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On Euclid's Lost *Porisms* and Its Arabic Traces

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In Book VII of the *Mathematical Collection*, Pappus of Alexandria quotes the first porism from Euclid's lost *Porisms*. This obscure quotation was explained by Robert Simson, first in a brief note [17] in 1723, and then in a different way in [18], published posthumously in 1776. Modern historians have followed Simson's 1776 interpretation, but in section 2 of the present paper we argue that the 1723 interpretation is more satisfactory. We also discuss some consequences of this interpretation for the historiography of Greek geometry.

In section 3 we show that the *Selected Problems* of Ibrahim ibn Sinan (died A.D. 946) and the *Geometrical Annotations* of Al-Sijzi (around A.D. 970) contain material related to Euclid's *Porisms*. Thus it seems that the first part of Book 1 of Euclid's *Porisms* was somehow transmitted to the Arabic-Islamic tradition. The relevant parts of the *Geometrical Annotations* have been edited and translated in an appendix. Our paper begins with a brief explanation of the concept of a porism.

1. Introduction

In modern historical investigations of the *Elements* Euclid sometimes emerges as a mediocre mathematician and compilator ([21] p.

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197). However, what we know about Euclid's lost *Porisms* sheds a more favorable light on his mathematical abilities.The most important ancient evidence about the *Porisms* is found in Book VII of the *Mathematical Collection* of Pappus of Alexandria. Book VII has now appeared in a new edition with translation and commentary by A. Jones, including an essay about the *Porisms* ([13] pp. 547-572).

In Book VII of the *Collection*, Pappus provides a vague and general discussion of the concept of a porism and of the *Porisms* of Euclid, and a classification of the porisms into 29 categories, according to the kind of thing that Euclid proved in each porism ([13] pp. 94-105, [11] pp. 648-661, [22] pp. 485-495). Pappus states that the *Porisms* consisted of three Books, containing a total of 171 porisms, and he proves 38 geometrical lemmas to the *Porisms* ([13] pp. 260-294, [11], pp. 866-914, [22] pp. 669-717). While some of Pappus' lemmas are auxiliary theorems, other lemmas turn out to be the converses of statements proved by Euclid in his *Porisms*. This is to say that if Euclid proved «if A then B», Pappus' lemma proves «if B then A» ([13] p. 460).

In Euclid's *Elements*, the Greek word πόρισμα means «corollary», but in connection with the *Porisms* the word has a different meaning. Pappus does not give a definition of a porism, but he says that a porism is for the «finding» (πορίζειν) of something, while a theorem is for the proof of something, and a problem is for the construction of something. On the basis of careful study of the ancient evidence in the *Collection*, Simson ([18] p. 323) gave a more precise definition [1], which can be rephrased in modern terms as follows: In an (Euclidean) porism one considers a geometrical figure in which certain parts are supposed to be constant, while others are variable. The porism states that a new element defined by means of the variables is constant (or: invariant) and can be determined (found) by means of the original constants. Simson used «given» instead of «constant», and «not given» instead of «variable», as the Greeks would probably have done. It would be very un-Greek to call any geometrical object «variable».

[1] Simson's definition of a porism is as follows: Porisma est Propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem datis, sed quae ad ea quae data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in Propositione descriptam ([18] p. 323). «A porism is a proposition in which it is proposed to prove that one or more things are given, for which and for innumerable things, that are not given themselves, but have the same relationship to the things that are given, it is agreed that they must display a common property described in the proposition» Compare [13] p. 550.

Loci can be considered as porisms, but Euclid's first porism, which we will discuss in section 2, is not a locus. Possible modifications of the Euclidean concept of a porism in the later Greek tradition do not concern us here [2].

Simson based his definition of a porism on the analysis of a number of passages in Pappus' Book VII, including the following, which we quote for later use. Pappus says:

> But at the beginning of the first Book, he (Euclid) placed some (porisms) of similar form, ..., to the number of ten. And hence, finding it to be possible to encompass these in one proposition, we (Pappus) have written: If in a *hyptios* or *paryptios* three points on one (line), or both the points on a parallel (line) are given, while (each of) the rest except one lies on a line given in position, then that one too will lie on a line *given* in position. This is enunciated only for four lines, of which not more than two are through the same point. It is not recognized that it is true for every number put forward if one states it thus... (quoted from [13] pp. 96-98, slightly modified, see also [11] pp. 652-654, [22] pp. 488-489, italics are mine)

Pappus then goes on to state his own generalization. Using this generalization and some of the 38 lemmas, Simson was able to make sense of the quoted passage ([18] pp. 348, 352-389). Jones has proposed a new definition of the *paryptios*, which is in my opinion more plausible than the definition given by Simson, but which does not affect Simson's interpretation of the theorem that Pappus states ([13] pp. 393 note 7). On the basis of Jones' new definition Simson's conclusions can be restated thus:

1. A hyptios is a figure like PQRVWX in Figure 1,

2. A paryptios is the analogous figure consisting of V, W, X and only two of the points P, Q and R, such that either VW||PQ or VX|| PR or WX||QR. We would say that the «missing point» is at infinity. Figure 2 displays the possibilities, but it should be kept in mind that the positions of WV and PQ may be interchanged, and that point X may also lie between WV and PQ, as in Figure 6 below.

3. In the quoted passage, Pappus summarizes the ten Euclidean porisms (henceforth called hyptios- and paryptios-porisms) in the following manner:

Let P, Q, R in the hyptios-figure be given points, and in the case of the paryptios let the two points among P, Q and R that occur in the

[2] Proclus' examples: «to find the centre of a given circle, or the greatest common measure of two commensurable magnitudes» are no good examples of Euclidean porisms ([14] p. 236).

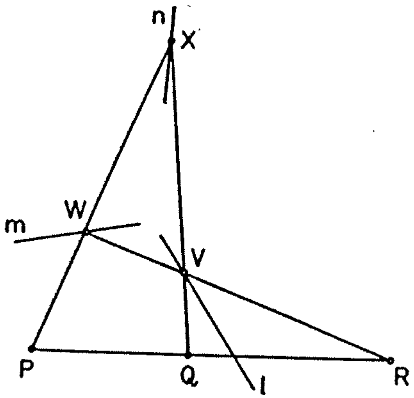


Figure 1

figure be given. If points V, W are on given lines l, m, then X is on a *given* line n ([13] pp. 392-393).

4. Euclid treated the paryptios and the hyptios separately, and at least in his treatment of the hyptios, he distinguished different cases, such as a). P, Q, R not on l, m; b). P on l, Q not on m; c). P on l, Q on m; etc. In each of the ten porisms he treated one case.

We note that in the passage quoted from Pappus, and in Greek geometrical texts in general, the word given is used in two different meanings:

1. assumed in the beginning
2. not assumed in the beginning, but determined (or constructible) by means of the things that are assumed in the beginning.

The context determines which of the two interpretations has to be chosen. In quotations I have italicized all words *given* which have (in my opinion) the second meaning.

2. The first porism

Pappus quotes one more porism in the beginning of his classification of the porisms in Euclid's Book 1:

In the beginning of the Book [3] is this diagram: if lines from two given points inflect on a line given in position, and one (line) cuts off (a segment)

[3] Before the publication of [7] in 1882, a corruption in the Greek text had not been recogni-

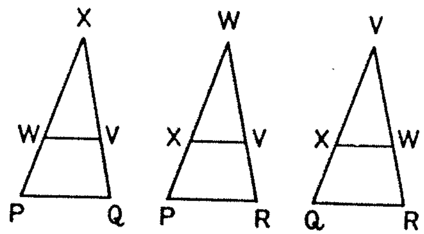


Figure 2

from a line given in position up to a point given on it, (then) the other (line) too will cut off from another (line) a (segment) having a *given* ratio (to the first segment) ([13] p. 100, [11] p. 656, [22] p. 490]; words in parentheses are not in the Greek, italics are mine).

In 1723, Simson interpreted the italicized word *given* as *determined* ([17] pp. 337-339). Then Pappus' quotation means the following (Figure 3):

Let lines RV and QV, drawn from two fixed points R, Q, meet at a variable point V on a fixed line l. Let a line m and a pont M_1 on m be fixed, and let RV meet m at W. Then one can find a line n, a point N_1 on it

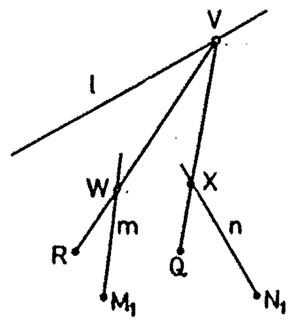


Figure 3

zed, with the result that the quoted passage «in the beginning of the Book is this diagram» had been misinterpreted as «in the beginning of the seventh is this diagram». This word «seventh» seems to have caused some confusion. Simson stated (correctly) that the quoted passage refers to the first porism in Euclid's book, but Chasles ([2] p. 66) thought that the ten hyptios- and paryptios porisms came first.

and a fixed ratio β_0 such that QV meets n at X and always $XN_1/WM_1 = \beta_0$ (Simson's β_0 is such that *l*, *m* and *n* intersect at the same point).

I will call this the 1723 porism. Simson added a simple proof ([17] pp. 337-339), to be discussed below.

Later Simson discovered that one can find *n* and N_1 also for arbitrary $\beta \neq \beta_0$ (the lines *n* for different β are parallel). He therefore withdrew his earlier interpretation, and stated in 1776, that *given* in Pappus' quotation means *assumed arbitrarily* ([18] pp. 350, 400). Thus Pappus' quotation means (Figure 3):

> Let lines RV and QV, drawn from two fixed points R, Q, meet at a variable point V on a fixed line *l*. Let line *m* and a point M_1 on *m* be fixed, and let RV meet *m* at W. Let a ratio β be assumed arbitrarily. Then one can find a line *n* and point N_1 on it such that QV meets *n* at X and always $XN_1/WM_1 = \beta$.

I will call this the 1776 porism.

Simson added a complicated reconstruction of the supposed Euclidean proof of the 1776 porism ([18] pp. 400-405; the reconstruction is also available in [13] p. 555). The reconstruction is related to, but much more complex than, the lemma for the first porism in Pappus' *Collection*.

Chasles, Zeuthen, Heath and Jones accepted the 1776 porism as Euclid's first porism (see [2] pp. 114-115, [23] pp. 152-154, [6] vol. 1, p. 437, [13] pp. 549, 552), and they offered different explanations of the relationship between the 1776 porism, Pappus' «lemma for the first porism», and the ten hyptios-paryptios porisms ([2] pp. 65-66, [23] pp. 169-171, [13] p. 560). No attention seems to have been paid to the possibility that Simson's 1723 interpretation is correct.

Simson apparently withdrew this interpretation because he believed that a mathematician of the calibre of Euclid must have proved the more general 1776 porism. However, it seems to me that there is an a priori methodological advantage in supposing that Euclid proved a result of less generality, whatever his mathematical abilities may have been. The 1723 interpretation is also supported by further arguments:

1. In theorems in Greek texts, the assumptions are usually listed before the consequences. This suggests that *given* in the quoted passage means *determined*. The same is true in the passage on the hyptios quoted in the first section, and throughout Euclid's *Data*, and Pappus' *Collection*.

2. Pappus' «lemma to the first porism» is related to the 1723 porism in a simple way. The lemma is as follows, in Pappus' notation ([13] pp. 260-261, [11] pp. 866-869, [22] pp. 669-671):

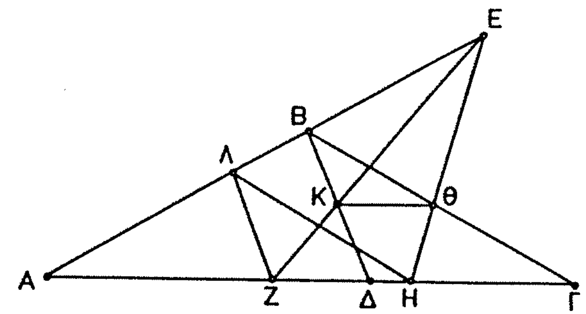


Figure 4

If in a figure ABΓΔEZH as in Figure 4 we have AZ:ZH = AΔ:ΔΓ, and if we join KΘ (K, Θ being the points of intersection of BΔ, EZ and BΓ, EH), then KΘ||AΓ. Pappus gives two proofs, the first of which is as follows. Draw ZΛ||ΔB to meet AE at Λ, and join ΛH. Then, since AZ:ZH=AΔ:AΓ, we have AZ:AΔ=AH:AΓ, but AZ:AΔ=AΛ:AB; so that AΛ:AB=AH:AΓ, whence ΛH||BΓ. Thus EK:KZ=EB:BΛ=EΘ:ΘH, so that KΘ||AΓ, as desired.

The correspondence with the 1723 porism is obvious if we put Z=R, H=Q, AE=*l*, K=W, Θ=X, E=V as in Figure 5, in which the same notation has been used as in Figure 3. Denoting the points of intersection of *l*, *m*, *n* and QR as L, M, N and the point of intersection of *l* and *m* as S, we have the further correspondences A=L, Δ=M, Γ=N, B=S. The lemma suggests the following proof of the 1723 porism (Figure 5): Let Q, R, 1 and m be given. Then S and M are also determined. Draw RT||*m* to meet *l* at T, join TQ, and let *n* be the line

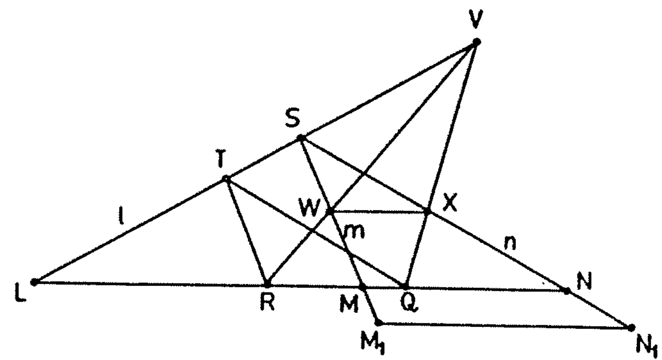


Figure 5

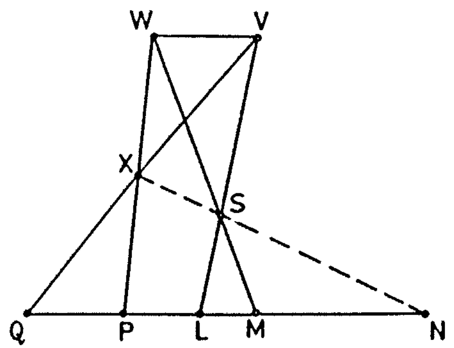


Figure 6

through S parallel to TQ. T corresponds to Λ in Figure 4. Then as in the lemma WX||QR. Hence, if M_1 on *m* is a given point, we let N_1 be on *n* so that M_1N_1 and QR are parallel. Then $N_1X{:}M_1W$ = NS:MS = QT:RT. Thus the porism is correct with β_0 = QT:RT, q.e.d..

If this was Euclid's proof, Pappus' lemma must have been intended to express N in a more abstract way in terms of the data of the problem. Simson defined N as the point on RQ such that LR:RQ = LM:MN ([17] pp. 337-339), but the present proof has the historical advantage that the lines RT, QT used in the definition of N occur as auxiliary lines in Pappus' lemma. We know that Pappus derived auxiliary lines in some of his lemmas to the works of Apollonius from the propositions to which the lemmas referred ([10] pp. 200-201, 211).

Pappus' lemma for the first porism is also related to the 1776 porism, but with the 1723 porism as intermediary. The hypothesis that Euclid's first porism was the 1723 porism is therefore of greater simplicity. Pappus' lemma is of the same difficulty as the 1723 porism itself, but this is not an argument against the 1723 interpretation. In the case of the paryptios- and hyptios-porisms, where we know the porisms and the lemmas, the lemmas have the same difficulty as the porisms to which they relate (see [13] p. 558, *contra* Heath in [6] vol. 1 p. 437).

3. The notation of Figures 2 and 5 shows that the 1723 porism stands in an interesting relationship to the paryptios-porism for the case where WX||QR. If we assume that this case of the paryptios-porism has been proven, and that line n in Figure 5 is the desired line, then NX:MW=NS:MS, because the lines *l*, *m* and *n* of the paryptios-porism intersect in one point, S. Thus the 1723 porism is immediately seen to be true for M_1=M, N_1=N. The case $M_1 \neq M$, $N_1 \neq N$ is now an easy generalization, which Euclid might have found interesting for reasons that are not clear to me. The important point is that the 1723 porism

and the case WX||QR of the paryptios-porism are so closely related that Euclid would probably treat them together.

Pappus' «lemma to the second porism» is of interest here (Figure 6, see [13] pp. 260-263, [11] pp. 868-871, [22] pp. 671-672, the notation is mine):

Let QPLMN, PXW, QXV, MSW, LSV be straight lines, such that VW||PQ and QP:PN=LM:MN. Then points X, S and N are also on a straight line. This lemma clearly belongs to the paryptios-porism for the case where WV||QP, so that this must have been Euclid's second porism ([13] p. 557). We now note that the 1723 porism and the second porism form a complete treatment of the paryptios-figure, as suggested by Figure 2 (the cases WX||RQ and VX||RP differ only in notation). Thus the 1723 porism and Euclid's second porism form a thematical unity.

The preceding arguments show to my mind that Euclid's «first porism» was the 1723 porism and not the more general 1776 porism.

This interpretation is in conflict with Zeuthen's theory ([23] p. 153) that Euclid used a generalization of the 1776 porism in his solution of the famous «locus of four lines». However, I do not find this conclusive. We know that Euclid solved the locus of four lines, because Apollonius says so in the preface to his *Conics* ([6] vol. 2, p. 129) but all we know about Euclid's solution is that the synthesis was incomplete, so that further speculation seems rather pointless.

If one accepts that the first porism was the 1723 porism, certain aspects of Chasles' reconstruction [2] of the *Porisms* become less likely. To make this clear it is necessary to say a few words about this reconstruction. According to Chasles, the most important theme of the Porisms was what he called «homographic correspondences» between points on two lines, that is the equivalent of the modern projectivities between two lines.

Chasles believed that many of Euclid's porisms stated relations between a variable point W on a line *m*, a point X on a line *n* related to W by a «homographic correspondence», and a number of suitably fixed points on *m* and *n*. For different choices of these fixed points and projectivities one obtains different types of relations and hence different porisms. Chasles stated and proved his reconstructed porisms in Greek style, and he therefore had to represent his projectivities geometrically. He sometimes used devices as in Figure 7, involving two given points P, Q and three lines *l*, *m* and *n* (cf. [2] pp. 129, 151). Any projectivity between *m* and *n* can be represented by such a device with suitably chosen *l*, P and Q. The only evidence that such configurations were actually studied in Greek geometry was Euclid's first porism in its

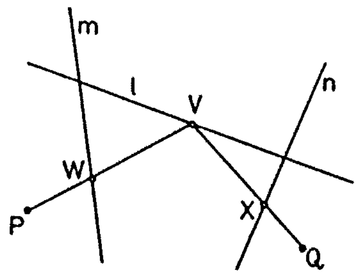


Figure 7

1776 interpretation, compare Figure 3. However, in the 1723 porism, *l*, *m* and *n* intersect in one point, and WX||PQ, so that W and X are related by a perspectivity with centre at infinity. This means that a number of Chasles' reconstructed porisms lose their historical basis.

3. Arabic traces of the Porisms

Yvonne Dold-Samplonius showed in 1977 ([4] p. 56) that some of Pappus' lemmas to the *Porisms* entered the Arabic tradition through the *Book of Assumptions,* that is an Arabic version of a lost Greek work by an unidentified author whose name was arabicized as Aqatun. The relationship with Pappus was further investigated by Jones ([13] pp. 602-605).

In this section we will discuss some additional Arabic material related to the *Porisms,* but probably independent of Pappus. This material consists of a few propositions of the *Geometrical Annotations* [4] of Al-Sijzi, who flourished around A.D. 970, and one proposition in the *Selected Problems* of Ibrahim ibn Sinan, who died in A.D. 946. A description of these works and biographical information on their authors can be found in [10] pp. 189-194. The relevant passages have been edited and translated in the appendix of the present paper, if not available elsewhere.

Neither Al-Sijzi nor Ibrahim ibn Sinan refer to the *Porisms*, but the

[4] *Geometrical Annotations* is an abbreviation used by Al-Sijzi himself. For the full title see the beginning of the appendix.

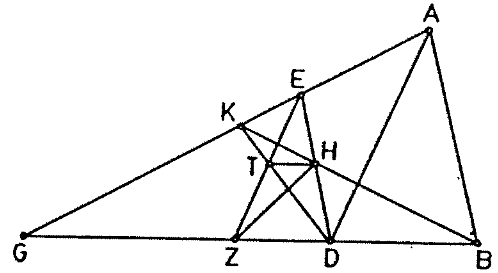


Figure 8

following summary will establish the mathematical relationship between the relevant propositions and the *Porisms.* We begin with Al-Sijzi's *Geometrical Annotations.*

In the first two propositions that are of concern to us (see the appendix, pp. 110-11), Al-Sijzi considers a given point D on the base of a given triangle ABG (Figure 8). He chooses E on AG and Z on BG such that DE||BA and EZ||AD. Al-Sijzi wishes to construct lines BK, DK that meet at some point K on AG and intersect ED, EZ at points H, T. The lines have to be such that in the first proposition

(3.1) triangle HTD has a given area

while in the second proposition

(3.2) the ratio between the areas of triangles HTK, HTZ is equal to a given ratio.

Al-Sijzi constructs H on ED in such a way that conditions (3.1) or (3.2) will eventually be satisfied. He then draws HT parallel to BG to meet EZ at T, and he proves that K, T and D are collinear as a result of the fact that H is on ED. He then proves that (3.1) or (3.2) hold. All further details can be found in the appendix.

The two propositions are related to a case of Euclid's first porism. Figure 8 corresponds to the special case of Figures 4 and 5 where in the notations of Figure 5 the given line m passes through the given point Q. The exact correspondences are as follows

Figure 8	B	D	AG	ED	EZ	E	A	K	H	T
Figure 5	R	Q	*l*	*m*	*n*	S	T	V	W	X

We note that the equivalent of HT||BD, used by Al-Sijzi, plays a fundamental role in Figures 4 and 5. His extra conditions (3.1) and (3.2) serve to make the paryptios BDHTK in Figure 8 completely determined, while its equivalent RQWXV in Figure 5 is variable. Thus

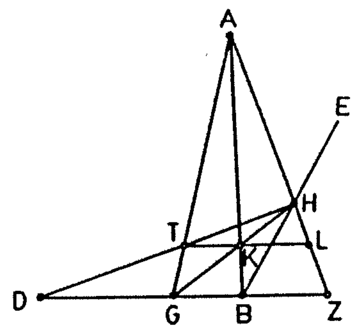


Figure 9

the porism or lemma is turned into a problem with a unique solution.

The *Geometrical Annotations* also contain a proposition related to Euclid's second porism (see appendix, p. 112). Al-Sijzi considers a given point D on the rectilinear extension of the base BG of a given triangle ABG (Figure 9), and a given line BE through B. He wishes to construct two lines DH, GH such that TK||GB and

(3.3) H is on BE.

Al-Sijzi constructs Z on GB extended such that ZB:BG=ZG:GD, and he joins AZ, to meet BE at H. That H is the desired point is proved as follows. Let HD meet AG at T, draw TKL||GB to meet AB

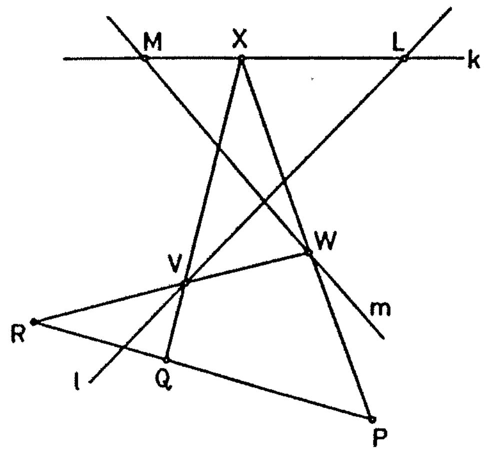


Figure 10

at K and AZ at L. Then points G, K and H are collinear, because LK:KT=ZB:BG=ZG:GD, q.e.d.

The proposition is obviously related to the second porism, again for the special case where m passes through the given point Q in the notation of Figure 6. The exact correspondences are as follows:

Figure 9	D	G	T	K	H	AB	AG	AZ
Figure 6	P	Q	W	V	X	*l*	*m*	*n*

(3.3) determines the paryptios DGKTH completely.

We now turn to a problem in the *Selected Problems* of Ibrahim ibn Sinan. The Arabic text of the problem and its solution is in [15] pp. 200-201, and [12] no. 6 pp. 17-18, an English translation is in [8] pp. 155-156. Here I render the problem in notation suggesting the relationship with the hyptios-configuration (Figure 10, compare Figure 1):

Let three collinear points P, Q, R and three lines *l, m, k* be given. One wishes to construct point X so that if QX, PX intersect *l, m* at points V, W, then W, V, R are collinear, and so that at the same time

(3.4) X is on *k*.

Thus the problem is about a hyptios as in Figure 1, with an extra condition (3.4) which determines the hyptios completely.

Let L, M be the common points of *l* and *k*, *m* and *k* respectively. Ibrahim does not construct line *n* in Figure 1, but he proves that the proportion XL:XM is determined by P, Q, R, *l*, *m*, which amounts to the same thing. The further details of the solution do not concern us here; we note only that Ibrahim gives an analysis, and that a synthesis of his solution is found in Al-Sijzi's *Geometrical Annotations,* for the Arabic text and an English translation of this see [8] pp. 154-155.

We now proceed to the historical analysis of the above-mentioned passages. Figures 8, 9 and 10 do not resemble other known material in the published or unpublished Arabic geometrical literature, and one wonders why Al-Sijzi and Ibrahim ibn Sinan were interested in the above-mentioned propositions at all. The simplest explanation is that the propositions and the *Porisms* are related not only mathematically, but also historically. This raises the question, what exactly did Ibrahim and Al-Sijzi take over from their sources.

It seems that the conditions (3.1)—(3.4) were added by Al-Sijzi and Ibrahim in order to create problems with unique solutions. One may compare this procedure with Ibrahim's *Treatise on the method of analysis, synthesis and the other procedures in geometrical problems* ([16] p. 294 no. 2). There it is explained how indeterminate problems (problems with infinitely many solutions) can be «corrected» (i.e. changed into problems with a finite number of solutions) through the addition of an extra condition ([15] pp. 80-82).

We note that as a general rule, Ibrahim ibn Sinan always makes acknowledgement when he takes a solution of a geometrical problem from another author. However, he does not find it necessary to give credit when he takes a problem, but not its solution, from another source (see for examples [10] pp. 191, 207, 219, 224). Since Ibrahim does not refer to other geometers in the present problem, we conclude that he found the hyptios-porism in some source, added the condition (3.4), and then solved the resulting problem by himself.

Al-Sijzi often copied a problem, made a trivial change in the solution, and then claimed authorship (see [9] pp. 252, 315, [10] pp. 221-222; there are many more examples in unpublished manuscripts). It is therefore plausible that Al-Sijzi adopted Figure 8 with the proof that K, T and D are collinear, and Figure 9 without line BE but with the proof that G, K and H are collinear. The changes made by Al-Sijzi are in this case the addition of conditions (3.1), (3.2) and (3.3).

It remains to discuss possible channels of transmission. There are forceful arguments against a transmission through (an Arabic version of) Book VII of the *Collection* of Pappus. If an Arabic translation of Book VII had been made, it would have been received by the Arabic geometers with the same enthusiasm as by their 16th-century European successors. The total absence of references to such a translation in the known Arabic medieval literature can therefore be taken as evidence that an Arabic version of Book VII never existed. But there is more. In the case of Ibrahim, it is difficult to see how he could take the hyptios-porism from an Arabic translation of the obscure passage quoted in section 1 of this paper, if we keep in mind that it took European scholars 150 years to understand the original Greek text. (There is no evidence that Ibrahim knew Greek himself). Regarding the propositions of Al-Sijzi it should be pointed out that they deal with a special case of the paryptios (Q on *m*) not treated by Pappus. One may also remark that point A in the triangle ABG in Figure 8, with which Al-Sijzi begins his first proposition, corresponds to the auxiliary point Λ in Figure 4, introduced by Pappus in the course of the proof of his lemma. Thus we can exclude the possibility that Ibrahim or Al-Sijzi used Book VII of the *Collection* or a source dependent on it.

The two remaining possibilities are in order of decreasing probability:

1. A lost Arabic translation of a lost Greek collection of lemmas or problems, depending on the *Porisms* but not on Pappus' *Collection*. We know that at least one such collection existed, namely the *Problems of the Greeks* (Masa'il li-l-Yunaniyyin), which was translated into Arabic in the 10th century by Yuhanna ibn Yusuf ([20] p. 14, e). Al-Sijzi's Figure 8 gives the impression of having been derived from a lemma to the first porism.

2. A lost Arabic translation of the *Porisms*, or perhaps, of a fragment of the *Porisms* containing the beginning of the work. The bibliographical work *Fihrist* of Ibn al-Nadim (late 10th c.) contains a reference to a *Book al-Fawa'id* by Euclid ([5] p. 266 line 16, translated as *Benefits* in [3], p. 636 line 11). The Arabic word *fa'ida* (plural: *fawa'id*) means *corollary*, which is, as mentioned in the beginning of this paper, the second meaning of the Greek word πόρισμα. Thus Suter ([19] pp. 49-50) assumed that Ibn al-Nadim referred to an Arabic version of the *Porisms*. Ibn al-Nadim remarks that the book in question is spurious - perhaps the *Porisms* was not considered as a genuine Euclidean work because of the mysterious character of its contents.

It is worth pointing out that all Arabic traces of the *Porisms* in [4] and in the present paper have been identified by means of the very incomplete Greek evidence about the work. Thus Arabic mathematical texts may well contain other traces of the *Porisms* that are no longer recognizable as such, because they are related to parts of the *Porisms* about which we are not informed by the Greek tradition.

Appendix

The following appendix contains English translations and Arabic texts of the passages discussed in section 3 from the *Book by Ahmad ibn Muhammad ibn ʿAbdaljalil (al-Sijzi) on the Selected Problems Which Were Currently Being Discussed by Him and the Geometers of Shiraz and Khorasan, and His (Own) Annotations* ([16] p. 333 no. 23). In the present paper, the title has been abbreviated to *Geometrical Annotations*.

Manuscripts:

D = Dublin, Chester Beatty 3562, f. 35a-52b (see [1] p. 59) dated A. D. 1215.

I = Istanbul, Resit 1191, 31b-62a, undated.

A more detailed description of these manuscripts can be found in [10] p. 228.

All words in parentheses do not belong to the text in the manuscripts.

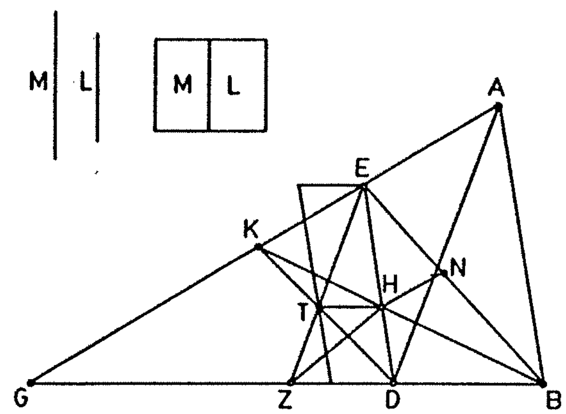


Figure 11

(D 50b:34-51a:10,I:59b:8-60a:7)

(Figure 11) [5] Also our (i.e. Al-Sijzi's) solution. Triangle ABG is assumed. From the assumed point D on side BG line DE has been drawn parallel to AB. We join AD, and we draw EZ parallel to AD. Area L is given.

We wish to draw lines BK, DK which meet at line AG, and which intersect DE, EZ at H, T in such a way that if we join HT, we obtain a triangle HDT equal to area L.

Thus we make LM twice L, and we apply to DE an area equal to area LM, deficient by an area similar to the double of triangle DEZ; this is parallelogram TD, I mean; the missing [6]. It is clear that angle T falls on line EZ. Thus we extend BH toward K, and we join DT, KT.

I say that triangle HDT is equal to area L, and that line DTK is one straight line.

Proof: Triangles ABD, DEZ are similar, so that the ratio of AB to DB is equal to the ratio of ED to DZ. The ratio of AB to EH is equal to the ratio of BK to KH. Line HT is parallel to line DZ, and the ratio of HT to HE is equal to the ratio of DZ to DE, that is, the ratio of DB to BA. Thus, permutando, the ratio of BD to HT is equal to the ratio of BA to HE. But the ratio of BA to HE was (shown to be) equal to the ratio of BK to KH. Therefore the ratio of BD to HT is equal to the ratio of BK to KH. But HT is parallel to BD, therefore line DTK is straight.

Since parallelogram DT is equal to area LM, triangle HDT, which is half of the parallelogram, is equal to area L. Thus we have constructed what we wanted, and that is what we wanted to demonstrate.

We repeat the figure of the triangles in exactly the same way, and we (now) want to draw lines BHK, DTK which meet at point K, and which intersect DE, ZE at H, T in such a way that if we draw HT, HZ we obtain triangles HTK, HTZ such that the ratio of triangle HTK to triangle HTZ is equal to a given ratio, I mean, equal to the ratio of L to M.

Thus let us join BE. We make the ratio of EN to NB equal to the ratio of L to M. We draw NH parallel to AG, we draw HT parallel to

[5] In the figure in the manuscript point N is at the intersection of EB and AD, and line NHZ is straight.

[6] For this procedure, which is called «application of areas» see for example [6] vol. 1, pp. 394-396, [21] pp. 118-124. The parallelogram THE is «missing» here, not TD.

Arabic Text

استخراجنا أيضا مثلث ا ب ج مفروض وقد خرج من نقطة د على ضلع ب ج المفروضة خط د هـ يوازى ا ب ونصل ا د ونخرج هـ ز يوازى ا د وسطح ل معطا

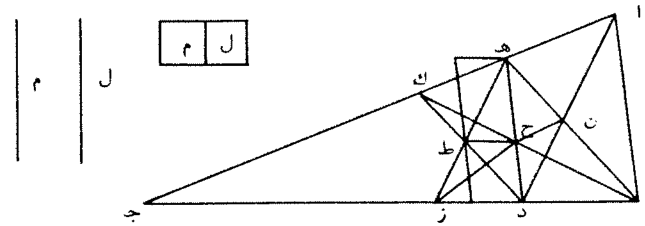
نريد أن نخرج كخطي ب ك د ك يجتمعان على خط ا ج ويقطعان د هـ هـ ز على ح ط إذا وصلنا ح ط يحدث مثلث ح د ط مساويا لسطح ل

فنجعل ل م ضعف ل ونضيف إلى د هـ سطحا مساويا لسطح ل م ينقص عن تمام سطحا شبيها بنصف مثلث د هـ ز وهو سطح ط د المتوازى أعني الناقص

فبين أن زاوية ط تقع على خط هـ ز فنخرج ب ح إلى ك ونصل د ط ك ط

أقول إن مثلث ح د ط مساو [1] لسطح ل وإن خط د ط ك خط واحد مستقيم

برهانه إن مثلثي ا ب د د هـ ز متشابهين فنسبة ا ب إلى ب د كنسبة هـ د إلى د ز ونسبة ا ب إلى هـ ح كنسبة ب ك إلى ك ح وخط ح ط يوازى خط د ز ونسبة ح ط إلى ح هـ كنسبة د ز إلى د هـ أعني كنسبة د ب إلى ب ا فبالإبدال نسبة ب د إلى ح ط كنسبة ب ا إلى ح هـ ونسبة (D 51a) ب ا إلى ح هـ قد كان كنسبة ب ك إلى ك ح فنسبة ب د إلى ح ط كنسبة ب ك إلى ك ح و ح ط يوازى ب د فخط د ط ك مستقيم ولأن سطح د ط [2] المتوازى الأضلاع مساو لسطح ل م يكون مثلث ح د ط الذى هو نصف المتوازى مساو لسطح ل [3] فقد عملنا ما أردنا وذلك ما أردنا أن نبين



ونعيد صورة المثلثات بأعيانها ونريد أن نخرج كخطي [4] ب ح ك د ط ك [5] يجتمعان على نقطة ك ويقطعان (I 60a) د هـ ز هـ على ح ط إذا أخرجنا ح ط ح ز يحدث مثلثي ح ط ك ح ط ز تكون نسبة مثلث ح ط ك إلى مثلث ح ط ز كنسبة مفروضة أعني كنسبة ل إلى م

١ مساويا (I) ٢ ز ط (D) ٣ ا (D, I) ٤ كخصى (I) ٥ د ط ز (D)

BG, and we join BHK, DT, TK. Then it is clear, as mentioned above, that DTK is a straight line. We draw HZ.

Then the ratio of DT to TK is equal to the ratio of BH to HK, that is, BN to NE, that is, M to L. But the ratio of TK to TD is equal to the ratio of triangle THK to triangle THD, which is equal to triangle ZHT. Thus the ratio of triangle TKH to triangle THZ is equal to the ratio of EN to NB, that is L to M. Thus we have constructed what we wanted, and that is what we wanted to demonstrate.

(D 52a:27-52b:1,I 62a:9-15)

(Figure 12) Our solution. Triangle ABG. BG has been extended rectilinearly toward D. Line BE is given in position. We want to draw from points G, D two lines which meet at line EB and which intersect AB, AG in such a way that if one draws from the points of intersection a straight line, it is parallel to line BG.

Thus let us extend GB toward Z so that the ratio of GB to BZ is equal to the ratio of GD to GZ. We join AZ to meet line EB at point H, and we draw DH, which intersects AG at T. We draw TKL parallel to GZ, and we join GK, HK.

I say that line GHK is one straight line.

Proof of this: the ratio of LK to KT is equal to the ratio of ZB to BG, but the ratio of ZB to BG is equal to the ratio of ZG to GD. Thus line HKG is one straight line. That is what we wanted to demonstrate.

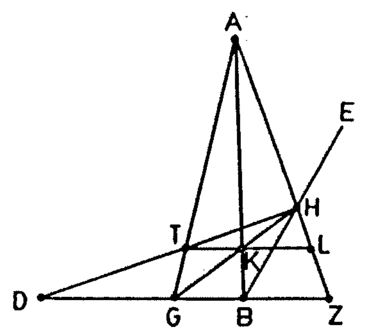
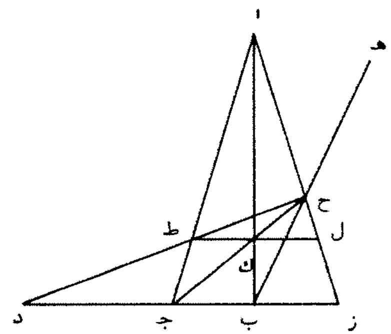


Figure 12

فلنصل ب هـ ونجعل نسبة هـ ن إلى ن ب كنسبة ل إلى م ونخرج ن ح يوازى ا ج
ونخرج ح ط يوازى ب ج ونخرج ب ح ك د ط ط ك فيبين كما قدمنا ذكره أن د ط ك
خط[6] مستقيم ونخرج ح ز
فنسبة د ط إلى ط ك كنسبة ب ح إلى ح ك أعني ب ن إلى ن هـ أعني م إلى ل
لكن نسبة ط ك إلى ط د كنسبة مثلث ط ح ك إلى مثلث ط ح د المساوى لمثلث ز ح ط[7]
فنسبة مثلث ط ك ح إلى[8] مثلث ط ح ز كنسبة هـ ن إلى ن ب أعني ل إلى م
فقد عملنا ما أردنا وذلك ما أردنا بيانه

٦ خطا (I) ٧ د حط (D) ٨ مثلث ط ك ح إلى : ناقص في I

استخراجنا مثلث ا ب ج قد خرج ب ج على استقامته إلى د وخط ب هـ معلوم الوضع
نريد أن نخرج من نقطتي ج د خطين يجتمعان على خط هـ ب ويقطعان ا ب ا ج
إذا أخرج من نقطة تقاطعهما[1] خط مستقيم يكون موازيا لخط ب ج
فلنخرج ج ب إلى ز وتكون نسبة ج ب إلى ب ز كنسبة ج د إلى ج ز
ونصل ا ز يقطع خط هـ ب على نقطة ح ونخرج د ح يقطع ا ج على ط
ونخرج ط ك ل يوازى ج ز ونصل ج ك ح ك
أقول إن خط ج ح ك خط واحد مستقيم
برهان ذلك إن نسبة ل ك إلى ك ط كنسبة ز ب إلى ب ج ونسبة ز ب إلى ب ج
كنسبة ز ج إلى (D 52b) ج د فخط ح ك ج خط واحد مستقيم وذلك ما أردنا
أن نبين



١ تقاطعهما (I)

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