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The Arabic version of Euclid's *On Divisions*

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1. Introduction

The Greek text of Euclid's work *On Divisions* (περὶ διαιρέσεων) is lost, and the only Greek references to this work are to be found in Proclus' commentary on the *Elements* of Euclid. Proclus tells us little more than that *On Divisions* is a work by Euclid which included the division of figures into "like and unlike" parts.[1] In the *Fihrist*, the 10th-century Arabic bibliographer Ibn al-Nadīm mentions Euclid's *Book of the division, corrected by Thābit* (= Thābit ibn Qurra, 836-901),[2] but as far as is known, no manuscripts of the Arabic translation or the revision by Thābit have survived. In 1851 F. Woepcke published[3] a French translation of an abstract of *On Divisions* which had been composed by the 10th-century Persian geometer Aḥmad ibn Muḥammad ibn ʿAbdaljalīl al-Sijzī.[4] Al-Sijzī rendered the first sentences of all propositions in *On Divisions* but the proofs of only four of them. He stated that he omitted the other proofs because he found them too easy. In an anonymous 10-th century Arabic text called *Various geometrical problems*, C. Schoy discovered a proposition of Euclid's *On Divisions* together with a geometrical construction by the anonymous translator of the work. Schoy published German translations of these passages in 1926.[5]

The modern studies of *On Divisions* (such as Archibald's reconstruction of 1915)[6] are based on Woepcke's French translation and on possible traces of

[1] Cf. J.L. Heiberg, *Litteraturgeschichtliche Studien über Euklid*. Leipzig (Teubner) 1882, pp. 36-37.

[2] Ibn al-Nadīm, *Kitāb al-Fihrist*. Mit Anmerkungen herausgegeben von Gustav Flügel, Leipzig 1871, vol. 1, p. 266 lines 15-16.

[3] F. Woepcke, Notice sur des traductions arabes de deux ouvrages perdus d'Euclide, *Journal Asiatique*, 4e série, 18 (1851), pp. 217-247. Reprinted in F. Woepcke. *Études sur les mathématiques Arabo-Islamiques*, ed. F. Sezgin, Frankfurt: Institut für Geschichte der arabisch-islamischen Wissenschaften, 1986, vol.1.

[4] See F. Sezgin, *Geschichte des arabischen Schrifttums*, Band 5, Mathematik bis ca. 430 H, Leiden: Brill, 1974, pp. 329-334. This work will henceforth be abbreviated as GAS V.

[5] C. Schoy, Graeco-Arabische Studien nach mathematischen Handschriften der Vicekö-niglichen Bibliothek zu Kairo. *Isis* 8 (1926), 21-40, see especially pp. 32-34. Reprinted in C. Schoy, *Beiträge zur arabisch-islamischen Mathematik und astronomie*, ed. F. Sezgin, Frankfurt: Institut für Geschichte der arabisch-islamischen Wissenschaften, 1988, vol. 2, pp. 589-591.

[6] R.C. Archibald, *Euclid's Book on division of Figures. With a restoration based on Woepcke's text and on the Practica Geometriae of Leonardo Pisano*. Cambridge (at the University Press) 1915.

On Divisions in medieval Latin texts (which do not contain explicit references to Euclid's work). No further attention has been paid to the Arabic texts themselves. Thus in Volume 8 of Euclid's *Opera Omnia*,[7] Heiberg and Menge printed Woepcke's translation of Euclid's *On Divisions* but not the Arabic text on which the translation is based.

The main purpose of this paper is to make available the Arabic texts translated by Woepcke and Schoy as well as literal English translations. Text **A** below is the abstract of Euclid's *On Divisions* by al-Sijzī, which was translated by Woepcke. Text **A** exists in a unique manuscript: Paris, Bibliothèque Nationale, Fonds Arabe, 2457, 53b-55b.[8] The manuscript was believed (by Woepcke and others) to be an autograph by al-Sijzī, written in 358 H./A.D. 968-9. However, on f. 216 there is a table of contents in the same hand, dated 657 H./A.D.1259, so the manuscript must be a copy of the manuscript written by al-Sijzī.[9]

Text **B** is part of the anonymous compilation entitled *Various geometrical problems*, which probably dates from the late 10th century.[10] The treatise exists in a unique manuscript in Cairo, Dār al-Kutub Muṣṭafā Fāḍil Riyāḍa 41m, 165b-170b,[11] copied in 1153 H./ 1740 A.D. Text **B** consists of props. 5 and 6 of the treatise. Proposition 5, which is said to be "from the Book of Euclid on division", deals with the construction of a segment of a circle contained between two parallel lines and equal in area to a given sector. The proposition is closely related to but not identical to al-Sijzī's proposition 28, on the construction of a segment of a circle contained between two parallel lines and equal in area to a given part of the circle. Prop. 6 is a trisection of the angle, which the text attributes to "the translator of it" (i.e. of *On Divisions*).

Euclid's *On Divisions* had a considerable influence on the Arabic geometrical tradition. For example, in Chapter 8, *On the division of triangles*, and Chapter 9, *On the division of quadrilaterals*, of the *Book on the geometrical constructions necessary for the craftsman* (GAS V, 324 no. 2), the 10-th century geometer Abu l-Wafā' al-Būzajānī included many solutions of

[7]Euclides, *Opera Omnia*, ed. J.L. Heiberg et H. Menge. Leipzig (Teubner) 1883-1916. 8 vols.

[8]See M. De Slane, *Catalogue des manuscrits arabes (de la Bibliothèque Nationale)*, Paris 1893-1895, p. 431.

[9]This was first noticed by F. Sezgin in *Geschichte des arabischen Schrifttums*, Band VI, Leiden 1978, p. 188 no. 1.

[10]See GAS V, op.cit., p. 396.

[11]On the manuscript see H. Suter, Der V. Band des Katalogs der arabischen Bücher der viceköniglichen Bibliothek in Kairo, *Zeitschrift für Mathematik und Physik, Historisch-litterarische Abteilung* 38 (1893), p. 24; reprinted in H. Suter, *Beiträge zur Geschichte der Mathematik und Astronomie im Islam*, Frankfurt: Institut für Geschichte der arabisch-islamischen Wissenschaften, 1986, vol. 1; D.A. King, *A catalogue of the scientific manuscripts in the Egyptian National Library* (in Arabic), Part 1. Cairo 1981 (published by the General Egyptian Book Organization), p. 447; D.A. King. *A survey of the scientific manuscripts in the Egyptian National Library*, Winona Lake: Eisenbrauns, 1986, p. 52 no. B 98.

problems in *On Divisions* (corresponding to *On Divisions* props. 1-9, 11-16 and 27-28). Text **C** below is one (unnumbered) proposition from Chapter 9. In this proposition Abu l-Wafā' constructs a segment of a circle contained between two parallel lines and equal to one-third of the circle. The wording of the solution is extremely close to that of al-Sijzī's abstract, prop. 28.[12] I have included text **C** here because it sheds light on the relationship between texts **A** and **B**.

In my edition of text **C** I have used the manuscripts: **F** = Istanbul, Aya Sofya 2753, f. 44:2-7, and **L** = Uppsala Tornberg 324, Vet. 27, f. 39b:6-16. Both manuscripts are undated. In the Uppsala manuscript the text is incorrectly attributed to al-Fārābī. Only the first and last page of the manuscript (f. 1b, 60a) belong to a text by al-Fārābī,[13] and the rest is a large continuous fragment of the treatise by Abu l-Wafā', which begins abruptly on f. 2a in the middle of a sentence. Text **C** can also be consulted in the edition by Ṣ. A. al-ʿAlī,[14], in the Russian translation by Krasnova[15] (based on **F**), and in al-Fārābī's *Mathematical works* [16] (based on **L**). The relationship with Euclid's *On Divisions* is not mentioned in these three publications.

I now comment on the relationship between the three texts **A**, **B** and **C**.

The following terminological argument shows that al-Sijzī's abstract (text **A**) was probably based on the revision of *On Divisions* by Thābit ibn Qurra. The rectangle contained by two segments AB and GD, which for Euclid is a geometrical concept, is rendered in al-Sijzī's abstract in an arithmetical way as *alladhī yakūnu min* AB *fī* GD, meaning "that which is (the result) of (the multiplication) AB times GD". This expression is unusual in Arabic geometrical texts, but Thābit ibn Qurra used similar expressions in his translations of Book V-VII of the Conics, namely *alladhī yakūnu min ḍarb* AB *fī* GD "that which is (the result) of the multiplication (of) AB times GD".[17]

The close resemblance between text **C** and al-Sijzī's abstract prop. 28 shows that Abu'l-Wafā' must have had a version of *On Divisions* in front of him when he wrote Chapters 8 and 9 of his own work. Because

[12]Cf. F. Woepcke. Analyse et extrait d'un recueil de constructions géométriques par Aboul Wafâ. Journal Asiatique, 5ᵉ série, 5 (1855), pp. 340-341. Reprinted in Woepcke, Études, op. cit. vol. 1.

[13]See GAS V, op. cit. p. 296 no. 1.

[14]See Ṣ.A. al-ʿAlī, *Ma yaḥtāju ilayhi al-ṣāniʿ min ʿilm al-handasa li-Abi l-Wafā'... al-Būzjānī*, Bagdad 1979, p. 126.

[15]S.A. Krasnova: Abu-l-Vafa al-Buzdzhani, Kniga o tom, chto neobchodimo remeslenniku iz geometricheskikh postroennii (translation with introduction and commentary) in: *Fiziko-Matematicheskie Nauki v stranakh vostoka, sbornik statei i publikatsii*, vypusk I (IV), Moscow (Nauka) 1966, pp. 42-159, see esp. p. 105.

[16]Al-Farabi, *Matematicheskie Traktaty*, Per. ... A. Kubesov, B.A. Rozenfeld, Alma Ata 1971, pp. 168-169.

[17]Cf. G.J. Toomer, *Apollonius' Conics Books V to VII. The Arabic translation of the lost Greek original in the version of the Banū Mūsā*, New York: Springer 1990. Sources in the History of Mathematics and Physical Sciences 9, e.g. p. 5 line 17, p. 11 line 1, index

Abu'l-Wafā' presents problems with solutions, he probably had the whole text of *On Divisions* and not the abstract of al-Sijzī, in which most constructions and proofs are omitted. Because al-Sijzī used the revision of *On Divisions* by Thābit ibn Qurra, this must also be the version which Abu'l-Wafā' used and copied. The coincidences between his Text **C** and al-Sijzī's text **A** suggest that al-Sijzī did not change the wording of the parts of *On Divisions* which he put together in his abstract.

There are considerable differences between prop. 28 of Text **A** and Text **C** on one hand and prop. 5 of Text **B** on the other hand. One can explain these differences by the assumption that Text **B** is based not on Thābit's revision but on the original translation of *On Divisions*. If this is true, Prop. 5 in text **B** was probably deleted by Thābit ibn Qurra in the course of his revision of the text. The trisection in prop. 6 in text **B** may have undergone the same treatment. This trisection is based on a *neusis*, but Thābit ibn Qurra preferred to trisect the angle by conic sections.[18]

By studying the Arabic tradition, and in particular the work of Abu'l-Wafā', one can improve the reconstruction of *On Divisions* published by Archibald in 1915. In his reconstruction Archibald took the problems from Al-Sijzī's abstract and the solutions (constructions and proofs) from Leonardo Fibonacci's *Practica Geometria*, written in 1220. Most constructions can also be taken from Abu l-Wafā'; these constructions are probably closer to the original *On Divisions* than those of Leonardo.

In the following editions and translations, passages in angular brackets are additions made by me in my attempt to restore the text, and my own explanatory additions are in parentheses or square brackets. The critical apparatus for the text is printed at the bottom of each page.

[18]Thābit's trisection can be consulted in W.R. Knorr, Textual studies in ancient and

Manuscripts:

Text A: MS. Paris, Bibliothèque Nationale, Fonds Arabe 2457, 53b-55b.

Text B: MS. Cairo, Dār al-Kutub Muṣṭafā Fāḍil Riyāḍa 41m, 167b-168a.

Text C: **F** = MS. Aya Sofya 2753, f. 44;
L = MS. Uppsala, Tornberg 324, f. 39.

Translation of text A

In the name of God, the Merciful, the Compassionate.

9[19] The Book of Euclid on the division.

1. We wish to demonstrate how we bisect a known triangle by a line parallel to its base.

2. We wish to demonstrate how we divide a known triangle into three equal parts by two lines parallel to its base.

3. We wish to bisect a known triangle by a straight line drawn from a known point which is on one of its sides.

4. We wish to bisect a known trapezium by a line parallel to its base. And[20] the known trapezium can be divided into three equal parts just as we divided the triangle, by a construction corresponding to that construction.

< 5 > We wish to bisect a parallelogram by a straight line drawn from a known point which is on one of its sides.

< 6 > We wish to demonstrate how we cut off from a known parallelogram an assumed part, any part we wish, by a straight line drawn from a known point which is on one of its sides.

< 7 > We wish to demonstrate how we bisect a known trapezium by a straight line drawn from a known point which is on the line of the highest[21] (part) of the trapezium.

8. We wish to cut off from a known trapezium an assumed part by a straight line drawn from a known point which is on the < line of the > highest (part) of the trapezium.

9. We wish to demonstrate how we bisect a parallelogram by a straight line drawn from a known point outside it.

10. We wish to demonstrate how we cut off from a parallelogram an assumed part by a straight line drawn from a known point outside it.

11. We wish to demonstrate how we bisect a known trapezium by a straight line drawn from a point beyond the highest (part) of that trapezium. It is necessary that the point is not beyond the place of the meeting of the two sides.[22]

12. We wish to demonstrate how we cut off from a known trapezium an assumed part by a straight line drawn from a known point beyond the line

[19] This is the ninth text in the manuscript Paris, Bibliothèque Nationale 2457.

[20] In the margin of the manuscript, this corollary is numbered "5", the next two propositions (my propositions < 5 > and < 6 >) are numbered "6" and "7", and the subsequent proposition (my proposition < 7 >) is not numbered. I assume that the corollary was numbered "5" by mistake, and that the scribe made up for his mistake by not assigning a number to my proposition < 7 >.

[21] Meaning: on the longer side of the trapezium.

[22] I do not understand this condition. The problem is solvable for all points outside the trapezium. Woepcke's interpretation (pp. 34-35) does not convince me.

Text A

بسم الله الرحمن الرحيم . ط كتاب اقليدس فى القسمة .

ا نريد ان نبيّن كيف نقسم مثلّثا معلوما بنصفين بخطّ يوازي قاعدته .

ب نريد ان نبيّن كيف نقسم مثلّثا معلوما بثلاثة اقسام متساوية بخطّين موازيين [1] لقاعدته .

ج نريد ان نقسم مثلّثا معلوما [2] بقسمين متساويين بخطّ مستقيم يخرج من نقطة معلومة تكون على احد اضلاعه .

د نريد ان نقسم منحرفا معلوما بقسمين متساويين بخطّ يوازي قاعدته

[3] وقد يقسم المنحرف المعلوم بثلاثة اقسام متساوية كما قسمنا المثلّث بعمل نظير ذلك العمل .

ه [4] نريد ان نقسم سطحا متوازى الاضلاع بقسمين متساويين بخطّ مستقيم يخرج من نقطة معلومة تكون على ضلع من اضلاعه .

و [5] نريد ان نبيّن كيف نفصل من سطح متوازي الاضلاع معلوم جزءا مفروضا اي جزء اردنا بخطّ مستقيم يخرج من نقطة معلومة تكون على ضلع من اضلاعه .

< ز > نريد ان نبيّن كيف نقسم منحرفا معلوما بقسمين متساويين بخطّ مستقيم يخرج من نقطة معلومة تكون على خطّ اعلى [6] المنحرف .

ح نريد ان نفصل من منحرف معلوم جزءا مفروضا بخطّ مستقيم يخرج من نقطة معلومة تكون على < خطّ > اعلى [7] المنحرف .

ط نريد ان نبيّن كيف نقسم سطحا متوازي الاضلاع بقسمين متساويين بخطّ مستقيم يخرج من نقطة معلومة خارجة عنه .

ي نريد ان نبيّن كيف نفصل من سطح متوازي الاضلاع جزءا مفروضا [54a] بخطّ مستقيم يخرج من نقطة معلومة خارجة عنه .

يا نريد ان نبيّن كيف نقسم منحرفا معلوما بقسمين متساويين بخطّ مستقيم يخرج من نقطة خارجة عن خطّ اعلى [7] ذلك المنحرف ويحتاج ان لا يكون النقطة خارجة [8] عن موضع ملتقى الضلعين .

يب نريد ان نبيّن كيف نفصل من منحرف معلوم جزءا مفروضا بخطّ مستقيم يخرج

1 موازيين : متوازيين ، موازيان (crossed out) . 2 مثلّثا معلوما : مثلّث معلوم . 3 ه (in margin)
4 و : ه . 5 ز : ر . 6 خطّ اعلى : حطا على . 7 اعلا . 8 الخارجة .

of the highest (part) of that trapezium. It is necessary that the point is not beyond the place of the meeting of the two sides of the trapezium.[23]

13. We wish to demonstrate how we bisect a known quadrilateral by a straight line drawn from a known angle of it.

14. We wish to demonstrate how we cut off from a known quadrilateral some assumed part by a line drawn from a known angle of it.

15. We wish to demonstrate how we bisect a known quadrilateral by a straight line drawn from a known point which is on one of its sides.

16. We wish to demonstrate how we cut off from a known quadrilateral an assumed part by a straight line drawn from a known point which is on one of its sides.

17. (Figure 1) We wish to apply to a straight line a rectangle equal to the area contained by lines AB, AG, and deficient from the completion of the line by a square area.[24] Since we have proved what we said, if someone says : why was it so that it was not possible for us to apply to AB an area equal to $AB \cdot AG$,[25] deficient from its completion by a square area, and equal to the area $AE \cdot EB$, we say: that is not possible because AB is greater than BE and AG is greater than AE, so $BA \cdot AG$ is greater than $AE \cdot EB$. So if there is applied to AB a parallelogram equal to $AB \cdot AG$ it is (like) area $AZ \cdot ZB$.[26]

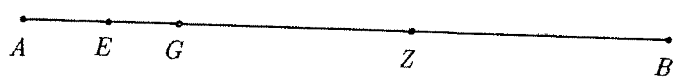

Figure 1

[23]See the preceding footnote.

[24]It would have been more correct to say: We wish to apply to *the straight line AB* a rectangle equal to the area contained by AB, AG and *deficient from its completion* by a square. The problem is mathematically equivalent to the following: to construct a point X on AB such that $AX \cdot XB = AB \cdot AG$. To explain the terminology, draw three perpendiculars AA', XX' and BB' to AB such that $AA' = XX' = BB' = XB$. Now rectangle $AXX'A'$ is equal to "the area contained by AB, AG", $XBB'X'$ is a square, and $ABB'A'$ is a "complete" rectangle with side AB. The problem is of a well-known type in Greek geometry, the so-called *application of areas*. See T.L. Heath, *Euclid. The thirteen Books of the Elements*. Vol. I, pp. 342-345, Vol. II, pp. 257-259.

[25]The notation $AB \cdot AG$ will be used to translate Arabic expressions like: *alladhī yakūnu min AB fī AG*, literally; "which is (the result) of AB times AG". The Arabic preposition *fī* is normally used in multiplications, so a multiplicative notation seems appropriate.

[26]Woepcke (p.36) assumed that the sentence is incomplete, but there is no good reason for this assumption. The text means that a point X such that $AX \cdot XB = AB \cdot AG$ cannot lie between A and G; hence it must lie between G and B. The text is not concerned with the necessary condition for the existence of point X.

من نقطة معلومة خارجة عن خطّ اعلى [9] ذلك المنحرف ويحتاج الّا يكون النقطة خارجة عن موضع ملتقى ضلعي المنحرف .

يج نريد ان نبيّن كيف نقسم شكلا ذا اربعة اضلاع معلوما بقسمين متساويين بخطّ مستقيم يخرج من زاوية معلومة منه .

يد نريد ان نبيّن كيف نفصل من ذى اربع اضلاع معلوم [10] جزءا ما مفروضا بخطّ يخرج من زاوية منه معلومة .

يه نريد ان نبيّن كيف نقسم شكلا ذا اربعة اضلاع معلوما [11] بقسمين متساويين بخطّ مستقيم يخرج من نقطة معلومة يكون على احد اضلاعه .

يو نريد ان نبيّن كيف نفصل من ذى اربع اضلاع معلوم جزءا مفروضا بخطّ مستقيم يخرج من نقطة معلومة تكون على احد اضلاعه .

يز نريد ان نضيف الى خطّ مستقيم سطحا قائم الزوايا مساويا للسطح الّذى يحيط به خطّا اب اج وينقص عن تمام الخطّ سطحا مربّعا . واذ قد بيّنّا ما قلنا فانّه ان قال قائل لم صار لا يمكننا ان نضيف الى اب سطحا مساويا للّذى يكون من اب فى اج ينقص عن تمامه سطحا مربّعا ويكون مثل سطح اه فى هب قلنا انّ ذلك لا يمكن لانّ اب اعظم من به وانّ اج اعظم من اه فالّذى يكون من با فى اج اعظم من الّذى يكون من اه فى هب فاذا اضيف الى اب سطح متوازى الاضلاع مساو للّذى يكون من اب فى اج كان سطح از فى زب .

< يح > نريد ان نبيّن كيف نقسم مثلّثا معلوما بقسمين متساويين بخطّ مستقيم يمرّ بنقطة معلومة داخل المثلّث . فليكن المثلّث المعلوم مثلّث ابج والنقطة الّتى فى داخله د ونريد ان نجيز على د خطّا مستقيما يقسم مثلّث ابج بقسمين متساويين . فنخرج من نقطة د خطّا موازيا لخطّ بج وهو ده . ونضيف الى ده سطحا مساويا لنصف الّذى يكون من اب فى بج وليكن طب فى هد . ونضيف [12] الى خطّ طب [13] سطحا متوازى الاضلاع مساويا [54b] للّذى يكون من بط فى به ينقص عن تمامه سطحا مربّعا وليكن السطح المضاف سطح بح فى حط . ونصل خطّ دح ونخرجه الى ز . فاقول انّه قد اخرج خطّ حدز [14] فقسم مثلّث ابج بقسمين متساويين وهما حبز حزجا .

برهان ذلك انّ الّذى يكون من طب فى به مساو للّذي يكون من طح فى حب

9 اعلا . 10 معلوما . 11 معلوم . 12 الى خطّ لب سطحا مساويا لنصف الّذى يكون من اب فى بج وليكن سطح طب فى هد ونضيف (added in MS.) 13 كب . 14 حبز .

< 18 > (Figure 2) We wish to demonstrate how we bisect a known triangle by a straight line which passes through a known point inside the triangle. Thus let the known triangle be triangle ABG, and (let) the point which is in its interior (be) D. We wish to let pass through D a straight line which bisects triangle ABG. Thus we draw from point D a line parallel to line BG, namely DE. We apply to DE an area equal to half $AB \cdot BG$, let it be $TB \cdot ED$. We apply to line TB a parallelogram equal to $BT \cdot BE$, deficient from its completion by a square area, let the applied area be area $BH \cdot HT$.[27] We join line DH and we extend it towards Z. I say that line DHZ has been drawn such as to divide triangle ABG into two equal parts, namely HBZ, $HZGA$.

Proof of this: $TB \cdot BE$ is equal to $TH \cdot HB$, so the ratio of BT to TH is equal to the ratio of HB to BE. Separando, the ratio of TB to BH is also equal to the ratio of BH to HE. But the ratio of BH to HE is equal to the ratio of BZ to ED. Thus the ratio of TB to BH is equal to the ratio of BZ to ED. So $TB \cdot ED$ is equal to $BH \cdot BZ$. But $TB \cdot ED$ is equal to half of $AB \cdot BG$, and the ratio of $BH \cdot BZ$ to $AB \cdot BG$ is equal to the ratio of triangle HBZ to triangle ABG, because the angles at point B are common. So triangle HBZ is half of triangle ABG. So triangle ABG has been divided into two equal parts, namely BHZ, $AHZG$. If we apply to TB a parallelogram equal to $TB \cdot BE$, deficient from its completion by a square, and if this area is $AB \cdot AT$,[28] then if we join AD and extend it towards K, we prove in the same way that triangle ABK is half of triangle ABD. That is what we wanted to demonstrate.

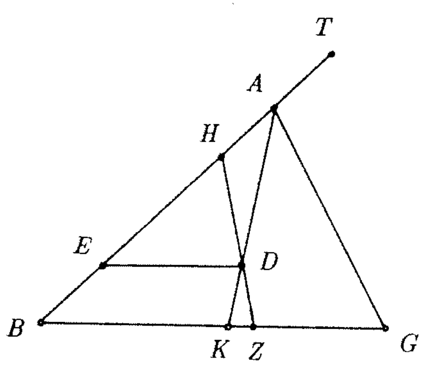

Figure 2

[27]The text does not discuss the possibility of this construction. Point H is between E and T as a consequence of the end of prop. 17.

[28]This means that point H coincides with A.

فيكون [15] نسبة بط الى طح كنسبة حب الى به [16] . واذا فصلنا ايضا يكون نسبة طب الى بح كنسبة بح الى حه . ولكن نسبة بح الى حه كنسبة بز الى هد . فنسبة طب الى بح كنسبة بز الى هد . فالّذى يكون من طب فى هد مساو للذى يكون من بح فى بز [17] . ولكن الّذى يكون من طب فى هد هو مثل نصف الّذى يكون من اب فى بج ، ونسبة ما يكون من بح فى بز الى الّذى يكون من اب فى بج كنسبة مثلّث حبز الى مثلّث ابج ، وذلك انّ الزوايا الّتى [18] عند ب مشتركة ، فيكون مثلّث حبز نصف مثلّث ابج . فقد قسم مثلّث ابج بقسمين متساويين وهما بحز احزج . فان كنّا اذا اضفنا الى طب سطحا متوازى الاضلاع يساوى الّذى يكون من طب فى به [19] ينقص عن تمامه مربّعا كان ذلك السطح هو الكائن من اب فى اط ، فاذا وصلنا خطّ اد ونفذناه الى ك بيّنّا بمثل هذه السبيل انّ مثلّث ابك هو نصف مثلّث ابد وذلك ما اردنا ان نبيّن .

< يط > نريد ان نبيّن كيف نفصل من مثلّث معلوم جزءا مفروضا بخطّ مستقيم يجوز على نقطة معلومة داخل المثلّث . فليكن المثلّث المعلوم مثلّث ابج والنقطة المعلومة الّتى داخله د ونريد ان نجيز على نقطة د خطّا مستقيما يفصل من مثلّث ابج جزءا مفرودا فليكن الجزء المفرود الثلث . ونخرج من نقطة د خطّا موازيا لخطّ بج وهو ده . ونضيف الى ده سطحا قائم الزوايا مساويا لثلث ما يكون من اب فى بج وليكن ذلك السطح بز فى هد . ونضيف الى زب [20] سطحا مساويا للّذى يكون من زب فى به ينقص عن تمامه سطحا مربّعا وليكن السطح المضاف سطح بح فى حز . ونصل خطّ حد ونخرجه الى ط . وبمثل هذا المسلك نبرهن انّ مثلّث حطب ثلث مثلّث ابج وبمثل هذا العمل نقسمه باىّ جزء اردنا وذلك ما اردنا ان نبيّن .

< ك > وليكن اربعة خطوط ا ب ج د وليكن ا فى د اعظم من ب فى ج فاقول انّ نسبة ا الى ب اعظم من نسبة ج الى د .

< كا > وايضا ان كان ا فى د اقلّ من ب فى ج فاقول انّ نسبة ا الى ب اقلّ من نسبة ج الى د .

< كب > وان كان خطّان مستقيمان عليهما اب ده [55a] ويكون نسبة اب الى بج اعظم من نسبة ده الى هز فاقول انّه يكون على جهة التفصيل نسبة اج الى جب اعظم من نسبة دز الى زه .

< كج > وعلى هذه الصورة بعينها [21] ليكن نسبة اج الى جب اعظم من نسبة دز الى زه فاقول انّه يكون على جهة التركيب نسبة اب الى بج اعظم من نسبة ده الى هز .

15 ويكون . 16 بد . 17 ز : هز . 18 الذى . 19 به : بد . 20 ب : ده . 21 بعينه .

< 19 > (Figure 3) We wish to demonstrate how we cut off from a known triangle an assumed part by a straight line which passes through a known point inside the triangle. Thus let the known triangle be triangle ABG and (let) the known point which is inside it (be) D. We wish to let pass through point D a straight line which cuts off from triangle ABG an assumed part, and let the assumed part be one-third. We draw from point D a line parallel to line BG, namely DE. We apply to DE a rectangle equal to one-third of $AB \cdot BG$, let this area be $BZ \cdot ED$. We apply to ZB[29] an area equal to $ZB \cdot BE$, deficient from its completion by a square area, let the applied area be area $BH \cdot HZ$.[30] We join line HD and extend it towards T. In a similar way we prove that triangle HTB is one-third of triangle ABG. By a similar construction we divide it into any part we wish. That is what we wished to demonstrate.

< 20 > Let there be four lines A, B, G, D and let $A \cdot D$ be greater than $B \cdot G$. I say that the ratio of A to B is greater than the ratio of G to D.

< 21 > Again, if $A \cdot D$ is less than $B \cdot G$, I say that the ratio of A to B is less than the ratio of G to D.

< 22 > (Figure 4)[31] If there are two straight lines AB, DE and the ratio of AB to BG is greater than the ratio of DE to EZ, I say that, separando, the ratio of AG to GB is greater than the ratio of DZ to ZE.

< 23 > In exactly the same figure, let the ratio of AG to GB be greater than the ratio of DZ to ZE. I say that, componendo, the ratio of AB to BG is greater than the ratio of DE to EZ.

< 24 > (Figure 5) Again, let us make the ratio of AB to BG less than the ratio of DE to EZ. Then, separando, the ratio of AG to GB is less than the ratio of DZ to ZE.

< 25 > We wish to demonstrate how we bisect a known triangle by a straight line drawn through a known point outside the triangle.

< 26 > We wish to demonstrate how we cut off from a known triangle an assumed part by a straight line drawn from a point outside the triangle.

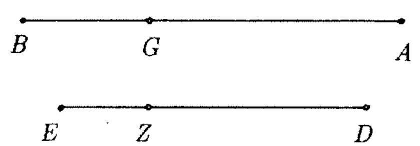

Figure 5

< 27 > (Figure 6) We wish to bisect a known figure contained by an arc and two straight lines which contain an angle. Thus let there be a known

[29] Instead of ZB the manuscript has DE by scribal error.

[30] The text does not say for which points D this construction is possible. Point H is between E and Z as a consequence of the remark at the end of proposition 17.

[31] Figure 4 in the manuscript contains points T, H and K, which were probably used in the proofs of props. 22 and 23.

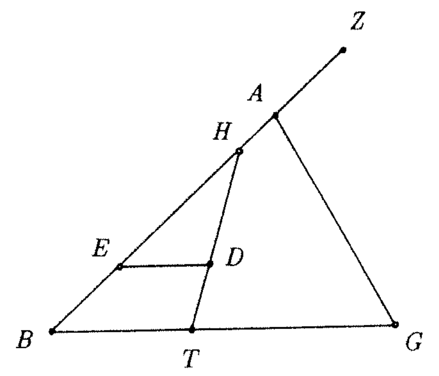

Figure 3

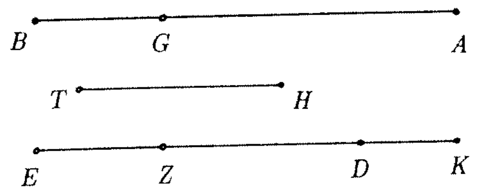

Figure 4

< كد > وايضا فلنجعل نسبة آب الى بج اقلّ من نسبة ده الى هز فانّه يكون على جهة التفصيل نسبة آج الى جب اقلّ من نسبة دز الى زه .

< كه > نريد ان نبيّن كيف نقسم مثلّثا معلوما بقسمين متساويين بخطّ مستقيم يخرج من نقطة معلومة خارجة عن المثلّث .

< كو > نريد ان نبيّن كيف نفصل من مثلّث معلوم جزءا مفروضا بخطّ مستقيم يخرج من نقطة خارجة عن المثلّث .

< كز > نريد ان نقسم شكلا معلوما يحيط به قوس وخطّان مستقيمان يحيطان بزاوية بقسمين متساويين .

فليكن شكل معلوم عليه ابج يحيط به قوس بج وخطّا با اج المستقيمان المحيطان بزاوية باج ونريد ان نخرج خطّا مستقيما يقسم شكل ابج بقسمين متساويين .

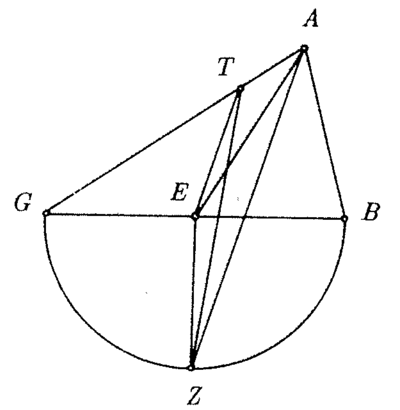

Figure 6

figure ABG, contained by arc BG and the straight lines BA, AG, containing angle BAG. We wish to draw a straight line which bisects figure ABG. Thus we join line BG, and we bisect it at point E. We draw from point E a line perpendicular to line BG, namely EZ. We join the straight line AE. Since line BE is equal to line EG, area BZE is equal to area EZG. But triangle ABE is equal to triangle AEG, so figure $ABZE$ turns out to be equal to figure $ZGAE$.

If line AE is on the rectilinear extension of line EZ, figure ABG has been divided into two equal parts, namely $ABZE$, $GAEZ$.

If line AE is not on the rectilinear extension of line ZE, we join line AZ and we draw from point E a line parallel to line AZ, namely ET. We join line TZ. I say that line TZ has been drawn so as to divide figure ABG into two equal parts, namely $ABZT$, ZGT. Since triangles TZA, EZA are on the same base, namely AZ, and between two parallel lines, namely AZ, TE, triangle ZTA is equal to triangle AEZ. Let a common addition be made to them, namely figure AZB. Then $TZBA$ turns out to be equal to $ABZE$, which is half of figure ABG. Thus the straight line TZ has been drawn so as to divide $BZGA$ into two equal parts, namely $ABZT$, TZG. That is what we wished to demonstrate.

< 28 > (Figure 7) We wish to draw in a known circle two parallel lines which cut off from the circle some assumed part. Thus let us make the part one-third and the circle ABG, and we wish (to do) what we have said. We make the centre of circle ABG <point D, and we make the> side of the triangle inscribed in this circle, namely AG, and we draw lines AD, DG. We let pass through point D a line parallel to line AG, namely DB. We join line GB. We bisect arc AG at point E. We draw from point E a line parallel to line BG, namely EZ. We draw line AB. I say that the parallel lines EZ, GB have been drawn so as to cut off from circle ABG one-third of it, namely figure $ZBGE$.

فنصل خطّ بج ونقسمه بنصفين على نقطة ه ونخرج من نقطة ه خطّا قائما على خطّ بج على زوايا قائمة وهو هز . ونصل خطّ اه المستقيم .

فلانّ خطّ به مساو لخطّ هج يكون سطح بزه مساويا لسطح هزج 22 . ومثلّث ابه مساو لمثلّث اهج فيصير شكل ابزه مساويا لشكل زجاه . فان كان خطّ اه على استقامة خطّ هز 23 فقد قسم شكل 24 ابج بقسمين متساويين وهما ابزه جاهز .

وان لم يكن خطّ اه على استقامة خطّ زه فانّا نصل خطّ از ونخرج من نقطة ه خطّا موازيا لخطّ از وهو هط فنصل خطّ طز فاقول انّه قد اخرج خطّ طز فقسم شكل ابج بقسمين متساويين وهما ابزط زجط . فلانّ مثلّثا طزا هزا على قاعدة واحدة وهي از وفيما بين خطّين متوازيين وهما از طه 25 يكون مثلّث زطا مساو لمثلّث اهز . وليزاد عليهما زيادة مشتركة وهي شكل ازب فيصير طزبا مثل ابزه وهو نصف شكل ابج . فقد اخرج خطّ طز المستقيم فقسم بزجا بقسمين متساويين وهما ابزط طزب 26 وذلك ما اردنا ان نبيّن .

< كح > نريد ان نخرج فى دائرة معلومة خطّين متوازيين يفصلان من الدائرة جزءا ما مفروضا .

فلنجعل الجزء الثلث و الدائرة ابج ونريد ما قلنا . فنجعل مركز دائرة ابج < نقطة د ونجعل > ضلع المثلّث الّذى يعمل فى هذه الدائرة وهو اج ونخرج خطّي اد دج ونجيز على نقطة د خطّا موازيا [55b] لخطّ اج وهو دب ونصل خطّ جب . ونقسم قوس اج بنصفين على نقطة ه ونخرج من نقطة ه خطّا موازيا لخطّ بج وهو هز

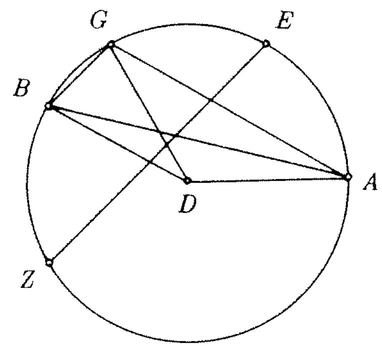

Figure 7

22 هزح . 23 هد . 24 + ابز مساويا (crossed out) . 25 + مثلّثان (crossed out) . 26 طزح .

Proof of this: Line AG is parallel to line DB, so triangle DAG is equal to triangle BAG. Let to them a common addition be made, namely the circular segment AEG. Then the whole figure $DAEG$ turns out to be equal to the whole figure $BAEG$. But $DAEG$ is one-third of circle ABG, so figure $BAEG$ is one-third of this circle.

Since EZ is parallel to GB, arc EG is equal to arc BZ. But EG is equal to EA, so EA turns out to be equal to ZB. We make arc EGB common. Then the whole arc AB is equal to the whole arc EZ, so the straight line AB is equal to the straight line EZ, and the circular segment $AEGB$ turns out to be equal to the circular segment $EGBZ$. We drop the common part, namely the circular segment GB, then by subtraction figure $EZBG$ is equal to figure $BAEG$.

But figure $BAEG$ is one-third of circle ABG. So figure $EZBG$ is one-third of circle ABG. That is what we wanted to demonstrate.

If we wish to cut off from the circle one-fourth of it or one-fifth of it or another assumed part by means of two parallel lines, we draw in this circle the side of the square or the pentagon which are inscribed in it, and we draw to it from the centre two lines as we have drawn them here, and we proceed as in this construction.[32]

< 29 > We wish to divide a known triangle into two parts such that the ratio between them is a known ratio, by means of a line parallel to its base.

< 30 > We wish to divide a known triangle by means of lines parallel to its base into parts in known ratios.

< 31 > We wish to divide a known trapezium into two parts such that the ratio between them is a known ratio, by a line parallel to its base.

< 32 > We wish to demonstrate how <we divide> a known trapezium into parts such that the ratios between them are equal to known ratios and such that the division is by lines parallel to the base of the trapezium.

< 33 > We wish to demonstrate how we divide a known quadrilateral into two parts such that the ratio between them is a known ratio, by a line drawn from a known angle of it.

< 34 > We wish to demonstrate how we divide a known quadrilateral by straight lines drawn from a known angle of it into parts in known ratios.

< 35 > If these things are known, it is possible for us to divide a known quadrilateral in a known ratio or known ratios by a straight line or straight lines drawn from a known point on one of its sides, if we bear in mind the conditions mentioned above.

The book is finished. We have restricted ourselves to the statements without proof, because the proof is easy.

[32]This is worked out in detail in the fragment of *On Divisions* preserved in the *Various geometrical problems*, see Text C below. Note that Thābit ibn Qurra wrote a short treatise on the problem of constructing a segment equal to one-sixth of the circle; this is published in J. Sesiano, Un complément de Ṯābit ibn Qurra au Περὶ διαιρέσεων d'Euclide, *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften* 4 (1988), 149-159.

ونخرج [27] خطّ اب فاقول انّه قد اخرج خطّا هز جب المتوازيين يفصلان [28] من دائرة ابج ثلثها وهو شكل زبجه . برهان ذلك انّ خطّ اج موازٍ لخطّ دب فمثلّث داج مساوٍ لمثلّث باج وليزاد عليهما زيادة مشتركة وهي قطعة دائرة اهج فيصير جميع شكل داهج مساويا لجميع شكل باهج . ولكن داهج هو ثلث دائرة ابج فشكل باهج هو ثلث هذه الدائرة . ولانّ هز يوازي جب يكون قوس هج مساوية لقوس بز ولكن هج مثل ها فيصير ها مثل زب . ونجعل قوس هجب مشتركة لجميع قوس اب مساوٍ لجميع قوس هز فخطّ اب المستقيم مساوٍ لخطّ هز المستقيم ويصير قطعة دائرة اهجب مساويا لقطعة دائرة هجبز . ونسقط المشترك وهو قطعة دائرة جب فيبقى شكل هزبج مساويا لشكل باهج ولكن شكل باهج هو ثلث دائرة ابج فشكل هزبج هو ثلث دائرة ابج وذلك ما اردنا ان نبيّن . فاذا اردنا ان نفصل من الدائرة ربعها او خمسها او جزءا آخر مفروضا بخطّين متوازيين فانّا نخطّ في هذه الدائرة ضلع المربّع او المخمّس الذى يعمل فيها ونخرج اليه من المركز خطّين مستقيمين كما اخرجناها هنا ونعمل مثل هذا العمل .

< كط > نريد ان نقسم مثلّثا معلوما بقسمين يكون نسبة احدهما الى الآخر كنسبة معلومة بخطّ يوازى قاعدته . < ل > نريد ان نقسم مثلّثا معلوما بخطوط توازى قاعدته اقساما على نسب معلومة [29] .

< لا > نريد ان نقسم منحرفا معلوما بقسمين يكون نسبة احدهما الى الآخر كنسبة معلومة بخطّ يوازى قاعدته . < لب > نريد ان نبيّن كيف < نقسم > منحرفا معلوما باقسام يكون نسبها بعضها الى بعض كنسب معلومة ويكون القسمة بخطوط موازية لقاعدة المنحرف .

< لج > نريد ان نبيّن كيف نقسم شكلا ذا اربعة اضلاع معلوم بقسمين يكون نسبة احدهما الى الآخر كنسبة معلومة بخطّ مستقيم يخرج من زاوية منه معلومة .

< لد > نريد ان نبيّن كيف نقسم ذا اربعة اضلاع معلوم بخطوط مستقيمة تخرج من زاوية منه معلومة باقسام على نسب معلومة .

فان كانت هذه الاشياء معلومة فقد يمكننا [30] ان نقسم ذا [31] اربعة اضلاع معلوم بنسبة معلومة او نسب معلومة بخطّ مستقيم او خطوط مستقيمة تخرج من نقطة معلومة على ضلع من اضلاعه اذا نحن اتّبعنا الشرائط الّتى [32] تقدّم ذكرها .

تمّ الكتاب اقتصرنا بالدعاوي دون البرهان لانّ البرهان عليه سهل .

27 ونعمل (crossed out) 28 يخرج 28 يفصلا . 29 معلوم . 30 يمكنا . 31 اذا . 32 الذى .

Text B

5.[33] (Figure 8) By Euclid, in the Book of the Division. Circle ADG[34] is known, and in it there is a known sector ABG. We wish to cut off from the circle a segment contained by two parallel lines and two arcs, such that the segment is equal to the sector. Thus we bisect arc AG at point E, and we join AG. We draw line BZ parallel to AG, and we join ZA. We draw line EH parallel to ZA. Then arc ZH is equal to arc AE, and this is clear. So arcs HZ (plus) AE are equal to arc AG. We make arc AZ common, then arc ZG is equal to arc EH. We join ZG. Then triangle AZG is equal to triangle ABG. So the segment contained by arc GA and lines AZ, ZG is equal to the sector. We make the segment contained by chord AZ common, then the segment of which the chord is GZ is equal to the sector plus the segment of which the chord is AZ. But the segment of which the chord is GZ is equal to the segment of which the chord is EH. We drop the common segment, of which the chord is AZ, then by subtraction the segment which is contained by arcs HZ, AE and the chords AZ, EH, which are parallel, is equal to the sector. That is what we wished to demonstrate.

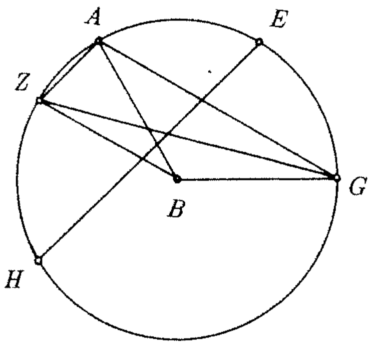

Figure 8

6.[35] (Figure 9) By the poor translator of it. We wish to trisect an acute rectilinear angle such as angle ABG. Thus we extend GB on the side of B rectilinearly and indefinitely. We draw with centre B and distance GB

[33]This is no. 5 of the *various geometrical problems*. The figure in the manuscript consists of a circle only, without letters. In Schoy's reconstruction, point B is on the circumference of the circle. It is more likely that B was the centre, because a "sector" (Arabic: qaṭṭāʿ) is usually part of a circle contained by an arc and two radii, and because the text is consistent with prop. 28 of Al-Sijzī's abstract if B is the centre.

[34]It is not clear where exactly point D is located.

[35]This is no. 6 of the *various geometrical problems*. A trisection of the angle is necessary if one wishes to apply prop. 5 of Text B and prop. 28 of Al-Sijzī's abstract to find the a segment equal to for example 1/9th of the circle.

Text B

ه . لاقليدس فى كتاب القسمة . دائرة ادج معلومة فيها قطّاع ابج معلوم ونريد ان نفصل من الدائرة قطعة يحيط بها خطّان متوازيان وقوسان فيكون القطعة مساوية للقطّاع . فنقسم قوس اج بنصفين على نقطة ة ونصل اج ونخرج خطّ بز يوازى اج ونصل زا ونخرج خطّ هح يوازي زا فقوس زح مثل قوس اه وذلك ظاهر فقوسا حز اه مثل قوس اج ونجعل قوس از مشتركة فيكون قوس زج مساوية لقوس هح ونصل زج فمثلّث ازج مثل مثلّث ابج فيكون القطعة الّتى يحيط بها قوس جا وخطّا از زج مثل القطّاع . ونجعل القطعة الّتى يحيط بها وتر از مشتركة فيكون القطعة الّتى وترها جز مثل القطّاع والقطعة الّتى وترها از . لكن القطعة الّتى وترها جز مثل القطعة الّتى وترها هح فنسقط القطعة الّتى وترها از مشتركة تبقى القطعة الّتى يحيط بها قوسا حز [1] اه ووترا از هح وهما متوازيان مثل القطّاع وذلك ما اردنا ان نبيّن .

و . لمترجمه الفقير . نريد ان نثلّث زاوية حادة مستقيمة [2] الخطّين كزاوية ابج . فنخرج جب من جهة ب على استقامته الى غير النهاية وندير على مركز ب ببعد جب نصف دائرة جاد ولتقطع ضلع با على نقطة < ا ونضع > طرف المسطرة على نقطة ا وطرفه الآخر . . . . . . [3] ونحرّكها حتّى يكون الخطّ < الّذى بين > خطّ بز وبين النقطة الواقعة تحت المسطرة من محيط نصف الدائرة كنقطة ه مساويا لنصف قطر الدائرة . فاذا وجدنا ذلك نصل خطّي اهز به . فلكون زاوية اهب الخارجة من مثلّث بهز مساوية لمجموع زاويتي هبز هزب المتساويتين يكون زاوية باه المساوية لزاوية بها ضعف زاوية بزه . ولكون زاوية ابج الخارجة من مثلّث ازب مساوية لمجموع زاويتي باز ازب يكون ثلاثة امثال زاوية ازب . فنعمل على نقطة ب من خطّ بج زاوية جبح مساوية لزاوية بزه فهي ثلث زاوية ابج وذلك ما اردنا ان نعمل .

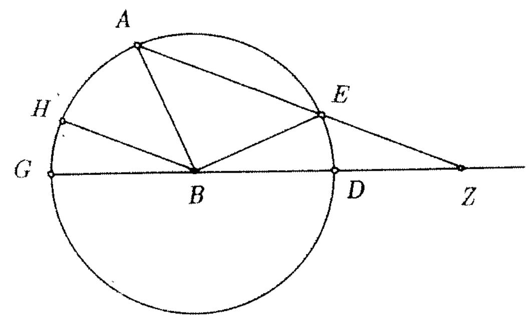

Figure 9

1 جز . 2 مستقيم . 3 illegible

semicircle GAD, let it intersect side BA at <point A. We put>[36] the end of the ruler on point A and its other end <......>[37] and we move it until the line segment <between> line BZ and the point under the ruler on the circumference of the semicircle, such as point E, is equal to the radius of the circle.[38] If we have found this, we join lines AEZ, BE. Because angle AEB, which is exterior to triangle BEZ, is equal to the sum of the equal angles EBZ, EZB, angle BAE, which is equal to angle BEA, is twice angle BZE. Because angle ABG, which is exterior to triangle AZB, is equal to the sum of angles BAZ, AZB, it is three times angle AZB. So we make on point B of line BG angle GBH equal to angle BZE. Then it is one-third of angle ABG. That is what we wished to do.

Text C

(Figure 7) If he said: how do we cut off from circle ABG one-third of it, or one-fourth of it, or any part we wish, by two parallel lines, then we make the centre of the circle point D, and we draw in the circle the chord of one-third of it, namely line AG, and we draw line BD parallel to AG. We join BG. We bisect arc AG at point E. We draw from point E line EZ parallel to BG. Then figure $ZBGE$, which is between the two parallel lines, is one-third of the circle. This is the figure for it.[39]

Arabic text:[40]

فان قال كيف نفصل من دائرة ابج ثلثها او ربعها [40a] او اي جزء شئنا بخطّين متوازيين فنجعل مركز الدائرة نقطة د ونخرج في الدائرة وتر ثلثها وهو خطّ اج ونخرج خطّ [40b] بد يوازي اج ونصل بج ونقسم قوس اج بنصفين على نقطة ه ونخرج من نقطة ه خطّ هز يوازي بج فيكون شكل زبجه الّذي فيما بين الخطّين المتوازيين ثلث الدائرة وهذه صورته [40c] .

[36]The manuscript is damaged by water. The words that could be read only faintly are in brackets.

[37]Two words are wiped out completely. The meaning must have been: we put the other end of the ruler on GB extended.

[38]This construction belongs to a type of construction called *neusis* in Greek geometry. A *neusis* is the insertion of a straight segment of given length between two given (straight or curved) lines such that the segment verges (Greek: νεύειν) to a given point, that is to say that the given point is on the rectilinear extension of the segment.

[39]The figure is the same as that in *On Divisions*, prop. 28 in the abstract of al-Sijzī (Figure 7 above). Line AB is also drawn in manuscripts **F** and **L**. Abu l-Wafā' does not give proofs, because they were supposed to be unnecessary for the craftsmen for whom the book was intended. The margin of the Istanbul manuscript contains a proof which I do not render here.