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Which version of Menelaus' *Spherics* was used by Al-Mu'taman ibn Hūd in his *Istikmāl*?

1. Introduction

Yūsuf al-Mu'taman ibn Hūd was the ruler of Saragossa from 1081 until his death in 1085. He was also a brilliant mathematician and astronomer, who probably did most of his scientific work during the reign of his father, Aḥmad al-Muqtadir ibn Hūd († 1081). Yūsuf al-Mu'taman ibn Hūd wrote a mathematical work, the *Istikmāl* (Perfection), which was well-known in his time. Large parts of this work have recently come to light in four anonymous manuscripts[1]. The *Istikmāl* is mainly a compilation made from Greek mathematical works in Arabic translation, of which the most important are Euclid's *Elements* and *Data, On the Sphere and Cylinder* of Archimedes, the *Conics* of Apollonius, the *Spherics* of Menelaus and the *Spherics* of Theodosius. Al-Mu'taman added material from treatises by Islamic mathematicians, namely the Banū Mūsā, Thābit ibn Qurra, Ibrāhīm ibn Sinān and Ibn al-Haytham, and also some discoveries of his own[2]. The extant parts of the *Istikmāl* contain nothing else than straight mathematics in the style of the *Elements* of Euclid, and no sources are mentioned. Therefore it is not easy to distinguish between original contributions by Al-Mu'taman and material taken from unidentified sources which may be lost. In any case, Al-Mu'taman's ability as a mathematician is already apparent from the skilful way in which he rearranged and simplified the material in his sources.

The *Istikmāl* was written in an interesting time and place. Less than a century after the death of Al-Mu'taman, a large part of Spain had been conquered by the Christians, including the kingdom of Saragossa, and many scientific works had been translated from Arabic into Latin in Spain. By analyzing the *Istikmāl* and its sources

1 On Al-Mu'taman see Djebbar. For the manuscripts see my paper, "Discovery ...". Dr. Djebbar independently identified the Cairo manuscript.
2 An analysis of the arithmetical part of the *Istikmāl* will be published by Dr. Djebbar. For the geometrical part see my paper, "The geometrical part of the *Istikmāl* of Al-Mu'taman ibn Hūd. An analytical table of contents."

one can discover interesting details about the scientific literature that was available at Saragossa shortly before this crucial period of transmission. In this paper I will analyze the relationship between the *Istikmāl* and one of its sources in order to determine what version of the source was used. For this purpose I have chosen the *Spherics* of Menelaus, because the complicated medieval tradition of this work has been relatively well researched by Bjørnbo in 1902 and Krause in 1936. It turns out that Al-Mu'taman used an Arabic version (Krause's version D) which is now lost, but the existence of which was predicted by Krause on the basis of the medieval Hebrew and Latin translations of the *Spherics*. By a consideration of scribal errors it will be shown that the manuscript of version D of the *Spherics* that Al-Mu'taman used was very closely related to the manuscript that was translated into Latin by Gerard of Cremona. The exact relationship between these two lost manuscripts cannot be established in the absence of a critical edition of Gerard's translation. It is also an open question whether similar relationships hold between other sources used by Al-Mu'taman and their 12th century Latin translations. Clearly, the *Istikmāl* opens interesting possibilities for research in the medieval tradition of some of the great classics of Greek mathematics.

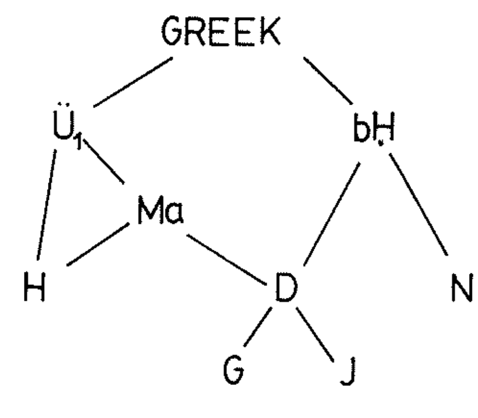
2. The Spherics of Menelaus and its medieval tradition

Menelaus of Alexandria[3] was a mathematician and astronomer in the late first century A. D. and the author of several mathematical works, most of which are lost. His *Spherics* was the most advanced text that was produced in antiquity on the geometry of the sphere. It is in three Books, and a small fragment of the first Book survives in a fourth-century redaction by Theon of Alexandria[4]. The work was translated into Arabic at least twice and there exist a large number of Arabic versions. In 1936 Krause published an edition with German translation of the Arabic revision of the *Spherics* by Abū Naṣr ibn ʿIrāq (10th century, GAS V, 338). Krause added an introduction in which he investigated the relationship between all medieval Arabic, Hebrew and Latin versions that were available to him.

3 On Menelaus see DSB IX, 296–302. For his works in Arabic see GAS V, 158–164.

4 On Theon see DSB XIII, 321–325. The fragment of the *Spherics* is discussed by Bjørnbo, *Studien,* 22–26.

I now summarize Krause's conclusions on the medieval tradition of the *Spherics,* as far as they are relevant here. Figure 1 is adapted from Krause, 86, and the same letters have been used to denote the versions. The numbering of the positions varies wildly among the different versions. I always follow the proposition numbers in the version of Abū Naṣr ibn ʿIrāq (which was edited by Krause).



The *Spherics* of Menelaus was translated into Arabic at least twice. The first translation ($Ü_1$) was made in the eighth century A. D. Nothing is known about the identity of the translator, and no manuscripts of the translation were available to Krause. Because the translation was defective, Al-Māhānī (ca. A. D. 860, GAS V, 260) undertook a revision of the text (Ma). He did not understand proposition 5 of Book III, which involves cross-ratios, so he did not go further, and thus his revision (Ma) was incomplete. The revision is also lost. A century later, Abū l-Faḍl al Harawī (GAS V, 329) revised the entire text on the basis of Ma and $Ü_1$. The version of al-Harawī (H) is extant in its entirety.

The *Spherics* was translated a second time by Isḥāq ibn Ḥunayn (9th century, GAS V, 272), and his translation (bḤ) was superior to $Ü_1$. The translation is now lost. It was slightly edited and provided with commentaries by Abū Naṣr ibn ʿIrāq, and the resulting version (N) was edited and translated by Krause.

The *Spherics* was translated into Latin by Gerard of Cremona (ca. 1114–1187, DSB XV, 173–192) and into Hebrew by Jacob ben Māḥir ben Tibbon (1236–1305, DSB XIII, 400–401). These two translations (indicated by G for Gerard and J for Jacob) are extant. Krause showed that they were made from a lost Arabic version (D) made by an unknown author. Version D consists of two parts, which Krause calls Da and Db, and which are clearly distinguishable on stylistic grounds. Part Da consists of the whole Book 1, Book 2, props. 1–17 and Book 3, prop. 1. Part Db consists of Book 2, props. 9–12 and Book 3, props. 2–25. Propositions 9–17 of Book 2 occur twice in the text, first in the part Da, and then in a different redaction in Db. Note that I use the proposition numbers of Abū Naṣr ibn ʿIrāq, in the versions G and J the propositions are of course numbered consecutively. Krause argued on good grounds that Db was adopted (almost) literally from the translation of Isḥāq ibn Ḥunayn (bḤ), and that Da was probably based on a revised form of the translation Ü_1. Krause showed that Da is not dependent on H, and he concluded that Da must be based on the revision by Al-Māhānī. One may well ask why the unknown author of D compiled a new version of the *Spherics* from two different versions (Ma and bḤ). This rather strange behaviour is perhaps best explained by the assumption that he had an incomplete and damaged manuscript of bḤ.

Some of the characteristics of D had already been mentioned before Krause by Bjørnbo in his *Studien,* 13–14, on the basis of G only. In this paper it will be shown that Al-Muʾtaman used a version D as described by Bjørnbo and Krause, so the conclusions of these eminent scholars will be confirmed.

Krause's scheme is not the final word on the medieval tradition of the *Spherics*. Naṣīr al-Dīn al-Ṭūsī (1201–1274, Krause p. 57, I 21) says that a third translation of the *Spherics* was made in the ninth century by Abū ʿUthmān al-Dimashqī (GAS V, 287), and revised in the tenth century by Yuḥannā ibn Yūsuf (GAS V, 298). Since 1936 new manuscripts and versions of the *Spherics* have been discovered by F. Sezgin (GAS V, 161). These manuscripts have yet to be studied in detail.

3. The propositions in the Istikmāl based on the Spherics of Menelaus

The *Istikmāl* is divided into five large parts, called "species" (anwāʿ). The first two species are divided into sections, the other species are divided into "species of species", which may, in turn, be divided into sections. Each section contains between 8 and 47 mathematical propositions (theorems or constructions). The propositions that are based on the *Spherics* of Menelaus occur in the "second section of the second species of the fourth species". I will abbreviate this section as section 422, and I will use a notation such as 422.5 for proposition 5 of section 422. Section 422 is entitled "On

the properties of circles of the sphere and their arcs and chords, according to their relation to one another." Only the beginning of the section is lost, including propositions 1, 2 and the beginning of proposition 3. The other propositions (most of 422.3 and 422.4–422.21) are extant in the Arabic manuscript Copenhagen[5], Royal Library, Or. 82, f. 79–90b:12.

Section 422 of the *Istikmāl* is essentially a summary of the *Spherics* of Menelaus, although Al-Muʾtaman also consulted other sources. In 422.10 he copied a proof of the spherical transversal theorem (*Spherics* III:1) from Thābit ibn Qurra's *Treatise on the Transversal theorem* (GAS V, 268 no. 1). Prop. 422.9 is a preliminary, taken from the same source. In propositions 422.18–19 Al-Muʾtaman proves theorems from Book III of Menelaus' *Spherics,* but his proofs use methods of the *Spherics* of Theodosius (ca. 200 B. C., see DSB XIII, 319–321). Propositions 422.16 and 422.17 are preliminary results directly based on Theodosius. The remainder of the section is exclusively based on the *Spherics* of Menelaus, and from now on, the term *Spherics* will always refer to the work of Menelaus. Al-Muʾtaman appears to have copied some propositions, such as 422.13 (edited and translated below) word for word from his manuscript of the *Spherics*. In other cases, such as 422.4, he put several theorems in the *Spherics* together in one proposition, and he condensed and modified the proofs.

I now render a table of the propositions with page and line numbers of the Copenhagen manuscript, and the corresponding propositions from the *Spherics*. Propositions indicated by an asterisk are edited and translated in Section 5 of this paper. Detailed descriptions of the other theorems in section 422 are to be found in my paper on the geometrical parts of the *Istikmāl*. In the cases where a proposition in the *Istikmāl* corresponds closely to the text in the *Spherics* in the version of Abū Naṣr ibn ʿIrāq, I have added page and line numbers of the corresponding Arabic text in the edition of Krause. A notation such as 63:15 refers to line 15 of page 63, and III:4 refers to proposition 4 of Book III of the *Spherics*.[6]

5 For a description of this manuscript see *COBRH* 64–67.

6 There is a curious difference between the text of *Spherics* III:2 and the *Istikmāl* 422.11. In the Copenhagen manuscript, 83b:16, 84a:7,12 Al-Muʾtaman says that two angles G and Z are right angles. In the edition of Krause, these angles are said to be equal. To get Al-Muʾtaman's text, change in Krause's edition 64:15, 65:8,13 *(al-)mutasāwiyatān* to *(al-) qāʾimatān.* Al-Muʾtaman was perhaps confused by a special case of III: 2, the so-called rule of four quantities, for which see Bjørnbo's Studien, 93 ($\angle C = \angle Z = 90°$).

Istikmāl (sec. 422) prop.	Copenhagen Or. 82	*Spherics*	Krause's edition (Arabic text)
3	79a:1–79b:1	I:2,3,7,6,10,11	–
4*	79b:2–80b:1	I:14,15,17,12,18,16	–
5*	80b:2–81b:2	I:19,20,21	≈ 15:8–17:3
6*	81b:2–17	I:22	≈ 17:10–20
7	81b:18–82a:14	I:29,32	–
8	82a:15–82b:22	I:34,35	–
10	83a:17–83b:11	III:1	–
11	83b:12–84a:12	III:2	64:11–65:14 (see note 6)
12	84a:13–84b:10	III:3	65:21–66:18
13*	84b:11–29	III:4	67:12–68:5
14	85a:1–21	III:6,7	72:23–73:16, 73:19–74:2
15	85a:22–85b:9	II:19	53:15–54:8
18	86b:24–87b:5	II:18,20	–
19	87b:6–88b:9	II:14,15	–
20	88b:10–89a:10	III:22	97:23–98:26
21	89a:11–90b:12	III:24,25	–

The table suggests that Al-Muʾtaman did not understand III:5, the proposition on cross-ratios which was also a stumbling block for his contemporaries (cf. Bjørnbo, *Studien*, 96–97, Samsó, *Estudios*, 66).

4. The Istikmāl and Krause's version D of the Spherics

The following arguments show that Al-Muʾtaman used the lost Arabic version D of the *Spherics*, which underlies the extant Latin translation by Gerard of Cremona (G) and the Hebrew translation by Jacob ben Māḥir (J).

1. Menelaus used the word τρίπλευρον, "trilateral", for a spherical triangle. The two Arabic translations Ü$_1$ and bḤ seem to have rendered this word differently, and as a result of this we find in G in the part corresponding to Da "triangulus ex arcubus circulorum magnorum super superficiem sphaerae" (triangle of arcs of great circles on the surface of a sphere), and in the part corresponding to Db "trilatera figura"[7]. In the same way, we find in the *Istikmāl* in propositions 422.4–8, based on Book I of the *Spherics*, the term "muthallath min qisī dawāʾir ʿiẓām ʿalā basīṭ kurā" (triangle of arcs of great circles on the surface of a sphere) and in 422.11–14, based on III:2–7, the term "shakl dhū thalātha aḍlāʿ" (figure having three sides). The spherical triangle is not mentioned in 422.3, 15, 18, 19, 20, 21.

7 Professor P. Kunitzsch (München) remarks that this translation is odd. One would expect Gerard to translate the Arabic "shakl dhū thalātha aḍlāʿ" as "figura habens tria latera."

2. There are interesting differences between Ü$_1$ and bḤ in the translations of I:13,22. These theorems were translated correctly by Isḥāq ibn Ḥunayn, but one condition was misunderstood in Ü$_1$, Ma, H, Da, and by Al-Muʾtaman. The details are as follows.

Consider two spherical triangles ABG, DEZ (Figure 2).

In I:13, one assumes $\angle A = \angle D$, arc(AB) = arc(DE), arc(BG) = arc(EZ) and one more condition, namely $\angle G + \angle Z \neq 180^\circ$ in the version of Abū Naṣr ibn ʿIrāq (hence also in the translation of Isḥāq ibn Ḥunayn), and $\angle G \neq 90^\circ$, $\angle Z \neq 90^\circ$ in H, G, J (therefore also Da) and the *Istikmāl*, 422.4 (see notes 16,19). One proves arc(AG) = arc(DZ).

The condition in H, Da and the *Istikmāl* is incorrect. It was criticized by Naṣīr al-Dīn al-Ṭūsī, who says that the version of Al-Māhānī also contained the same incorrect condition (Krause p. 56). The error must have been caused by an incorrect translation from the Greek in Ü$_1$. In the Greek summary of I:13 by Theon of Alexandria, the relevant passage is: Ἐὰν δύο τρίπλευρα … ἔχῃ, …, τὰς δὲ λοιπὰς γωνίας ἅμα δυσὶν ὀρθαῖς μὴ ἴσας, …" "If two triangles are such that … the remaining angles (G, Z in Figure 2) are not equal to two right angles …" (Bjørnbo, *Studien*, 22–23). The phrase can be understood in two different ways, as $\angle G \neq 90^\circ$, $\angle Z \neq 90^\circ$ and

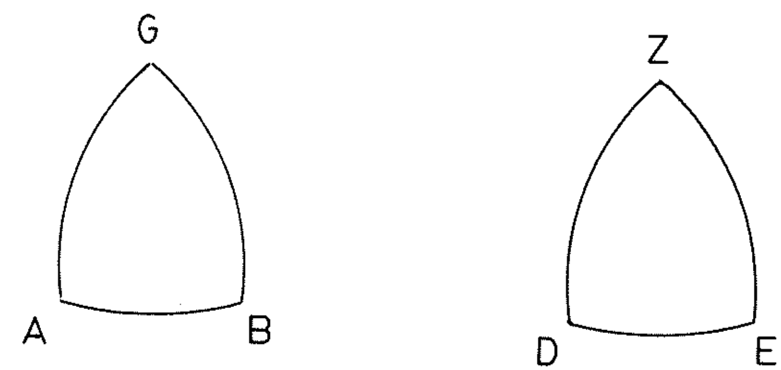


Fig. 2

as $\angle G + \angle Z \neq 180^\circ$. One can only be certain of the correct interpretation if one understands the mathematics. The Greek is sometimes ambiguous in rendering the sum of two mathematical quantities, and this also caused problems to the translator of a Greek text on the regular heptagon, attributed to Archimedes (see my "Greek and Arabic constructions of the regular heptagon", 211–212).

In I:22, one assumes in Figure 2 $\angle Z > G$, $\angle D < \angle A$, and one more condition, namely $\angle E + \angle B \geq 180^\circ$ in the version of Abū Naṣr ibn ʿIrāq (hence also in bḤ) and $\angle E \geq 90^\circ$, $\angle B \geq 90^\circ$ in G, J (and therefore Da), H, and the *Istikmāl* (422.6, see notes 32, 33). The difference must be due to a similar misinterpretation of the Greek in $\ddot{U}_1$. Naṣīr al-Dīn al-Ṭūsī says that Ma and H have the condition $\angle E \geq 90^\circ$, $\angle B \geq 90^\circ$, which is "correct, but more special than is necessary." (Krause, 58)

3. We now turn to the relationship between the *Istikmāl* and the last part of the *Spherics*, corresponding to Db (II:9–21 and III:2–25). The text of 422.21, corresponding to III:24, is different from that in the *Spherics*, because Al-Muʾtaman modified the proof. Propositions 422.11–15 and 422.20 correspond almost word for word to the parts of the version of Abū Naṣr ibn ʿIrāq that are displayed in the table above. These parts all belong to the main text of the *Spherics*, not to the comments by Abū Naṣr ibn ʿIrāq (which are clearly distinguished from the main text). Krause argued (p. 20) that part Db was directly based in bḤ and that Abū Naṣr ibn ʿIrāq made only minor changes to the main text of the *Spherics* in the translation of Isḥāq ibn Ḥunayn. Therefore the close correspondence can be explained by assuming that Al-Muʾtaman used the version D of the *Spherics*.

4. There is a striking terminological resemblance between the *Istikmāl* and Db. In Book III of the *Spherics*, Menelaus often uses the chord of twice a given arc in a circle with the same diameter as the sphere which one considers. For example, if we write Ch(2AH) for the chord of twice arc AH, Menelaus proves in III:4 (edited and translated below) that Ch(2AH) : Ch(2GH) = Ch(2DT) : Ch(2TZ) in Figure 6 below. As a technical term, he used ἡ ὑπὸ τὴν διπλῆν AB for Ch(2AB) (Krause 43). A natural Arabic translation of this term is "watar ḍaʿf al-qaws" ("the chord of twice the arc"). This term was used by Thābit ibn Qurra in his *Treatise on the transversal figure* in his own proof of the spherical transversal theorem, which Menelaus had proved in a more complicated way in *Spherics* III:1. In 422.9 and 422.10 Al-Muʾtaman copied the proof of Thābit ibn Qurra, including the term "watar ḍaʿf AB" for the chord of twice arc AB.

In propositions 422.11 and further, Al-Muʾtaman suddenly calls the same quantity "naẓīr qaws AB", the (line) corresponding to arc AB[8]. The same term must have been used in Db, because Gerard of Cremona uses the translation *nadir arcus.*

In his revision of the *Spherics*, Abū Naṣr ibn ʿIrāq changed the chord of twice arc AB everywhere into the sine (Arabic: jayb) of arc AB in a circle with the same diameter. The two functions are not the same, because Ch(2AB) = 2Sin(AB) (in a circle with the same radius), but the factors 2 cancel out in all proportions, so the theorems in Book III of the *Spherics* remain valid.

The preceding four arguments prove that Al-Muʾtaman used version D of the *Spherics*. One can even go a little further. Krause presents in his "Beilage 2" the text of III:4 in all medieval editions which were available to him, including the Latin translation (G) and the Hebrew (J). I have edited the corresponding proposition 422.13, and I have appended the Latin text in G. Krause points out that a large part of the proof was omitted in G, probably as a result of scribal error. The missing part of the proof is mathematically necessary, and it is found in J, so the proof in D must have been complete. Exactly the same part is also missing from the *Istikmāl* (see footnote 36). Similarly, in *Spherics* I:22, a passage which belonged to D (because it is mentioned in J) is missing in G, again as a result of scribal error, and exactly the same passage is also missing in the corresponding proposition 422.6 in the *Istikmāl* (see footnote 33). Thus Al-Muʾtaman and Gerard of Cremona used manuscripts of D in which two identical scribal errors occur[9]. I find it

8 Al-Muʾtaman introduces the term immediately after the enunciation of 422.11 (corresponding to *Spherics* III: 2) in the following way: "wa-aʿnī bi-qawlī naẓir al-qaws al-khaṭṭ al mustaqīm alladhī yuwattiru ḍaʿfaha" (and I mean by expression "naẓīr of the arc" the straight line which subtends the double of it). Compare Gerard of Cremona in III:2 (cited by Krause, 31). Professor Samsó points out to me that the term *naẓīr qaws AB* occurs in quotations from the *Spherics* in a treatise by Abū Naṣr ibn ʿIrāq which he wrote before he composed his own revision of the *Spherics* (Samsó, *Estudios*, pp. 134–150, 159). Thus the term *naẓīr qaws AB* was introduced by the translator Isḥāq ibn Ḥunayn and not by the author of D, as Krause believed (pp. 31, 43).

9 Krause (p. 11) mentions a few more differences between G and J, and two of these differences correspond to passages which occur in the *Istikmāl*. In *Spherics* III:2 (Krause, Arabic text 64:22), N and J have "and we complete the figure", but G has "et protraham duos arcus *gb*, *th* in parte *b*, donec concurrant. Ergo concurrant super punctum *k*." The *Istikmāl* is similar to G: wa-nakhruju qawsay GB, HT ḥattā yal-

hard to believe that this can be attributed to mere accident. I therefore conclude that the manuscripts of D used by Al-Muʾtaman and Gerard of Cremona were very closely related.

The *Istikmāl* is the first Arabic text which appears to be based on the lost version D. Part Db is closely related to the extant version of Abū Naṣr ibn ʿIrāq, so the corresponding textual material in the *Istikmāl* is not really exciting. However, Da is based on a lost revision of a lost translation $\ddot{U}_1$, and the revision and translation survive only in the drastically revised version of Al-Harawī (Krause 15–19, 35 note 2, 37). It is therefore interesting to look for traces of Da in the *Istikmāl*. In propositions 422.3, 4, 7 and 8 Al-Muʾtaman rearranged the material in Da, and he modified the proofs and the notation. Only propositions 422.5 and 6 have the same structure as *Spherics* I:19–22, and therefore these propositions are as close as one can get to a fragment of Da in the *Istikmāl*. They have therefore been edited and translated below. It is interesting to compare the notation in the figure in 422.5, which is ultimately dependent on the old translation $\ddot{U}_1$, and the corresponding notation in the edition of Abū Naṣr ibn ʿIrāq, which depends on the translation of Isḥāq ibn Ḥunayn. It is natural to assume that neither the author of Da nor Abū Naṣr ibn ʿIrāq changed the notation in the geometrical figures. Therefore the letters in the *Istikmāl* and the version of Abū Naṣr are probably the same as those in $\ddot{U}_1$ and bḤ respectively, and we can now plausibly reconstruct the Greek letters from which they were transcribed. Isḥāq seems to have transcribed the Greek letters by Arabic letters with the same numerical value, see the table below. The transcriptions of H, Θ and K in $\ddot{U}_1$ show that $\ddot{U}_1$ was translated from a Syriac version[10]. Thus Krause's conjecture (p. 85) is confirmed.

N (=bḤ)	ا	ب	ج	د	ه	ز	ح	ط	ك	ل	م
Istikmāl (= $\ddot{U}_1$)	ا	ب	ج	د	ه	ز	ي	ث	ق	ل	م
Greek	A	B	Γ	Δ	E	Z	H	Θ	K	Λ	M

taqiyā ala nuqṭa K" (ms K 83b:22 and we extend arcs GB, HT until they meet at K). There is another difference in *Spherics* III:22 (Krause, Arabic text 98:17–19), but Al-Muʾtaman seems to have made a little change in the corresponding passage in prop. 422.20 to make the proof clearer.

10 This was pointed out to me by Professor H. Daiber. See Daiber, *Aetius Arabus.*

5. *Text and translation of selected propositions*

The following pages contain edited Arabic texts and translations of propositions 422.4–6 and 422.13 of the *Istikmāl*. The text is found in the manuscript K: Copenhagen, Royal Library, Or. 82, 79b:2–81b:17 and 84b:11–29. I am grateful to the Royal Library for permission to publish this text.

The scribe of K always wrote in the text the full names of letters denoting points in geometrical figures. Thus he writes "triangle alīf bāʾ jīm", not "triangle ABG". This rather unusual procedure renders scribal errors less likely. In my edited text I have written the letters themselves instead of their full names, but in the apparatus I write them as the scribe does. Trivial errors in the text which involve diacritical points only (such as yakūnu vs. takūnu) have not been noted in the apparatus. All mathematically significant errors have been noted, even if they involve diacritical points only. In propositions 422.4 and 422.5 a point in the figure is called tā' in many cases and thā' in many other cases. I have assumed that thā' is the original reading, because it is easier to explain tā' (with two diacritical points) as a scribal error of thā' (with three diacritical points) than the other way around. I did not want to burden the apparatus with indications of all passages where the manuscript has tā', because no confusion with other points in the figure is possible. In establishing the text I have used marginal corrections in the manuscript, as indicated in the apparatus. Passages in angle brackets ⟨ ⟩ have been restored to the text by me. Words in square brackets [ ] occur in the manuscript but have to be deleted from the text. All explanatory additions of my own are in parentheses. In the Arabic text, numbers between parentheses refer to the apparatus at the end of the text, and a notation such as K 80b indicates the beginning of page 80 b in the manuscript.

In the translation I have transcribed thā' as Θ, ṭā' as T and yā' as Y. All other transcriptions are standard in the scholarly literature in English on the history of Arabic science. The translations are followed by notes on the mathematical meaning of the text and the relation with the *Spherics* of Menelaus. The text also presupposes a familiarity with basic facts on spherical geometry as explained in Books I and II of the *Spherics* of Theodosius, but I have not given explicit references to this work. I have attempted to draw the figures as they appear in the manuscript, and therefore I have not used a mathematically consistent projection of spherical triangles on a plane. The figures are to be found at the end of this paper: Figure 3 is for 422.4, Figure 4 for 422.5, Figure 5 for 422.6 and Figure 6 for 422.13.

د كل مثلثين من قسي دوائر عظام على بسيط كرةٍ تساوي زاويتان من أحدهما زاويتين من الآخر كل زاوية لنظيرتها والقوسان اللتان عليهما الزوايا المتساوية متساويتين أو (١) تكون إحدى القسي الموتَّرة أحدهما مساوية لنظيرتها من المثلث الآخر وتكون كل واحدة من القوسين الباقيتين مع نظيرتها من المثلث الآخر غير مساويتين لنصف دائرة إن لم تكن الزاويتان المتساويتان على القوس المساوية، أو كانت زاويتان منهما قائمتين وزاويتان أُخريان منهما (٢) مساويتين غير قائمتين وكانت القوسان اللتان (٣) توتّران الزاويتين القائمتين متساويتين (٤)، أو تساوت زاويتان منهما وتساوت القسي المحيطة بزاويتين أُخريين منهما كل قوس لنظيرتها وكانت كل واحدة من الزاويتين الباقيتين ليست بقائمة، أو كانت زوايا أحدهما مساوية لزوايا المثلث الآخر كل زاوية لنظيرتها، فإن المثلثين متساويان وأضلاعهما متساوية كل ضلع لنظيره وكذلك زواياهما كل زاوية لنظيرتها.

مثال ذلك: مثلثا ا ب جـ و د هـ ز زاوية ا مساوية لزاوية د وزاوية جـ مثل زاوية ز وقوس ا جـ مساوية لقوس د ز. فاقول إن زاوية ب مساوية لزاوية هـ وقوسا ا ب ب جـ مساويتان (٥) لقوسي د هـ هـ ز كل واحدة لنظيرتها. برهان ذلك: إنه لا يمكن غيره فإن أمكن فليكن أحدهما أعظم أو أصغر وليكن ا ب أصغر. ولنزد فيها حتى تكون مثل هـ د وهو ا ح. ولنرسم على نقطتي ح جـ قوساً من دائرة عظيمة. فيكون لذلك قوسا ح ا ا جـ مساويتين لقوسي هـ د د ز وزاوية ا مثل زاوية د. فقوس ح جـ مساوية لقوس هـ ز، وسائر الزوايا كسائر الزوايا. فزاوية ح جـ ا مساوية لزاوية ز التي هي مثل زاوية ا جـ ب. فزاوية ح جـ ا مساوية لزاوية ب جـ ا وهي أعظم منها. هذا خلف لا يمكن. فليست قوس ب ا بأصغر ولا بأعظم من قوس د هـ فهي مساوية لها. وأقول أنه إن كانت زاوية ا مثل زاوية د وزاوية ب مثل زاوية هـ وقوس ا جـ مثل قوس د ز وقوسا ب ا هـ د غير مساويتين لنصف دائرة وكذلك قوسا ب جـ هـ ز، فإن زاوية جـ مثل زاوية ز وقوس ا ب مثل د هـ وكذلك ب جـ مثل هـ ز. لا يمكن غيره (٦) فإن أمكن فليكن قوس ا ب أصغر أو أعظم من د هـ، وليكن أوّلا أصغر. ولنزد فيها حتى يكون مثل د هـ وهي ا ح. ونرسم على نقطتي ح جـ قوساً من دائرة عظيمة. فلأن ح ا ا جـ مثل هـ د د ز وزاوية ا مثل زاوية د فقوس ح جـ مثل قوس هـ ز وزاوية ح مثل زاوية هـ. وزاوية هـ قد كانت مساوية لزاوية ا ب جـ، فهي إذا مثل زاوية ح الداخلة. فقوسا ب جـ جـ ح مساويتان لنصف دائرة وقد فرضنا أقلّ. هذا خلف لا يمكن. فليست قوس ب ا بأصغر من قوس هـ د، ولا بأعظم منها فهي مساوية لها. ولذلك يكون

Translation of propositions 422.4, 5, 6, 13

4. (If there are) two triangles (consisting) of arcs of great circles on the surface of a sphere, such that (1,2) two angles of one of them are equal to two angles of the other, each angle to the (angle) corresponding to it, and (1) the two arcs at which are the equal angles, are equal, or (2) one of the arcs subtending one of them (sc. the angles) is equal to the (arc) corresponding to it in the other triangle, and each of the two remaining arcs plus the arc corresponding to it in the other triangle is not equal to a semicircle[11], if the two equal angles are not on the equal arc[12], or (if the triangles are) such that (3) two angles of them are right (angles), and two other angles are equal, not right (angles), and the arcs subtending the right angles are equal, or (4) such that two angles of them are equal and the arcs containing two other angles are equal, each arc to the (arc) corresponding to it, and each of the remaining angles is not (a) right (angle), or (5) such that the angles of one of them are equal to the angles of the other triangle, each angle to the (angle) corresponding to it, then the triangles are equal, and their sides are equal, each side to the (side) corresponding to it, and similarly their angles, each angle to the (angle) corresponding to it[13].

Example of this (Figure 3): Triangles ABG and DEZ (are such that) angle A is equal to angle D, angle G is equal to angle Z and arc AG is equal to arc DZ. I say that angle B is equal to angle E, and that arcs AB, BG are equal to arcs DE, EZ, each one to the (arc) corresponding to it.

Proof of this: It cannot be otherwise. For if that were possible, let one of them be greater or less; let AB be less. Let us add to it until it is equal to ED, namely (i. e. let the result be) AH. Let us draw through points H, G an arc of a great circle. Arcs HA, AG are therefore equal to arcs ED, DZ, and angle A is equal to angle D.

11 Al-Mu'taman states conditions for "each of the two remaining arcs", but only one of these conditions are necessary, see note 15.

12 It is unnecessary to say that "the two equal angles are not on the equal arc", because it has already been said that the two equal arcs "subtend" one pair of equal angles. Therefore the two equal arcs do not lie between the two pairs of equal angles.

13 Parts (1), (2), (3), (4) and (5) of this complicated sentence correspond to I:14–15, 17, 12, 13 and 18 respectively, using the proposition numbers of Abū Naṣr ibn ʿIrāq. In Bjørnbo's *Studien,* the corresponding proposition numbers are I:14, 16, 12, 13, 17 respectively. Al-Mu'taman's proofs differ from those in the *Spherics*. It seems that Menelaus did not like indirect proofs (Krause, 118).

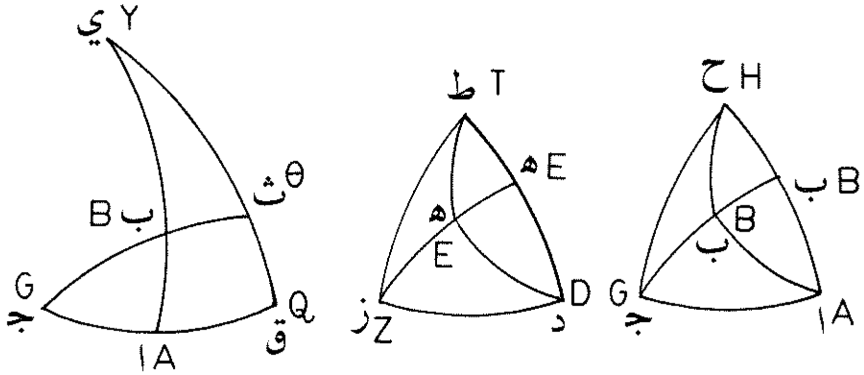


Fig. 3

Thus[14] arc HG is equal to arc EZ, and the remaining angles (are equal) to the remaining angles. Thus angle HGA is equal to angle Z, which is equal to angle AGB. Thus angle HGA is equal to angle BGA, but it is greater than it. This is absurd and impossible. Thus arc BA is neither less nor greater than arc DE, so it is equal to it.

I say that if angle A is equal to angle D, angle B is equal to angle E, arc AG is equal to arc DZ and arcs BA, ED are not equal to a semicircle and similarly arcs BG, EZ, then angle G is equal to angle Z, arc AB is equal to DE and similarly BG is equal to EZ.

It cannot be otherwise. For if this were possible, let arc AB be less or greater than DE; thus let it first be less. Let us add to it until it is equal to DE, namely (i. e. let the result be) AH. Let us draw on points H, G an arc of a great circle. Since HA, AG are equal to ED, DZ, and angle A is equal to angle D, hence arc HG is equal to arc EZ, and angle H is equal to angle E. But angle E was (supposed to be) equal to angle ABG, and therefore it is equal to the interior angle H. Therefore arcs BG, GH are equal to a semicircle, but we supposed (them to be) less[15]. This is absurd and impossible. Thus arc BA is not less than arc ED, and not greater than it, so it is equal to it. Therefore arc BG is equal to arc EZ, and the angle which is at G is equal to the angle which is at Z.

I say also: if angle A is (a) right (angle) and equal to the right angle D, (K 80a), and angles G, Z are equal, not right (angles), and

14 *Spherics* I:4. In the *Istikmāl* this must have been proved in one of the missing propositions 422.1 or 422.2.

15 We have arc(BG) + arc(GH) = 180° by *Spherics* I:10 (proved in 422.3). Note that Al-Mu'taman supposed arc(BG) + arc(EZ) $\neq$ 180°, not arc(BG) + arc(EZ) < 180°. The condition arc(BA) + arc(ED) $\neq$ 180° is not necessary.

arc BG is equal to arc EZ, then angle B is equal to angle E, AB is equal to DE and AG is (equal) to DZ.

Proof of this: We extend AB and DE to the poles of circles AG and DZ, let them be H, T. We join TZ, HG. Since angle BGA is equal to angle EZD, if we subtract them from the right angles G, Z, by subtraction angle HGB is equal to angle TZE. But the arcs containing them are equal. Thus angles H, T are equal and the arcs containing them are equal. Thus arc AG is equal to arc DZ, and AB (is equal) to DE, and angle B is equal to angle E.

I say also: if angle A is equal to angle D and arcs AB, BG are equal to arcs DE, EZ, each one to the (arc) corresponding to it, and none of the angles G, Z is (a) right (angle)[16], then arc AG is equal to arc DZ.

Proof of this: Let the poles of circles AG, DZ be points H, T. If each of the angles A, D is (a) right (angle), arc BH is equal to arc ET. The figure is the same as before, so[17] triangles BHG, ETZ have equal sides and angles. Thus[18] AG is equal to DZ. If not each of the angles A, D is (a) right (angle), we draw through the points A, H, H, G, H, B arcs of great circles, and similarly through the points T, E, T, Z, T, D. Then, if we subtract the equal angles BAG, EDZ from the right angles A, D, by subtraction angle HAB is equal to angle TDE. But the arcs containing them are equal. Thus arc HB is equal to arc ET, and angle AHB is equal to angle DTE. Thus the sides of triangles HBG are equal to the sides of triangle TEZ, so angle BHG is equal to angle ETZ. Thus[19] the whole angle AHG is equal to the whole angle DTZ, so arc AG is equal to arc DZ.

16 The conditions $\angle G \neq 90°$, $\angle Z \neq 90°$ are incorrect. The correct condition is $\angle G + \angle Z \neq 180°$. See note 19.

17 We have arc(BH) = arc(ET), arc(HG) = (TZ) = 90° and arc(BG) = arc(EZ), so *Spherics* I:4 can be used.

18 Because $\angle BHG = \angle ETZ$.

19 Here Al-Mu'taman tacitly assumes $\angle AHG = \angle AHB + \angle BHG$ and $\angle DTZ = \angle DTE + \angle ETZ$ (Figure 4, right and middle; points B and E are not on AH and DT). These assumptions are not always correct. If $\angle G$ is obtuse and $\angle A$, $\angle D$ and $\angle Z$ are acute, we have $\angle AHG = \angle AHB - \angle BHG$ and $\angle DTZ = \angle DTE + \angle ETZ$, so the proof is invalid. In this case also $\angle AGB = 90° + \angle HGB$ and $\angle DZE = 90° - \angle TZE$, $\angle HGB = \angle TZE$, so $\angle AGB + \angle DZE = 180°$. Therefore the proof can be rescued if we assume $\angle G + \angle Z \neq 180°$. In his *Studien*, 42, Bjørnbo "corrects" the theorem in the version of Gerard of Cremona by requiring $\angle G < 90°$, $\angle Z < 90°$ or $\angle G > 90°$, $\angle Z > 90°$. However, Gerard said $\angle G \neq 90°$, $\angle Z \neq 90°$ just like Al-Mu'taman, see the quotation in *Studien*, 22.

قوس ب جـ مساوية لقوس هـ ز والزاوية التي عند جـ مساوية للزاوية التي عند ز.

وأقول أيضاً أنه إن كانت زاوية ا قائمة مساوية لزاوية د القائمة (ك ٨٠ ا) وزاويتا جـ ز متساويتان غير قائمتين وقوس ب جـ مثل قوس هـ ز، فإن زاوية ب مساوية لزاوية هـ وإن ا ب مثل د هـ و ا جـ <مساوية> لـ د ز. برهان ذلك: إنا نخرج ا ب و د هـ إلى قطبي دائرتي ا جـ و د ز وليكونا ح ط ونصل ط ز ح جـ. فلأن زاوية ب جـ ا مساوية (٧) لزاوية هـ ز د فإذا طرحناهما من زاويتي جـ ز القائمتين تبقى زاوية ح جـ ب مساوية لزاوية ط ز هـ. والقسي المحيطة بهما متساوية. فزاويتا ح ط متساويتان والقسي المحيطة بهما متساويتان. فقوس ا جـ مساوية لقوس د ز و ا ب لـ د هـ وزاوية ب مثل زاوية هـ.

وأقول أيضاً أنه إن كانت زاوية ا مساوية لزاوية د وقوسا ا ب ب جـ مساويتان لقوسي د هـ هـ ز كل واحدة لنظيرتها وليست واحدة من زاويتي جـ ز قائمة فإن قوس ا جـ مساوية لقوس د ز. برهان ذلك: ليكن قطبا دائرتي ا جـ د ز نقطتى ح ط. فإن كانت كل واحدة من زاويتي ا د قائمة كانت قوس ب ح مثل قوس هـ ط. والصورة على حالها فكان (٨) مثلثا ب ح [ح] جـ هـ ط ز متساويي الأضلاع والزوايا فيكون ا جـ مثل د ز. وإن لم تكن كل واحدة من زاويتي ا د قائمة فإنا نرسم على نقط ا ح ح جـ [جا] ح ب قسي دوائر عظيمة وكذلك على نقط ط هـ ط ز ط د. فإذا نقصنا زاويتي ب ا جـ هـ د ز المتساويتين من زاويتي ا د القائمتين تبقى زاوية ح ا ب مساوية لزاوية ط د هـ. والقسي المحيطة بهما متساوية، فقوس ح ب مساوية لقوس هـ ط وزاوية ا ح ب لزاوية د ط هـ. فأضلاع مثلث ح ب جـ مساوية لأضلاع مثلث ط هـ ز فزاوية ب ح جـ مساوية لزاوية هـ ط ز. فجميع زاوية ا ح جـ مثل جميع زاوية د ط ز فقوس ا جـ مساوية لقوس د ز.

وأقول أيضاً أنه إن كانت زاوية ا مثل زاوية د وزاوية ب مثل زاوية هـ وزاوية جـ مثل زاوية ز فإن قوس ا ب مساوية لقوس د هـ وقوس ب جـ لقوس هـ ز وقوس ا جـ لقوس د ز. برهان ذلك: إنا نخرج قوس ا ب (٩) إلى ي ونفصل ب ي مثل د هـ. ونخرج قوس جـ ب إلى ث ولتكن ب ث مثل هـ ز. ونخرج قوس ي ث من دائرة عظيمة ونخرجها ولتلق قوس جـ ا إذا أخرجت علي نقطة ق. فيكون قوس ث ي مساوية لقوس د ز ويكون زاوية ي (١٠) المساوية لزاوية د مساوية لزاوية ا وزاوية ب ث ي مساوية لزاوية ز المساوية لزاوية جـ. فزاوية ب ث ي الخارجة عن مثلث ث ق جـ مساوية لزاوية جـ المقابلة لها

I say also: if angle A is equal to angle D, angle B is equal to angle E and angle G is equal to angle Z, then arc AB is equal to arc DE, arc BG is equal to arc EZ, and arc AG is equal to arc DZ. Proof of this: We extend arc AB to Y and we cut off BY equal to DE. We extend arc GB to Θ and let BΘ be equal to EZ. We draw arc YΘ of a great circle and we extend it, let it meet arc GA if it is extended at point Q. Then[20] arc ΘY is equal to arc DZ, and angle Y, which is equal to angle D, is equal to angle A, and angle BΘY is equal to angle Z, which is equal to angle G. Thus the exterior angle BΘY of triangle ΘQG is equal to the opposite interior angle G. Thus[21] arcs ΘQ, QG are equal to a semicircle. Since the exterior angle A of triangle AQY is equal to the opposite interior angle, namely angle Y, arcs AQ, QY are equal to a semicircle. Thus arcs GQ, QΘ are equal to arcs AQ, QY. We drop the common (arcs) AQ, QΘ, then, by subtraction, arc AG is equal to arc ΘY, which is equal to arc DZ. Thus arc DZ is equal to arc AG. But angle A is equal to angle D and angle G (is equal) to angle Z. Thus[22] arcs AB, BG are equal to arcs DE, EZ, each one to the (arc) corresponding to it.

It has become clear from what preceded that if there are two triangles of arcs of great circles on the surface of a sphere, such that two angles of one of them are equal to two angles of the other, each angle to the (angle) corresponding to it, and (such that) the arcs containing the two remaining angles are equal, each arc to the (arc) corresponding to it, and (such that) the poles of the two remaining arcs are not at the two remaining angles, then the remaining arcs are equal[23]. That is (clear) if we draw to the angles of the triangles from the two poles of the bases arcs of great circles (K 80b). Then it will be clear as we have demonstrated previously. That is what we wanted to demonstrate.

5. (In) every two triangles of arcs of great circles on the surface of a sphere, such that two angles of one of them are equal to two angles of the other, each angle to the (angle) corresponding to it, and (such that) the remaining angle of one of them is greater than the remaining angle of the other triangle, the arc which subtends the greater angle is greater than the corresponding (arc) subtending the lesser angle. If one of the two remaining arcs of one of the triangles plus

20 *Spherics* I:4.
21 *Spherics* I:10, proved in 422.3.
22 *Spherics* I:14–15, proved in the beginning of 422.4.
23 This corollary is equivalent to *Spherics* I:16 (I:15 in Bjørnbo's *Studien*). Al-Mu'taman sketches a proof which is different from that in the *Spherics*.

الداخلة. فقوسا ث ق ق جـ مساويتان لنصف دائرة. ولأن زاوية ا الخارجة عن مثلث ا ق ي مساوية للداخلة المقابلة لها وهي زاوية ي فإن قوسي ا ق ق ي مساويتان لنصف دائرة. فقوسا جـ ق ق ث مساويتان لقوسي ا ق ق ي. فنلقي ا ق ق ث المشتركتين فتبقى قوس ا جـ مساوية لقوس ث ي المساوية لقوس د ز. فقوس د ز مساوية لقوس ا جـ. وزاوية ا مساوية لزاوية د وزاوية جـ لزاوية ز. فقوسا ا ب ب جـ مساويتان لقوسي د هـ هـ ز كل واحدة لنظيرتها.

وتبيّن مما تقدم أنه إذا كان مثلثان من قسي دوائر عظيمة على بسيط كرة تساوي زاويتان من أحدهما زاويتين من الآخر كل زاوية لنظيرتها ويكون القسي المحيطة بالزاويتين الباقيتين متساوية كل قوس لنظيرتها ولم يكن قطبا القوسين (١١) الباقيتين عند الزاويتين الباقيتين فإن القوسين الباقيتين متساويتان وذلك بأن نخرج إلى زوايا المثلثين من قطبي القاعدتين قسي دوائر عظيمة (ك ٨٠ ب) فتبيّن كما بيّنا فيما تقدم. وذلك ما أردنا أن نبين.

هـ كل مثلثين من قسي دوائر عظام على بسيط كرة تساوى زاويتان من أحدهما زاويتين من الآخر كل زاوية لنظيرتها وتكون الزاوية الباقية من أحدهما أعظم من الزاوية الباقية من (١٢) المثلث الآخر، فإن القوس التي توتر الزاوية العظمى أعظم من نظيرتها الموترة للزاوية الصغرى. فإن كانت إحدى القوسين الباقيتين من أحد المثلثين مع نظيرتها من المثلث الآخر مساويتين (١٣) لنصف دائرة فإن القوس الباقية مساوية لنظيرتها الباقية. وإن كانتا أعظم من نصف دائرة فإن القوس الباقية من المثلث الذي زاويته أصغر تكون أعظم من النظيرة لها من المثلث الآخر. وإن كانتا أصغر من نصف دائرة فإنها تكون أيضاً أصغر.

مثال ذلك : مثلثا ا ب جـ د هـ ز على بسيط كرة وزاوية ب مثل زاوية د وزاوية جـ مساوية لزاوية ز وزاوية هـ أعظم من زاوية ا. فأقول إن قوس د ز أعظم من قوس ب جـ. وإن كانت قوسا جـ ا هـ ز مساويتين لنصف دائرة كانت قوس ب ا مثل قوس هـ د ، وإن كانتا قوسا جـ ا هـ ز أصغر من نصف دائرة كانت قوس ب ا أصغر من هـ د ، وإن كانتا أعظم كانت أعظم.

برهانه: إنا نخرج جـ ا إلى ث (١٤) ولتكن قوس جـ ث مثل قوس هـ ز. ونخرج ب جـ إلى ي ونجعل جـ ي مثل د ز ونخرج ث ي (١٥) من دائرة عظيمة فتكون مثل قوس

the (arc) corresponding to it of the other triangle is equal to a semicircle, the remaining arc is equal to the remaining (arc) corresponding to it. If they are greater than a semicircle, the remaining arc of the triangle of which the angle is less, is greater than the (arc) corresponding to it in the other triangle. If they are less than a semicircle, it is also less[24].

Example of this (Figure 4): Triangles ABG, DEZ are on the surface of a sphere. Angle B is equal to angle D, angle G is equal to angle Z and angle E is greater than angle A. I say that arc DZ is greater than arc BG. If arcs GA, EZ are equal to a semicircle, arc BA is equal to arc ED. If arcs GA, EZ are less than a semicircle, arc BA is less than arc ED, and if they are greater, it is greater.

Its proof: We extend GA to Θ and let arc GΘ be equal to arc EZ. We extend BG to Y and we make GY equal to DZ. We draw ΘY of a great circle, then it is equal to arc ED.

Since GA, EZ are equal to a semicircle, AGΘ is a semicircle. Thus arc AB ends at Θ. Then, since angle Y is equal to the exterior angle B of triangle BΘY, arcs BΘ, ΘY are equal to a semicircle[25]. But arc ABΘ is a semicircle, so arc AB is equal to arc ΘY, which is equal to DE. Since angle GΘY, which is equal to angle E, is greater than angle A, if we make angle YΘQ equal to angle A, it meets arc GY, let it meet it at Q. Then, since angles Y, Θ are equal to angles A, B and arc YΘ is equal to arc AB, triangle ABG is equal to triangle YΘQ, and arc YQ is equal to arc BG[26]. Therefore arc YG, which is equal to arc DZ, is greater than arc BG.

Again, let arcs GA, EZ be less than a semicircle. I say that arc BA is less than ED. Its proof: We draw GΘ, GY as in the first figure, and we draw YΘ. Then it is equal, as we have proved, to arc ED. Since arcs GA, EZ are less than a semicircle, arc AGΘ is less than a semicircle. Thus if AB is extended, it does not meet point Θ. Thus let us extend BA, YΘ, and they meet at Q. Since

24 This sentence is equivalent to the enunciation of *Spherics* I:19 (Krause, Arabic text 15:8–18), but in the *Spherics*, the two arcs are first supposed to be less than the semicircle, and then to be greater. The proof in the *Istikmāl* is similar to that in the *Spherics*, in Krause's Arabic text 15:19–17:3. Abū Naṣr ibn ʿIrāq divided the proof into three propositions numbered 19 ($\mathrm{arc}(GA) + \mathrm{arc}(EZ) = 180^\circ$), 20 ($\mathrm{arc}(GA) + \mathrm{arc}(EZ) < 180^\circ$) and 21 ($\mathrm{arc}(GA) + \mathrm{arc}(EZ) > 180^\circ$). In the version of Gerard of Cremona, the proof is given in one proposition (numbered 18, see Krause, 7).

25 *Spherics* I:10.

26 *Spherics* I:14–15 (proved in 422.4).

هـ د . ولأن جـ ا هـ ز مثل نصف دائرة تكون ا جـ ث نصف دائرة فقوس ا ب تنتهي إلى ث (١٦). فلأن زاوية ي (١٧) مثل زاوية ب (١٨) الخارجة عن مثلث ب ث ي فقوسا ب ث ث ي (١٩) مثل نصف دائرة. وقوس ا ب ث نصف دائرة فقوس ا ب [ث] مثل قوس ث ي (٢٠) المساوية لـ د هـ . ولأن زاوية جـ ث ي المساوية لزاوية هـ أعظم من زاوية ا فإذا عملنا زاوية ي ث ق مثل زاوية ا لقيت قوس جـ ي فتلقيها على ق . فلأن زاويتي ي ث مساويتان لزاويتي ا ب وقوس ي ث مثل قوس ا ب يكون مثلث ا ب جـ مساوياً لمثلث ي (٢١) ث ق ويكون قوس ي ق مساوية لقوس ب جـ. فقوس ي (٢٢) جـ المساوية لقوس د ز أعظم من قوس ب جـ .

وأيضاً فليكن قوسا جـ ا هـ ز أصغر من نصف دائرة. فاقول إن قوس ب ا أصغر من هـ د. برهانه : إنا نخرج جـ ث جـ ي على ما في الصورة الأولى ونخرج ي ث فتكون مساوية على ما بيّنا لقوس هـ د . فلأن قوسي جـ ا هـ ز أصغر من نصف دائرة يكون قوس ا جـ ث أصغر من نصف دائرة. فإذا أخرج ا ب لم تلق نقطة ث ، فلنخرج ب ا ي ث ويلتقيان على ق. فلأن زاوية ي مثل زاوية ب الخارجة يكون قوسا ب ق ق ي مساويتان لنصف دائرة. ولأن زاوية جـ ث ي الخارجة من مثلث ا ق ث مثل زاوية هـ التي هي أعظم من زاوية ا تكون قوسا ا ق ق ث أصغر من نصف دائرة. ونسقط ب ق ق ث المشتركتين تبقى قوس ا ب أصغر من ث ي المساوية لقوس هـ د . (ك ٨١ ا) فإذا فصلنا من ي ث مثل ا ب على ل وأخرجنا ا ل من دائرة عظيمة قطعت قوس جـ ي فلتقطعها على م . ونجعل ب ق <ق> ل مشتركتين فتكون قوسا ا ق ق ل مساويتان لنصف دائرة لأنهما مثل قوسي ب ق ق ي المساويتين (٢٣) لنصف دائرة. فزاوية ق ا ل الداخلة مساوية لزاوية م ل ي (٢٤) المساوية للخارجة . فزوايا مثلث ا ب م مساوية لزوايا مثلث ي م ل فهما متساويان. فقوس ب م مساوية لقوس م ي (٢٥) . فلذلك قوس ب جـ أصغر من قوس جـ ي المساوية لقوس ز د ا وقوس ب جـ مساوية لقوس لقوس د ز ا.

وأيضا فليكن قوسا جـ ا هـ ز أعظم من نصف دائرة. فاقول إن ب ا أعظم من هـ د. برهانه: إنا نترك الأشياء على حالها في هاتين الصورتين. فلأن قوسي جـ ا هـ ز أعظم من نصف دائرة يكون قوس ا جـ ث أعظم من نصف دائرة. فإذا أخرجنا قوس ا ب حتى تلقى قوس ا جـ ث على نقطة ق كانت نقطة ق تحت نقطة ث مما يلي نقطة جـ. فلتُخرج حتى تلقى

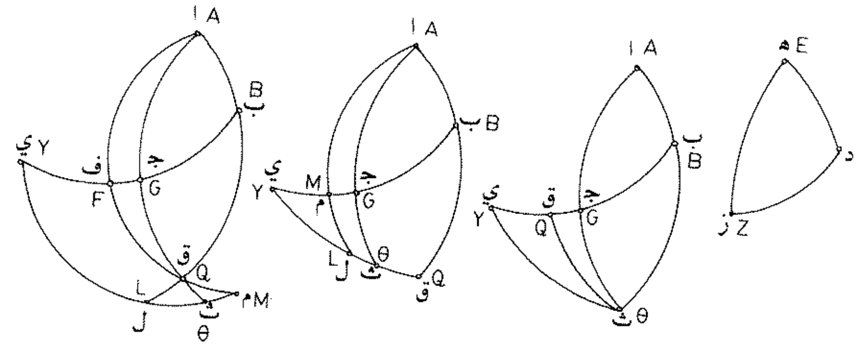


Fig. 4

angle Y is equal to the exterior angle B, arcs BQ, QY are equal to a semicircle. Since the exterior angle GΘY of triangle AQΘ is equal to angle E, which is greater than angle A, arcs AQ, QΘ are less than a semicircle[27]. We drop the common arcs BQ, QΘ, (hence) by subtraction arc AB is less than ΘY, which is equal to arc ED. (K 81a) Thus if we cut off from YΘ (an arc) equal to AB at L, and (if) we draw AL, (arc) of a great circle, it intersects arc GY, thus let it intersect it at M. We make BQ, ⟨Q⟩L common. Then arcs AQ, QL are equal to a semicircle since they are equal to arcs BQ, QY which are equal to a semicircle. Thus[28] the interior angle QAL is equal to angle MLY which is equal to the exterior angle. Thus the angles of triangle ABM are equal to the angles of triangle YML, so they (the triangles) are equal[29]. Thus arc BM is equal to arc MY. Therefore arc BG is less than arc GY, which is equal to arc ZD.

Again, let arcs GA, EZ be greater than a semicircle. I say that BA is greater than ED. Its proof: We leave the things as they are in these two figures. Then, since arcs GA, EZ are greater than a semicircle, arc AGΘ is greater than a semicircle. Thus if we extend arc AB to meet arc AGΘ at point Q, point Q is below point Θ, on the side of point G. Thus let it be extended until it meets arc YΘ at point L. Then, since angle B is equal to angle Y, arcs BL, LY are equal to a semicircle. Thus if we drop the common arc BQ, by subtraction arc AB is equal to arcs YL, LQ. Since angle Θ, which is equal to angle E, is greater than angle A, and angle A is equal to angle Q, hence angle Θ is greater than angle ΘQL, so arc QL is

27 *Spherics* I:10.
28 *Spherics* I:10.
29 *Spherics* I:18 (proved in 422.4).

قوس ي (٢٦) ث على نقطة ل . فلأن زاوية ب مساوية لزاوية ي تكون (٢٧) قوسي ب ل ل ي مساويتين لنصف دائرة. فإذا ألقينا قوس ب ق المشتركة بقيت قوس ا ب مساوية لقوسي ي ل ل ق . ولأن زاوية ث التي هي مثل زاوية هـ أعظم من زاوية ا وزاوية ا مثل زاوية ق فإن زاوية ث أعظم من زاوية ث (٢٨) ق ل فقوس ق ل أعظم من قوس ل ث . فقوسا ث ل ل ي (٢٩) المساويتان لقوس د هـ أصغر من قوسي ق ل ل ي اللتين هما مساويتان لقوس ا ب . ولنجعل قوس ي ث م مثل قوس ا ب. ونخرج على نقطتي م ق قوساً من دائرة عظيمة فهي تجوز على نقطة ا وتقطع قوس جـ ي فلتقطعها (٣٠) على نقطة ف . فلأن قوسي ي ل ل ق مساويتان لقوس ا ب وقوس ي (٣١) ل م فُصلت مساوية لقوس ا ب فإذا أسقطنا قوس ي ل المشتركة بقيت قوس ق ل مساوية لقوس ل م (٣٢). فزاوية ل ق م المساوية لزاوية ا مساوية لزاوية م. وزاوية ب مساوية لزاوية ي فزوايا مثلث ا ف ب مساوية لزوايا مثلث ي ف م. فقوس (ك ٨١ ب) ي ف مساوية لقوس ف ب. فقوس ي ف جـ المساوية لقوس د ز أعظم من قوس ب جـ. وذلك ما أردنا أن نبين.

و إذا كان على بسيط كرة مثلثا ا ب جـ د هـ ز وكانت قوس ا جـ مساوية لقوس د ز وكانت زاوية ا أعظم من زاوية د وزاوية جـ أصغر من زاوية ز وزاويتا ب هـ كل واحدة منهما ليست أصغر من قائمة، فأقول إن قوس ب جـ أعظم من قوس هـ ز وقوس هـ د أعظم من قوس ب ا . برهان ذلك: لأن زاوية جـ أصغر من زاوية ز فإنا نجعل على نقطة جـ من قوس ا جـ زاوية ا جـ ط مثل زاوية ز، ونجعل قوس جـ ط مثل قوس هـ ز. ونجيز على نقطتي ا ط قوساً من دائرة عظيمة. فيكون قوس ا ط مثل قوس هـ د وزاوية ا ط جـ مثل زاوية هـ . فلأن كل واحدة من زاويتي ب هـ ليست أصغر من قائمة فإنا اذا أخرجنا قوس ب ط من دائرة عظيمة تكون كل واحدة من زاويتي ا ب ط جـ ط ب أعظم من كل واحدة من زاويتي ا ط ب جـ ب ط . فلذلك زاوية ا ب ط أعظم من زاوية ا ط ب فقوس (٣٣) ط ا أعظم من قوس ب ا . ولأن زاوية جـ ط ب أيضاً أعظم من زاوية جـ ب ط فقوس جـ ب أعظم من قوس جـ ط. ولكن قوس ط ا مثل قوس هـ د فقوس هـ د أعظم من قوس ب ا. وقوس جـ ط مثل قوس هـ ز فقوس (٣٤) ب جـ أعظم من قوس هـ ز.

greater than arc LΘ[30]. Thus arcs ΘL, LY, which are equal to arc DE, are less than arcs QL, LY, which are equal to arc AB.

Let us make arc YΘM equal to arc AB. We draw through points M, Q an arc of a great circle; then it passes through point A, and it intersects arc GY, thus let it intersect it at point F. Then, since arcs YL, LQ are equal to arc AB and arc YLM has been cut off equal to arc AB, hence if we drop the common arc YL, by subtraction arc QL is equal to arc LM. Therefore[31] angle LQM, which is equal to angle A, is equal to angle M. But angle B is equal to angle Y. Thus the angles of triangle AFB are equal to the angles of triangle YFM. Thus arc (K 81b) YF is equal to arc FB. Thus arc YFG, which is equal to arc DZ, is greater than arc BG. That is what we wanted to demonstrate.

6. (Figure 5) If there are on the surface of a sphere triangles ABG, DEZ, and arc AG is equal to arc DZ, angle A is greater than angle D and angle G is less than angle Z, and each of the angles B, E is not less than (a) right (angle)[32], I say that arc BG is greater than arc EZ and arc ED is greater than arc BA. Proof of this: Since angle G is less than angle Z, we make on point G of arc AG angle AGT

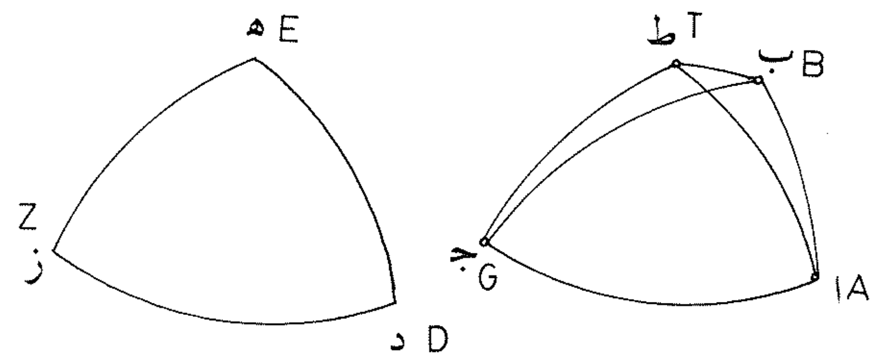


Fig. 5

30 In the version of Abū Naṣr ibn ʿIrāq part of the argument is missing. Krause's reconstruction of the missing steps, which is based on the text of Gerard of Cremona, is consistent with the *Istikmāl*. Arc(QL) > arc(LΘ) by *Spherics* I:7 (proved in 422.3).

31 *Spherics* I:2 (proved in 422.3).

32 The conditions $\angle B \geq 90^\circ$, $\angle E \geq 90^\circ$ are sufficient but not necessary. The version of Abū Naṣr ibn ʿIrāq has $\angle B + \angle E \geq 180^\circ$, and this must have been what Menelaus intended. The theorem 422.6 corresponds to *Spherics* I:22.

وذلك ما أردنا أن نبين.

يج إذا كان شكلان ذوا ثلاثة (٣٥) أضلاع وكانت زوياهما التي على القاعدة متساوية كل زاوية لنظيرتهما ولم تكن زاوية منها قائمة وأخرج عمودا الشكلان من نقطتي رأسيهما (٣٦) فإن نظائر القسي التي تنفصل من القاعدتين تكون متناسبة. فليكن شكلان ذوا ثلاثة أضلاع عليهما ا ب جـ د هـ ز ولتكن الزاوية التي عند ا مساوية للزاوية التي عند نقطة د والزاوية التي عند جـ مثل الزاوية التي عند ز ولا تكون واحدة من هذه الزوايا قائمة. ولنخرج من نقطتي ب هـ عمودين على قاعدتي ا جـ د ز وهما ب ح هـ ط. فأقول إن نسبة نظير قوس ا ح إلى نظير قوس ح جـ كنسبة نظير قوس د ط إلى نظير قوس ط ز. برهان ذلك: إنا نجعل قطبي قوسي ا جـ د ز نقطتي ك ل. فلأن الزاويتين اللتين عند نقطتي ح ط قائمتان والزاويتان اللتان عند نقطتي جـ ز متساويتان وليستا بقائمتين، تكون نسبة (٣٧) نظير قوس جـ ح إلى نظير قوس ز ط مؤلفة (٣٨) من نسبة نظير قوس ب ح إلى نظير قوس هـ ط (٣٨) ومن نسبة نظير قوس هـ ل إلى نظير قوس ب ك. وتكون لذلك نسبة نظير قوس ا ح إلى نظير قوس د ط كنسبة نظير قوس جـ ح إلى نظير قوس ز ط. وإذا بدّلنا أيضاً (٣٩) تكون متناسبة. وذلك ما أردنا أن نبيّن.

(١) أو تكون:وتكون. (٢) وزاويتان أخريان منهما:وزاويتين أخريين منها. (٣) القوسان اللتان: القوسين اللتين. (٤) متساويتين :مساويتين. (٥) مساويتان: مساويتين. (٦) غيره:غيرها. (٧) مساوية:المساوية. (٨) فكان:كان. (٩) با:زاي (نص)، با (حاشية). (١٠) يا:با. (١١) القوسين: (حاشية). (١٢) أحدهما أعظم من الزاوية الباقية من:(حاشية). (١٣) مساويتين:متساويتين. (١٤) ثا:با. (١٥) ثا يا: تا با. (١٦) ثا: فوّ (نص)،تا (حاشية). (١٧) يا: با. (١٨) با:يا. (١٩) فقوسا با ثا ثا يا:(حاشية). (٢٠) يا: با. (٢١) يا:با. (٢٢) يا:با. (٢٣) المساويتين: المتساويتين. (٢٤) يا:با. (٢٥) يا:با. (٢٦) يا:با. (٢٧) تكون: فلأن. (٢٨) ثا: با. (٢٩) يا: با. (٣٠) فلتقطعها:فلتقطعه. (٣١) وقوس يا :(حاشية). (٣٢) ميم:جيم. (٣٣) فقوس:وقوس. (٣٤) فقوس:وقوس. (٣٥) ذوا ثلاثة: ذوا ثالثة. (٣٦) رأسيهما: رأسهما. (٣٧) (حاشية: هذا هو الوجه التاسع من وجوه الثمانية عشر في تأليف النسبة.) (٣٨) مؤلفة...هـ ط (مكرر في المخطوط). (٣٩) وإذا بدلنا أيضاً: (حاشية).

equal to angle Z. We make arc GT equal to arc EZ. We let pass through points A, T an arc of a great circle. Then arc AT is equal to arc ED and angle ATG is equal to angle E. Since each of the angles B, E is not less than (a) right (angle), hence if we draw arc BT of a great circle, each of the angles ABT, GTB (*)[33] is greater than each of the angles ATB, GBT. Therefore angle ABT is greater than angle ATB, so arc TA is greater than arc BA. Since angle GTB is also greater than angle GBT, arc GB is greater than arc GT[34]. But arc TA is equal to arc ED. Thus arc ED is greater than arc BA. Arc GT is equal to arc EZ, so arc BG is greater than arc EZ. That is what we wanted to demonstrate.

13. If there are two trilateral figures, and if their angles at the base are equal, each angle to the (angle) corresponding to it, and no angle among them is (a) right (angle), and if the two altitudes of the figures are drawn from their vertices, then the (lines) corresponding to the arcs which are cut off from the two bases are proportional[35].

(Figure 6) Thus let there be two trilateral figures ABG, DEZ, and let the angle which is at A be equal to the angle which is at point D, and let the angle which is at G be equal to the angle which is

33 At (*) the Hebrew version (J) adds a helphul passage (see Krause 29, I 19), which is *italicized* in the following translation: "Hence if we draw arc BT a great circle, each of the angles *ATB, GBT is less than a right angle and therefore each of the angles* ABT, GTB is greater than each of the angles ATB, GBT." The Latin version (G) has "tunc cum nos extraxerimus arcum bg (read bp) circuli magni, erit unusquisque duorum angulorum abp, gpb maior unoquoque duorum angulorum apb, gbp." Here the letter p corresponds to T in Arabic. The underlined passage was obviously extant in D. It must have been left out in (an Arabic archetype of) G as a result of scribal error, because the eye of the scribe slipped from the first passage "each of the angles" to the second "each of the angles". The same passage was omitted in the *Istikmāl* at (*).
The proof in D, G, H and the *Istikmāl* assumes $\angle B \geq 90^\circ$, $\angle E \geq 90^\circ$. If one assumes $\angle B + \angle E \geq 180^\circ$ like Abū Naṣr ibn ʿIrāq and Menelaus in the original version, one can also prove $\angle GBT < \angle GTB$ and $\angle ATB < \angle ABT$. We leave the details to the reader as an exercise (cf. Krause 141 note 3).

34 *Spherics* I:7.

35 This theorem is *Spherics* III:4. The "line corresponding to" an arc x is the chord of 2x (in modern terms 2R sin(x), R denotes the radius of the sphere on which one is working).

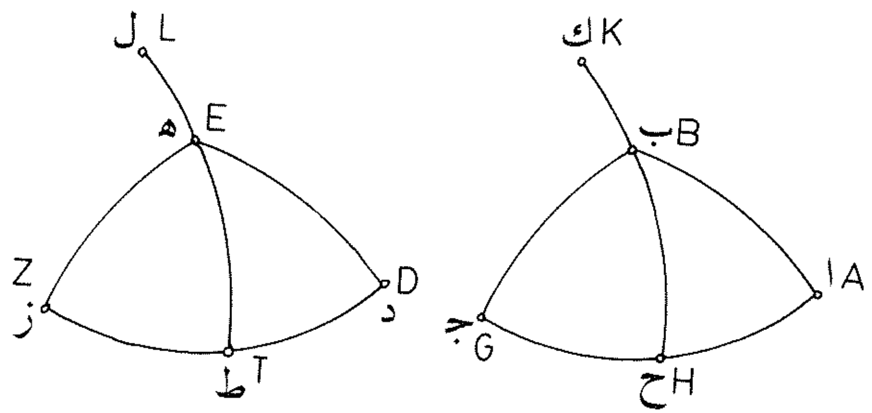


Fig. 6

at Z, and not one of these angles is (a) right (angle). Let us draw from points B, E altitudes to the bases AG, DZ, namely BH, ET.

Then I say that the ratio of the (line) corresponding to arc AH to the (line) corresponding to arc HG is equal to the ratio of the (line) corresponding to arc DT to the (line) corresponding to arc TZ.

Proof of this: We make the poles of arcs AG, DZ points K, L. Then, since the angles at points H, T are right (angles) and the angles at points[36] G, Z are equal and not right (angles), the ratio of the (line) corresponding to arc GH to the (line) corresponding to arc ZT is composed of the ratio of the (line) corresponding to arc BH to the (line) corresponding to arc ET and of the ratio of the (line) corresponding to arc EL to the (line) corresponding to arc

36 A comparison with the version of Abū Naṣr ibn ʿIrāq and the Hebrew version (J) shows that the following passage in version D was left out here: ⟨ D, A are equal and not (right) angles, and because points K, L are the poles of arcs AG, DZ, the ratio of the (line) corresponding to arc AH to the (line) corresponding to arc DT is composed of the ratio of the (line) corresponding to arc BH to the (line) corresponding to arc ET and of the ratio of the (line) corresponding to arc EL to the (line) corresponding to arc BK. Again, the angles at points ⟩. See Krause, Arabic text 67:23-26, translation, 20, and his introduction p. 91 note 2 and p. 93, and my note 39. In the missing passage *Spherics* III:3 is used. The missing passage is necessary, see my note 38. The passage was probably omitted by scribal error, because the eye of the scribe moved from the first "angles at points" (D, A) to the "angles at points" (G, Z) in the very monotonous text.

BK[37]. Therefore [38] the ratio of the (line) corresponding to arc AH to the (line) corresponding to arc DT is equal to the ratio of the (line) corresponding to arc GH to the (line) corresponding to arc ZT. Thus, permutando, they are also equal. That is what we wanted to demonstrate.

Menelaus, Spherics *III:4, in the Latin translation of Gerard of Cremona, taken from Krause, 91.*

Cum fuerint due figure trilatere, et fuerint duo anguli ipsarum, qui sunt super basim equales, scilicet omnis angulus suo relatiuo, en non fuerit angulus earum rectus. Et protrahentur due perpendiculares duarum figurarum ex duobus punctis duorum capitum ipsarum, tunc nadir arcuum qui separantur ex duabus basibus erunt proportionales.

Sint ergo due figure trilatere, super quas sint *abg, dez,* et sit angulus, qui est apud punctum *a* , equalis angulo, qui est apud punctum *d* , et angulus, qui est apud punctum *g* equalis angulo, qui est apud punctum *z,* et non sit aliquis horum angulorum rectus, et protraham ex duobus punctis *b, e* duas perpendiculares super duas bases *ag, dz,* que sint *bh, et*. Dico ergo, quod proportio nadir arcus *ah* ad nadir arcus *gh* est sicut proportio nadir arcus *dt* ad nadir arcus *tz*.

Quod sic demonstratur. Ponam duos polos duorum arcuum *ag, dz* duo puncta *k, l*. Et quoniam duo anguli, qui sunt apud duo puncta *h, t* sunt recti, et duo anguli, qui sunt apud duo puncta[39] *g, z* sunt equales et non recti, tunc erit proportio nadir arcus *gh* ad nadir arcus *zt* composita ex proportione nadir arcus *bh* ad nadir arcus *et* et ex proportione nadir arcus *el* ad nadir arcus *bk,* et erit propter illud proportio nadir arcus *ah* ad nadir arcus *dt* sicut proportio nadir arcus *gh* ad nadir arcus *zt* . Et cum permutaverimus, etiam erunt proportionales. Et illud est, quod demonstrare voluimus.

37 *Spherics* III:3 (proved in 422.12).
38 This conclusion is based on the missing passage mentioned in note 36.
39 Exactly the same passage as in note 36 is also missing here (compare Krause, 91 footnote 2).

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