In this paper I present some new information on the Geometrical Elements of Menelaus and its influence in the medieval Islamic tradition.

My sources are two unpublished texts by al-Sijzi, Abū Os'ād Ahmad ibn Muhammad ibn al-Abdalājīl al-Sijzī. He was a very productive geometer and astrologer, who originated from Sijistan in South-Eastern Iran, and who was active in the second half of the tenth century A.D.

Section 3 is about a new trace of the Geometrical Elements of Menelaus. In al-Sijzi's hitherto unpublished Treatise on the Properties of the Perpendiculars Drawn From a Given Point to a Given Equilateral Triangle, By Way of Determination, this text shows that in the beginning of the Geometrical Elements, Menelaus studied the sum (or difference) of perpendiculars drawn from a given point to the sides of a given equilateral triangle.

In Section 4, I discuss the relation between the Geometrical Elements II.2 and al-Sijzi's hitherto unpublished Letter to Abū'Alī Naṣr ibn Yumn the Physician on the Construction of an Acute-Angled Triangle From Two Different Straight Lines. In this letter al-Sijzi presents a ruler-and-compass construction of certain triangles, as an improvement over the construction by his contemporary Abū Sa'ād al'Ala' ibn Sahl, who had used an ellipse. This construction is to be found in Text 3.

Thus, the Geometrical Elements II.2 was related to a series of studies by geometers in the Arabic-Islamic tradition.

In Section 5, I will use the new evidence to disprove a suggestion by the late Wilbur Knorr that the Geometrical Elements contained a trigonometric formula for the Angle by means of conic sections. The Geometrical Elements seem to have been a collection of elementary theorems and problems on straight lines, circles, and triangles in the plane. The Arabic-Islamic geometers were very interested in such theorems and problems, and the influence of the Geometrical Elements on such works is reflected in the many references to them in the works by Menelaus and his contemporaries.

For example, in the book on the Measurement of Plane and Spatial Figures, the Elements contains a passage on the measurement of figures.

The influence of the Geometrical Elements of Menelaus on such works is reflected in the many references to them in the works by Menelaus and his contemporaries.
1. The bio-bibliographer Ibn al-Nadim (ca. 970) says in the Fihrist that Menelaus wrote:

"The book on the Elements of Geometry, which Thābit ibn Qurra made (i.e., edited) in three treatises (i.e., books); the Book of the Triangles, of which a small part came out in Arabic." In a passage quoted below, al-Biruni says that Thābit ibn Qurra commented on the Geometrical Elements. Thus it is likely that Thābit revised the existing translation of the three Books of the Geometrical Elements and added his own commentary. The reference to the Book on Triangles shows that Menelaus wrote other works on elementary plane geometry as well.

2. Around AD 1000, Abu Nasr ibn Irazi wrote a letter to al-Biruni in which he corrected some errors in the Astronomical Handbook of the Plates (Zij al-Safâ'īh) by Abū Ja'far al-Khâzin (ca. AD 950). Abu Nasr says: "Even excellent mathematicians can make mistakes, and he gives the following example:

"And Abū Ja'far al-Khâzin himself corrected a mistake or omission which Menelaus happened to make in his book called the Geometrical Elements." Unfortunately, Abu Nasr does not give us more information on Abu Ja'far's criticisms.

3. In the Extraction of Chords, al-Biruni mentions the Geometrical Elements of Menelaus in the following passage:

"..."
In other works, al-Sijzi discusses the three perpendiculars.

In the rest of Text 1, al-Sijzi discusses the three perpendiculars.

From the diagram, we can see that the perpendiculars intersect at point E. The text explains that this point is important in solving the problem. The text also notes that the perpendiculars can be used to divide the triangle into smaller parts, which can then be solved individually.

The text concludes by discussing the importance of the perpendiculars and how they can be used to solve a wide range of geometrical problems. It is clear that al-Sijzi was a highly skilled mathematician, and his work has had a lasting impact on the field of geometry.
the Geometrical Elements of Menelaus

We have discussed this case, so let us move on to the next case discussed, which is the case where $P$ is outside the triangle. This is the case where $P$ is outside the triangle, as shown in Figure 4.

In Figure 4, we have a line segment $AB$ with a point $P$ outside the triangle. The problem is to construct a line segment $CP$ such that $CP = AP$. This is a difficult problem in Euclidean geometry, as shown in Figure 5.

**Figure 5**

\[ A \quad B \quad C \]

\[ P \]

To solve this problem, we can use the following steps:

1. Draw a line segment $AB$.
2. Draw a line segment $AP$.
3. Draw a line segment $CP$ such that $CP = AP$.

This problem is also discussed in the book of As-sumption of Aqatun.19 My conclusion that the Geometrical Elements of Menelaus contained the theorem for Figure 4, agrees with the conclusion of Yvonne Dold-Samplonius20 that Menelaus is not the same author as Aqatun or Pseudo-Archimedes, who did give the theorem.

Ibn al-Haytham (ca.960—ca.1040) proved the theorem for points $P$ inside the equilateral triangle and he generalized the theorem for points $P$ inside an isosceles triangle.21 Ibn al-Haytham did not discuss the case where $P$ is outside the triangle, so it is likely that he did not know the Geometrical Elements of Menelaus.

**Figure 6**

\[ A \quad B \quad C \]

\[ P \]

\[ \text{Given: } \angle A \text{ and } \angle B \text{ and } \angle C \text{ and } \angle D \text{ and } \angle E \text{ and } \angle F \text{ and } \angle G \]

\[ \text{Prove: } \angle A = \angle B = \angle C = \angle D = \angle E = \angle F = \angle G \]

**Figure 7**

\[ A \quad B \quad C \]

\[ P \]

\[ \text{Given: } \angle A \text{ and } \angle B \text{ and } \angle C \text{ and } \angle D \text{ and } \angle E \text{ and } \angle F \text{ and } \angle G \]

\[ \text{Prove: } \angle A = \angle B = \angle C = \angle D = \angle E = \angle F = \angle G \]

**Figure 8**

\[ A \quad B \quad C \]

\[ P \]

\[ \text{Given: } \angle A \text{ and } \angle B \text{ and } \angle C \text{ and } \angle D \text{ and } \angle E \text{ and } \angle F \text{ and } \angle G \]

\[ \text{Prove: } \angle A = \angle B = \angle C = \angle D = \angle E = \angle F = \angle G \]
The two circles intersect at two points \( X \) and \( Y \), but it is also possible

This paper is Text 2 in the Appendix, see the Appendix, Figures 11-13.

This information will be used in Section 5 of this paper.

To construct this triangle, consider the isosceles triangle with base \( AB \) and height \( AC \). Then, we can construct triangle \( AB \) by this method. This triangle is isosceles, but it is not acuteangled triangle.
In Textual studies in Ancient and Medieval Geometry, pp. 219-289, the late Wilbur Knorr suggests that the Geometrical Elements of Menelaus contained a trisection of the angle by means of a circle and a hyperbola. Knorr argued that this trisection was adapted from this work by Thabit ibn Qurra in his own treatise on the trisection of the angle. I will argue that is suggested by Texts 1 and 2. The Geometrical Elements of Menelaus did not contain a trisection of the angle by means of conic sections. In the notation of ML, the idea is $B'$, $A'$, $C$, $D$. The problem is solved by the construction of the circle. The solutions are $L$, $M$, $N$, $O$, $P$, $Q$, $R$, $S$. By Thabit's theorem, the construction is equivalent to the solution of the problem of Menelaus to construct a "broken line" $ALB$ equal to $c$ in the semicircle. Again, we see that al-'Alã dealt with the construction of all triangles $AZB$ such that $AZ + ZB = c$. Thabit ibn Qurra and Abu Sahila-Kabi in the same text, al-Sijzi's solution of the problem of Menelaus did not contain a trisection by means of conic sections. I will argue that this trisection was adapted from this work by Thabit ibn Qurra. It follows that al-Sijzi knew the Geometrical Elements of Menelaus at some stage in his career, and Text 2 strongly suggests that he knew this work in 970/359. In the treatise on the trisection of the angle, al-Sijzi says that the ancient geometers were unable to trisect the angle, and that the only modern geometers who could solve the problem were Abu Sahila-Kabi and Abu Usman al-Kabi. In the same text, al-Sijzi's solution of the problem of Menelaus to construct a "broken line" $ALB$ equal to $c$ in the semicircle is equivalent to the solution of the problem of Menelaus. The relations between Text 2, al-Sijzi's solution of the problem of Menelaus, and the text of al-'Alã suggest that the problem "to construct the (i.e. every) acute-angled triangle from two different straight lines" may have occurred in the Geometrical Elements as well, and that Geometrical Elements I:11:2 was a preliminary to the solution. Al-Biruni informs us that the solution by Menelaus (probably by ruler and compass) was complicated, and therefore al-'Alã must have preferred his own elegant solution by means of the ellipse. His solution to the problem is as follows (see Figure 5). The intersections of the ellipse with $AL$ and $BL$ are $Z$ and $M$. If you extend the semicircle to the point $Q$, then two different points $L$ and $M$ are obtained. The intersections $L$, $M$ of the ellipse with the semicircle $ALB$ are obtained by extending the semicircle to the point $Q$. Points $L$ and $M$ are obtained by extending the semicircle to the point $Q$. The problem is solved by the construction of the circle. The problem is solved by the construction of the circle.

Figure 5:

- $ALB$ is the broken line.
- The construction is equivalent to the solution of the problem of Menelaus.
Appendix: Arabic texts and translations

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The following edition and translation is based on manuscripts in Dublin, Chester Beatty Library 3562, ff. 66b—67b (Arberry vol. 3, p. 60), and Istanbul, Suleymaniye Library, Reit 1191, ff. 124b—125b (GASV 333 no. 19). The Dublin manuscript is dated Friday 7 Ramadän 611 H. = January 9, 1215 A.D.

The manuscripts contain numerous scribes, and some passages had to be restored to make mathematical sense. Such passages appear in angular brackets <...> in the text and translation. The text is followed by a critical apparatus, in which I have indicated the manuscripts by the symbols € (for Dublin) and € (for Reit). I have made some changes in the orthography, but I have not corrected grammatical errors in the text. Only the Dublin manuscript indicates by sas the letters in the text referring to points in geometrical figures.

My own explanatory additions to the translation are in parentheses.

The manuscript contains 11 figures (my Figures 9—I to 9—XI), displaying special cases of the theorem in the text. The Roman numbers I, II, IX, XI which appear in these figures in the text are my transcriptions of the Arabic alphabetical numbers in the figures in the Dublin manuscript. These 11 figures are derived from the composite figure at the end of the text (Figure 10). This figure is special because no less than 46 points are labelled by a letter, Al-Sijzi first used single letters of the Arabic alphabet to label these points. He used these letters roughly in order of increasing numerical values, with € = 6 and € = 10 as exceptions. The last few points are labelled by the letters € = 50, € = 70, € = 80, € = 90 because the corresponding Greek letters € and € are not in Arabic. The letters were indicated in diagrams in Arabic texts, probably.


<table>
<thead>
<tr>
<th>Arabic Alphabet</th>
<th>Numerical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>آ</td>
<td>1</td>
</tr>
<tr>
<td>ب</td>
<td>2</td>
</tr>
<tr>
<td>ج</td>
<td>3</td>
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<tr>
<td>د</td>
<td>4</td>
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<td>5</td>
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<td>ح</td>
<td>6</td>
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</tr>
<tr>
<td>ق</td>
<td>8</td>
</tr>
<tr>
<td>ك</td>
<td>9</td>
</tr>
<tr>
<td>ل</td>
<td>10</td>
</tr>
</tbody>
</table>

The letters € and € were infrequently used in diagrams in Arabic texts, probably because the corresponding Greek letters € and € are not in Arabic. The letters were indicated in diagrams in Arabic texts.


The text follows the system of E. Kennedy. I have transcribed the Arabic script in the manuscript, while preserving the original layout. The text is followed by a critical apparatus, in which I have indicated the manuscripts by the symbols € (for Dublin) and € (for Reit). I have made some changes in the orthography, but I have not corrected grammatical errors in the text. Only the Dublin manuscript indicates by sas the letters in the text referring to points in geometrical figures.

My own explanatory additions to the translation are in parentheses.
If $P$ is the linear $p$-sum of $X$ and $Y$, then $P$ is also the $p$-sum of $X$ and $Y$.

Moreover, $P$ is the $p$-sum of $X$ and $Y$ if and only if

\[ P = \frac{1}{p^2} \left( \frac{1}{p} X + \frac{1}{p} Y \right) \]

is the $p$-sum of $X$ and $Y$.

Let $\mathbb{R}$ be the real numbers. Then $P$ is the $p$-sum of $X$ and $Y$ if and only if

\[ P = \frac{1}{p^2} \left( \frac{1}{p} X + \frac{1}{p} Y \right) \]

is the $p$-sum of $X$ and $Y$.

Theorem 3.1. Let $p > 0$ and $X, Y$ be $p$-sums.

If $X$ and $Y$ are $p$-sums, then $P$ is also a $p$-sum.

Proof. We have

\[ P = \frac{1}{p^2} \left( \frac{1}{p} X + \frac{1}{p} Y \right) \]

is the $p$-sum of $X$ and $Y$.

This completes the proof.

\[ \square \]
equivalent triangle. By way of determination (of all possible cases),
properties of the perpendiculars drawn from a given point to a given
line are used in the solution of the problem, the solution of which
requires the use of all the methods of the previous chapter.

In the name of God, the Merciful, the Compassionate. Lord, make

Translation of Text 1

Figure 10

Figures 9-11
Mayawen mentioned in the beginning of his book on the Geonier neat Elements the property of equality (resulting) from drawing the perpendiculars in an equilateral triangle to its perimeter. He followed in this method of division (of the problem into special cases) but then the necessary division was not exhaustive. So I intended to follow in this method of division (into cases) according to the (different) positions of the points, and to prove that in an easy way, in one figure (Figure 10), so that the positions of the points can be seen by the eyes, and the properties of the perpendiculars which occur according to the (different) positions of the points become easy for someone who studies this (matter).

Success is with God. On Him I count, verily, He is the Trustworthy.

Thus let the given equilateral triangle be ABC and let the perpendicular (amid) drawn from angle A to the base BG be perpendicular AD.

Required (to prove): If from any given point inside the triangle, or at an angle (i.e. angular point) of it, or on one of its sides, perpendiculars are drawn to the three sides of the triangle, they (i.e. their sum) are equal to perpendicular AD. If the point is assumed outside the triangle and the three perpendiculars are drawn to the sides of the triangle or their rectilinear extensions, then the two perpendiculars drawn to the two sides of the equilateral triangle or their extensions exceed perpendicular AD by the perpendicular drawn to the base of the triangle or its rectilinear extension.

Thus let us enumerate the positions of the points. We say the point can fall with respect to the triangle at an angle (i.e. an angular point, Figure 9—I), or on a side (Figure 9—II) of it, or inside the triangle (Figures 9—I—III, 9—I—IV), or outside it (Figures 9—V to 9—XI).

If it (the point) falls inside the triangle, it falls on perpendicular AD (Figure 9—I—III) or not on perpendicular AD (Figure 9—I—IV).

If it (the point) falls outside the triangle, it falls (i.e. the feet of the three perpendiculars are) either on its sides (Figures 9—V, 9—VI, 9—VII), or on its rectilinear extensions (Figure 9—X), or some of them fall on the sides and the other on the rectilinear extensions of the sides (Figure 9—XI), or some on the rectilinear extensions and some on the sides (Figure 9—I—VII). If it (the point) falls inside the triangle, it falls on perpendicular AD (Figure 9—I—III) or not on perpendicular AD (Figure 9—I—IV).

Thus let us enumerate the positions of the points. We say the position of the points can be seen by the eyes, and the properties of the perpendiculars which occur according to the (different) positions of the points become easy for someone who studies this (matter).

This (method) is that the given equilateral triangle is ABG, and the perpendiculars are AB, AD, AG (Figure 9—II) and let us consider the positions of the points.

Thus let us consider the positions of the points. We say the point can fall with respect to the triangle at an angle (i.e. an angular point, Figure 9—I), or on a side (Figure 9—II) of it, or inside the triangle (Figures 9—I—III, 9—I—IV), or outside it (Figures 9—V to 9—XI).

If it (the point) falls inside the triangle, it falls on perpendicular AD (Figure 9—I—III) or not on perpendicular AD (Figure 9—I—IV).

If it (the point) falls outside the triangle, it falls (i.e. the feet of the three perpendiculars are) either on its sides (Figures 9—V, 9—VI, 9—VII), or on its rectilinear extensions (Figure 9—X), or some of them fall on the sides and the other on the rectilinear extensions of the sides (Figure 9—XI), or some on the rectilinear extensions and some on the sides (Figure 9—I—VII). If it (the point) falls inside the triangle, it falls on perpendicular AD (Figure 9—I—III) or not on perpendicular AD (Figure 9—I—IV).

Thus let us enumerate the positions of the points. We say the position of the points can be seen by the eyes, and the properties of the perpendiculars which occur according to the (different) positions of the points become easy for someone who studies this (matter).

This (method) is that the given equilateral triangle is ABG, and the perpendiculars are AB, AD, AG (Figure 9—II) and let us consider the positions of the points. We say the point can fall with respect to the triangle at an angle (i.e. an angular point, Figure 9—I), or on a side (Figure 9—II) of it, or inside the triangle (Figures 9—I—III, 9—I—IV), or outside it (Figures 9—V to 9—XI).

If it (the point) falls inside the triangle, it falls on perpendicular AD (Figure 9—I—III) or not on perpendicular AD (Figure 9—I—IV).

If it (the point) falls outside the triangle, it falls (i.e. the feet of the three perpendiculars are) either on its sides (Figures 9—V, 9—VI, 9—VII), or on its rectilinear extensions (Figure 9—X), or some of them fall on the sides and the other on the rectilinear extensions of the sides (Figure 9—XI), or some on the rectilinear extensions and some on the sides (Figure 9—I—VII). If it (the point) falls inside the triangle, it falls on perpendicular AD (Figure 9—I—III) or not on perpendicular AD (Figure 9—I—IV).

Thus let us enumerate the positions of the points. We say the position of the points can be seen by the eyes, and the properties of the perpendiculars which occur according to the (different) positions of the points become easy for someone who studies this (matter).
Ist2

The perpendiculars fall parallel to BG, as in the equilateral triangle with perpendicular (i.e., altitude) AK, so they exceed perpendicular AD by perpendicular KD. And if the point falls on line GT, such as point Y, we draw perpendiculars YU, YG. Then lines YU, YG fall perpendicular to AB, for they exceed perpendicular AD by perpendicular KD. Line YG is equal to line KD, so they exceed line AD by line YO.

VIII. (Figure 9—VIII) If the point falls between lines EF, the resulting perpendiculars meet sides AB, AG, that is, BY, GI; for example point a and perpendiculars as S, ay, af. The following edition and translation of Text 2 is based on the manuscript: Paris, Bibliothèque nationale, I. O. A. 136b—sides AB, AG, that is, BY, GI; for example point a and perpendiculars as S, ay, af. The following edition and translation of Text 2 is based on the manuscript: Paris, Bibliothèque nationale, I. O. A. 136b.

IX. (Figure 9—IX) If the point falls between (i.e., inside) triangle TG, the resulting perpendiculars meet the base BG, side AB and the rectilineal extension of side AG, that is, GI; for example point a and perpendiculars as S, ay, af. I have restored a lacuna in the manuscript, which must have occurred because the eye of the scribe moved from one passage in the equilateral triangle with perpendicular AK to the next. I have also restored perpendicular YH in Figure 9—VI. I assume that AD is greater than YK and GB are parallel, so that YOKD = AD. The Dublin manuscript ends with the remark: the first volume of geometrical propositions of the Dauli manuscript ends with the remark: the first volume of geometrical propositions. If the point falls on line EF, such as point a, we draw perpendiculars as S, ay, af. Then lines S, ay, af are equal to AD, so they exceed AD by line DT.
In the name of God, the Merciful, the Compassionate.

Letter by Ahmad Thu Muhammad ibn 'Abdal al-Talib ibn 'All Nazif ibn Yumn Al Phattani on the constitution of the angled triangle from two different straight lines.

You asked, may God make your happiness and last it.

Angled triangle from two different straight lines.

In the name of God, the Merciful, the Compassionate.

Translation of Text 2

Apparatus for Text 2

Traces of the 3rd Geometrical Elements of Mechanics
Traces of the lost Geometrica Elementa of Menelaus

158JANP.HOOENDIJK

Traces of the lost Geometrica Elementa of Menelaus

158JANP.HOOENDIJK

Traces of the lost Geometrica Elementa of Menelaus

Traces of the lost Geometrica Elementa of Menelaus
Traces of the Lost Geometric Elements of Menelaus

drawn from point A toward the side of C contains with line AB an obtuse angle, and similarly the line drawn from B toward the side of D contains with AB an obtuse angle. Therefore the limit of drawing them is in between the parallel lines AG, BD, that is what we wanted.

This is what we have presented by way of division (into cases) and determination (of possible and impossible cases), in a general method, easy to grasp and easy to follow, concisely phrased, according to what suits your mind and understanding. So profit from it, and may God make you happy with it. The letter has ended, with the praise to God and His grace. I wrote this on Thursday, the day Dey of the month Aban of the year 339 of the Yazdgerderac. The text means: a line AZ drawn on the other side of line AH, where point H is the second point of intersection of AG and circle ABD.

A construction by AbuSazda-Alã ibn Sahl, from al-Sijzi’s treatise On the Selected Problems Which Were Discussed by Him and the Geometers of Shiraz and Khorasan, and His Annotations.

A construction by Abu Sazda-Alã ibn Sahl, from al-Sijzi’s treatise On the Selected Problems Which Were Discussed by Him and the Geometers of Shiraz and Khorasan, and His Annotations.

The text means: a line AZ drawn on the other side of line AH, where point H is the second point of intersection of AG and circle ABD.

The solution is not mentioned in Kardzil’s edition of the works of Abu Sahl, although another solution appears on p. 354 of al-Karaji’s Al-Fakhri, in a different context. The solution is found in MS. Chester Beatty 3652. The solution is found in MS. Chester Beatty 3652.

The solution has been used in this text.
The two straight lines AR, GD are different, and AR is the longer of them. We want to construct from the man acute angled triangle. Let us place the (shorter) line at the middle of the greater line and let us make AL times DB equal to the square of EC. We place KE perpendicular at the midpoint of line AR and we make on diameters AR, TK the ellipse AKBT. Then it is tangent to the ellipse, or it intersects it, or it falls inside the ellipse. We draw perpendiculars GZ, DI to AR. If the semicircle falls inside the ellipse, then any two lines drawn from points G, D to line KG on the boundary of the ellipse contain an acute angle, and the two remaining angles are acute, and their sum, that is of the two lines drawn from points G, D, is equal to line AR, so we have constructed what we wanted. If it is tangent to the ellipse at point K, then any two lines drawn from points G, D to any point on line KG except point K contain an acute angle, and the two remaining angles are acute. And if it intersects it at points L, M, then any two lines drawn from points G, D to lines ZL, BM contain an acute angle, and the two remaining angles are acute, and their sum, that is of the two lines drawn from points G, D, is equal to line AR, so we have constructed from lines AR, GD an acute angled triangle, and that is what we wanted to demonstrate.
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