this information to Mr. Rahim Rezazadeh Malek and Mr. Mohammad Bagheri, Tehran.

1191, ff. 31b–62a. In this treatise al-Sijzī discusses many problems with solutions by himself or by other geometers, including the problem of Text 2 with the solution by al-ʿAlāʾ. This solution is found in MS. Chester Beatty 3652, ff. 37b:37–38a:9, Reşit 1191, ff. 36b:17–37a:6. The solution has not been published before.[47] In the apparatus the same notation has been used as in Text 1.

للعلاء بن سهل خطي اب جد المستقيمين مختلفان و اب أطولهما. فنريد أن نعمل منهما مثلثًا حاد الزوايا. فلنركب الخط على وسط الخط الأعظم ولنجعل اد في دب يعدل مربع هك ونقيم كهط على نصف خط اب ونعمل على قطري اب طك قطعًا ناقصًا عليه اكبط وندير على قطر جد نصف دائرة جكد فإمّا أن تماس القطع وإمّا أن تقطعه[١] وإمّا أن تقع داخل القطع. ونخرج عمودي جز دح على اب فإن وقع نصف الدائرة داخل القطع فإن الخطين المخرجين من نقطتي جـ د إلى خط دكجـ[٢] من محيط القطع الناقص يحيطان بزاوية حادة وتكون الزاويتين اللّتين على جنبتي جـ د حادتان ويكون مجموع الخطين يعدلان خط[٣] اب الأطول المعطى فقد عملنا ما أردنا. وإن ماس القطع الناقص على نقطة ك فالخطين المخرجين من نقطتي جـ د إلى خط دكجـ سوى نقطة ك يحيطان بزاوية حادة والزاويتين الباقيتين حادتان. وإن قطعه على نقطتي ل م فالخطين المخرجين من نقطتي جـ د إلى خطي زل مح يحيطان بزاوية حادة وتكون الزاويتان الباقيتان حادتان ويكون مجموعهما أعني الخطين المخرجين من نقطتي جـ د يعدلان < خط اب فعملنا من >[٤] خطي اب جد مثلثا حاد الزوايا وذلك ما أردنا أن نبيّن.

Apparatus for Text 3

١ تقطعه: في مخطوط ر : نقطة ‖ ٢ دكجـ : في مخطوط ر : د ‖ ٣ خط : في مخطوط ر خطى ‖ ٤ خط اب فعملنا من : ناقص في المخطوطين يقتضيه السياق.

[47] The solution is not mentioned in Rashed's edition of the works of al-ʿAlāʾ, although another solution by al-ʿAlāʾ, which occurs one page earlier in the same text of al-Sijzī, appears on p. 190 of Rashed's edition. In Hogendijk 2, p. 193, I mentioned the fact that a reference to al-ʿAlāʾ occurs in the text *On the Selected Problems* .... There I did not mention that the text contains *two* solutions by al-ʿAlāʾ and this may have misled Rashed.

Translation of Text 3

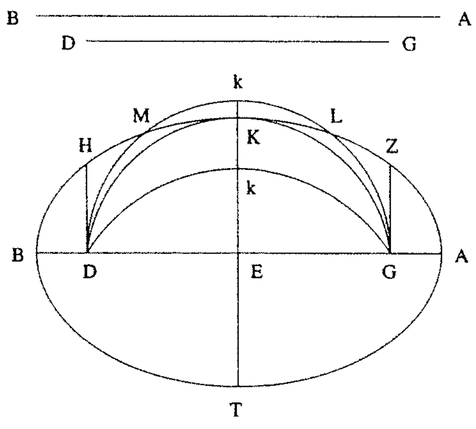

Figure 14

By al-ʿAlāʾ ibn Sahl. (Figure 14) The two straight lines AB, GD are different, and AB is the longer of them. We want to construct from them an acute-angled triangle. Let us place the (shorter) line at the middle of the greater line and let us make AD times DB equal to the square of EK. We place KET perpendicularly at the midpoint of line AB and we make on diameters AB, TK the ellipse $AKBT$.[48] We describe on diameter GD semicircle GkD.[49] Then it is tangent to the ellipse, or it intersects it, or it falls inside the ellipse. We draw perpendiculars GZ, DH to AB. If the semicircle falls inside the ellipse, the (i.e. any) two lines drawn from points G, D to line DKG on the boundary of the ellipse contain an acute angle, and the two angles on the sides of G, D are acute, and the sum of the two lines is equal to the longer line AB,[50] which is given, so we have constructed what we wanted. If it is tangent to the ellipse at point K, the (i.e. any) two lines drawn from points G, D to (any point on) line DKG except point K contain an acute angle, and the two remaining angles are acute. And if it intersects it at points L, M, then the (i.e. any) two lines drawn from points G, D to lines ZL, MH contain an acute angle, and the two remaining angles are acute, and their sum, that is of the two lines drawn from points G, D, is equal to < line AB, so we have constructed from > lines AB, GD an acute-angled triangle, and that is what we wanted to demonstrate.

48 Apollonius, *Conics*, I:56, Ver Eecke p. 106–109.

49 Point K had already been defined as endpoint of EK, but here al-Sijzī defines K a second time in a way which is inconsistent with the earlier definition. To avoid confusion I write k for the point defined according to the second definition. In al-Sijzī's text and figure, these points K and k are indicated by three letters زك.

50 Apollonius, *Conics* III:52, Ver Eecke p. 271–2.

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