B. A. Rosenfeld's Contributions to the History of Mathematics in Medieval Islamic Civilization.

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1 Introduction

The total scientific output of Professor B.A. Rosenfeld to date consists of more than 400 publications. Roughly one-half of these publications concern history of mathematics, and more than 70 % of the historical work is related to mathematics and astronomy in medieval Islamic civilization or closely allied topics (such as Hebrew mathematics and ancient mathematics in Arabic translation). Rosenfeld‘s earliest article on history of Arabic mathematics appeared in 1951, when he was 34 years old; at that time he was working at the University of Baku in Azerbaijan. Since that time, he has produced a continuous stream of articles and books, and he has also made important contributions in the form of commentaries to Russian translations of medieval Arabic and Hebrew works (included in the list of publications in this Festschrift).

For my personal research, Rosenfeld's books and papers have been very important. These papers and my correspondence with him were the main reason why I took a course "Russian for Reading" at Brown University, Providence RI. This has been a very rewarding activity, for my research in the history of Arabic mathematics has been much facilitated by my ability to read some Russian.

It is not possible to present an adequate description of Rosenfeld's work in the history of science in Islamic civilization in a short paper. I will therefore discuss some specific examples in detail, in order to give the reader a feeling for his work and his contributions. I do not assume that the reader is familar with the history of Arabic mathematics, and the paper therefore begins with brief historical survey.

2 Historical survey of Arabic mathematics and astronomy

The religion of Islam was founded by the prophet Muḥammad around 620. In the first century of Islam, the Muslims (i.e., those who have surrendered to God) established an enormous empire, reaching from the Atlantic ocean to India. The early Muslims were not interested in science, but there were living scientific traditions in areas which had been conquered, notably astronomy in pre-Islamic Iran. Around 750 a new Muslim dynasty came to power, and the new capital of Baghdad was founded. Under the influence of the caliphs (the heads of state), Baghdad soon became the scientific center of the world. First, the astronomical knowledge of the Iranian tradition was assimilated. Around 775, a delegation of Indian astronomers was received at the court of the caliph, and some Sanskrit astronomical works were translated into Arabic. Soon afterwards Greek scientific texts became the focus of attention. Many important mathematical and astronomical works were translated from Greek into Arabic, sometimes several times, and these translations were supported by the caliphs. In this translation process, an Arabic vocabulary was developed for technical mathematical and astronomical concepts. In the middle of the ninth century there existed good Arabic versions of Euclid's *Elements*, a few works of Archimedes, the *Conics* of Apollonius, the *Almagest* of Ptolemy, etc. Scholars soon began to produce creative works in mathematics and astronomy, written in Arabic. This is the beginning of the "Arabic" mathematical and astronomical tradition, which is named after the scientific language Arabic rather than the nationality of its authors. In the beginning the centre of this tradition was the Middle East (Iraq and Syria), but from the tenth century onwards Iran and adjacent areas (such as Uzbekistan), Egypt and Islamic Spain were also important. One of the main reasons why the exact sciences flourished in Islamic civilization was the support which scientists and schools often received from kings and the wealthy.

Arabic science started to decline in the Eastern part of the Islamic world after 1300. In the 15th century there was a brief but splendid renaissance at the court of Ulugh Beg in Samarkand. In the West, most of present-day Spain was conquered in the 12th century by the Christians. Science also declined there, but many Arabic manuscripts were translated into Latin in the Christian parts of Spain (i.e., texts by Arabic authors or Arabic versions of Greek texts, such as Euclid's *Elements*). In this way, Western European scholars first came in contact with geometry as a deductive science, with axioms, theorems and proofs. However, the transmission to medieval Christian Europe was very incomplete. This has become apparent since the middle of the 19th century, when orientalists and mathematicians began to study Arabic manuscripts in libraries in Europe. It turned out that many of the discoveries made by the "Arabic" scholars (i.e. the scholars who wrote in Arabic) did not reach medieval Europe. Many of Rosenfeld's papers deal with such hitherto unknown Arabic contributions.

3 The sources

I have divided Rosenfeld's contributions to the history of Arabic mathematics and astronomy into four categories. The first category is "sources".

The most important sources for the history of Arabic mathematics and astronomy are medieval Arabic manuscripts in libraries all over the world. These are in most cases not the original manuscripts written by the mathematicians themselves, but later copies, which may have been written several hundred years after the death of the author. Most manuscripts which existed are lost; we have today perhaps a couple of thousand mathematical manuscripts, some ten thousand astronomical manuscripts, and a couple of thousand manuscripts on related areas important for the history of mathematics (for example, the mathematical principles of astrology or optics). To say that "we have" these manuscripts is somewhat misleading. Many manuscripts are in libraries in Turkey, the Middle East or India, which are closed to the general public, and it is often extremely difficult or impossible to obtain microfilms.

Because many manuscripts have not been studied at all in modern times, our picture of Arabic science is still far from complete. In particular, there are large numbers of texts dealing with "applied mathematics" (astronomy, astrology, etc.) which are interesting for the history of mathematics, and which have been systematically ignored by most historians of mathematics, with few exceptions such as Rosenfeld.

The first thing one needs to know is where the manuscripts are. Together with Professor Matvievskaya from Tashkent, Professor Rosenfeld has produced an important bibliography, which is modelled after the work of the Swiss historian of mathematics Heinrich Suter. In 1900, Suter published a book in German on *The mathematicians and astronomers of the Arabs and their works* (see the bibliography). In this book he listed all the information on medieval Arabic authors which he could find in Arabic bibliographies, and all known manuscripts containing texts by these authors. Suter collected his information from the (often incorrect) descriptions in catalogues of libraries in Europe. Suter listed 528 mathematicians and astronomers who lived between the eighth and the seventeenth centuries.

In 1974 and 1978, Professor F. Sezgin (Frankfurt) published volumes 5 and 6 of his *Geschichte des arabischen Schrifttums* (History of Arabic literature), on mathematics and astronomy before the year 430 of the Islamic chronology (i.e., 1050 A.D.). In these volumes, Sezgin summarized the modern literature on Arabic mathematics and astronomy, and he listed all manuscripts of texts by authors who lived before 1050. He based his information not only on library catalogues but also on his own very extensive research in libraries in Turkey and other Islamic countries (in particular, the very rich libraries in Istanbul). Sezgin's volumes contained a wealth of new titles of medieval Arabic works which had never been studied. Sezgin also listed the Arabic translations of ancient (Greek and Sanskrit) works, as well as the commentaries, written between 1050 and 1700, on ancient works and on works written before 1050. He did not include original treatises written after 1050.

In 1983, Matvievskaya and Rosenfeld published their collective work *Mathematicians and astronomers from the Muslim Middle Ages and their works (8th-17th centuries).* This work does not replace Sezgin's volumes, but Matvievskaya and Rosenfeld make important additions. They included the original treatises in Arabic written in the later period (after 1050), as well as the results of modern research published between 1974 and 1983. There is an enormous bibliography, including a large number of Russian books and articles. Many of these publications are not well known in Western Europe and the United States, even though they contain valuable information on unpublished and often inaccessible Arabic manuscripts.

The work of Matvievskaya and Rosenfeld is the most comprehensive bibliographical tool available to date for the history of Arabic mathematics and astronomy. The work is in Russian, but it is even useful for researchers who know only the letters of the Russian alphabet, since the names of the authors, the manuscript numbers, the transcription of the titles and the references to the bibliography can be understood without any further knowledge of the Russian language.

Professor Rosenfeld sometimes referred to his bibliography as his "neo-Suter", because he followed the same numbering as Suter. However, the number of Arabic mathematicians and astronomers went up from 528 in Suter's work to more than 1000 in the "neo-Suter". Unlike Sezgin, Matvievskaya and Rosenfeld do not deal with Arabic translations of ancient Greek and Sanskrit sources. Together, Sezgin's volumes and the bibliography of Matvievskaya and Rosenfeld cover the whole history of Arabic mathematics and astronomy.

The compilation of bibliographical works such as those of Sezgin and Matvievskaya-Rosenfeld requires a great deal of effort and is very frustrating. One has to sift through an enormous amount of published literature, and because it is not possible to see the thousands of manuscripts themselves, one has to depend on (usually unreliable) descriptions of these manuscripts in the library catalogues. One never has the satisfaction that the work is finished or perfect, because new information arrives almost every day and errors turn up all the time. Because historical research is essentially based on these bibliographies, Sezgin, Rosenfeld and Matvievskaya have done a great service to the field. After volumes 1 - 3 were published, Rosenfeld started immediately to work on volume 4, including corrections and new additions. An English version of the bibliography (including the hitherto unpublished volume 4) is currently being prepared.

This chapter would not be complete without at least one more example of a contribution of Rosenfeld concerning manuscript sources. This example is his discovery, announced in 1975, of an Arabic manuscript which was kept in a mosque in Kuybishev (now Samarra) - I use the word "discovery" in the sense that Rosenfeld was the first to realize the importance of the manuscript. The manuscript contains a list of works by Ibn al-Haytham, a geometer who lived in Egypt around the year 1000, as well as a number of treatises by him. Some of these treatises were unknown, for example, a very interesting treatise entitled "on the structure (hay'at) of the motions of the seven stars [meaning: the sun, moon and five visible planets]". This treatise, in about 100 pages, is entirely devoted to the following phenomenon. Assume that an observer is located on the Northern hemisphere of the earth, north of the Tropic of Cancer. During the apparent daily rotation of the universe around the earth, all fixed stars culminate exactly in the south for this observer. However, the planets have a very small proper motion, which Ibn al-Haytham chooses not to ignore, and he proves that as a result of this proper motion, their point of culmination is in most cases not exactly in the south. A similar situation holds for the lower culmination of the planets, and Ibn al-Haytham concludes that (in localities north of the Arctic Circle) a planet can set in the East, and then rise again in the East (very close to the North point of the horizon), or it can set in the West and then rise in the West. Ibn al-Haytham believed that these phenomena, which were ignored by earlier astronomers, could explain the differences in their observations, and hence the differences between the parameters in the astronomical handbooks. It can easily be shown, even by ancient and medieval mathematical methods, that the phenomena which Ibn al-Haytham described (i.e. culminations east or west of the meridian, etc.) are so imperceptible that they cannot be observed by means of ancient and medieval instruments. Ibn al-Haytham was a theoretical mathematician, not a practical astronomer.

4 The translations

Rosenfeld translated, or assisted in the translation of a large number of Arabic mathematical and astronomical texts into Russian. Here follows a partial list (see the bibliography for references):

1. Thābit ibn Qurra (836-901, Baghdād), 35 mathematical and astronomical works (half of these works not edited before), 1 volume [Sabit *1984*].

2. Al-Fārābī (ca. 870-ca. 950, Middle East), mathematical works, 1 volume [Al-Farabi *1972*].

3. Al-Bīrūnī (973-1048, Iran, Afghanistan, Central Asia), selected mathematical and astronomical works, 4 volumes [Al-Bīrūnī *1973, 1975, 1976a, 1987*].

4. Al-Khayyām (1048-1131, Iran), mathematical and astronomical works, 1 volume (this is the famous poet-mathematician, who wrote his mathematics in Arabic and his poetry in Persian) [Khayyām *1961*].

5. Al-Kāshī (Iran, Samarkand, died ca.1430), mathematical works, 1 volume [Al-Kāshī *1956*]. The translated works include his *Key to Arithmetic*, the treatise on the circumference of the circle (in which he showed that $2\pi = 6.2831853071795865$ correct to 16 decimals.), and a later treatise on the method of al-Kāshī to determine $\sin 1°$ correct to 9 sexagesimal places.

Prof. Rosenfeld also translated a number of other treatises (by Ibn al-Haytham, Naṣīr al-Dīn al-Ṭūsī, al-Khwārizmī, the Arabic Diophantus etc., for which see the list of publications.) Many of these works have never been edited or translated into any language except Russian.

When translating Arabic mathematical and astronomical texts, one is confronted with the following problems. Arabic manuscripts were often copied by people who did not understand their contents, so there may be scribal mistakes. The following two examples are taken from the Qānūn al-Mas^{c}ūdī of al-Bīrūnī (973-1048) which Rosenfeld translated together with Bulgakov and Ahmedov in [Al-Bīrūnī *1973, 1976a*]. This is a handbook of astronomy of around 1000 pages, which al-Bīrūnī wrote for sultan Mas^{c}ūd of Ghazna in Afghanistan. The Arabic text was published in an unreliable edition in Hyderabad in India in [Al-Bīrūnī *1954*], but there are a few Arabic manuscripts of good quality. A small part on trigonometry was translated into German by Carl Schoy, a German orientalist, in 1927. Most of the work was never translated in any language except Russian. A table of contents has been published in English by E.S. Kennedy in 1971.

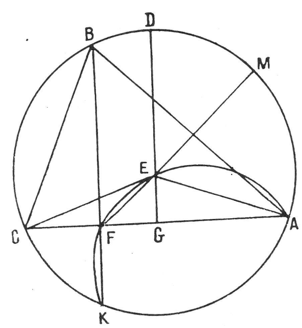

Figure 1

My first example is a computation which was misrepresented in the Hyderabad edition of the Arabic text [Al-Bīrūnī *1954*, 2:655] and which was corrected by Rosenfeld and Akhmedov in their Russian translation [Al-Bīrūnī *1976a*, 46]. The computation concerns the solar apogee. The Babylonian astronomers knew that the sun moves with a slightly variable speed against the background of the fixed stars. Ptolemy and most Arabic astronomers explained this irregularity by the assumption that the sun moves uniformly on a circle of which the centre E does not coincide with the earth F (Figure 1, taken from [Al-Bīrūnī *1976a*, 46]). Ptolemy put the radius of the circle equal to 60 units, and he determined the eccentricity EF and the position of E as seen from the Earth F (that is, the position of the solar apogee M on the ecliptic) on the basis of observations of the lengths of the seasons. The procedure was repeated by Arabic astronomers, who found more accurate results than Ptolemy. In Book VI of al-Qānūn al-Mas^{c}ūdī, al-Bīrūnī considers the ecliptic ABC; the points A, B and C are the positions of the sun at the beginning of spring, summer and fall respectively. These points are 90 degrees apart from each other, as seen from the earth F, so $\angle AFB = 90°$, etc. Al-Bīrūnī says that he had observed the length of the spring and summer, and because the sun moves uniformly on the circle with centre E and the length of the solar year was known, he could transform these observations into arcs. Hence he obtained arc $AB = 92°7'11''2'''$, arc BC= $91°47'31''30'''$ (the supposed accuracy of these observations does not concern us here).

Al-Bīrūnī assumes that the radius EM of the circle is unity, and he then performs a (needlessly difficult) computation of the magnitude of $EF : EM$ and the position of the solar apogee M in the ecliptic, based on his pet trigonometric theorems. I now present some steps in the computation which were misrepresented in the Hyderabad edition. The correct values, restored by Rosenfeld and Akhmedov, appear in brackets.

We have from a table of chords:
AB = 2;24,24,27,39 [1;26,24,27,39]
BC = 1;26,10,9,4
AC = 1;59,55,47 [1;59,55,47,44]
$AF = \frac{1}{2}(\frac{AB^2-BC^2}{AC} + AC) = 1;0,8,11,28$
$FC = AC - AF$ = 0;19,46,36,16 [0;59,47,36,16]
etc.

The Arabic astronomers did not use the modern Hindu-Arabic system of numbering. Instead they used the Arabic alphabetical numbers for integers and for sexagesimal fractions. Thus the numbers 3 and 8 are very similar to one another, 19 and 59 differ by only a dot, and so on. As a result, confusion can very easily occur. The editors of the Hyderabad edition clearly did not check the computations. Before translating the text, Rosenfeld and Akhmedov had to correct all the errors in the Arabic.

A similar situation arises in complicated geometrical figures in which many points occur. These points are indicated by letters of the Arabic alphabet, and because some of these letters are very similar (for example B and Z, B and I, I and N, N and D, etc.), mistakes are frequent. If there is more than one manuscript, there are usually different readings, and one can only decide between the readings by means of the mathematical context. An example of this situation occurs in another passage on the solar apogee in Book VI of al-Qānūn al-Mas^{c}ūdī. [Al-Bīrūnī *1976a*, 62] (Figure 2). Here al-Bīrūnī performs another computation of the solar apogee in a ludicrously complicated way. The argument does not concern us here; I only note that in the manuscripts and in the Hyderabad edition [Al-Bīrūnī *1954*] there exist many confusions between the letters B, D and I (a letter which occurs in the text but not in the figure), and between L and X (Arabic: sīn), which Rosenfeld and Akhmedov had to sort out. In the text in [Al-Bīrūnī *1954*, 679], we find the statements $DK \cdot KL + BK^2 = BI^2$ and $\frac{1}{2}(DK + KL) = DL$ (or BL in one of the manuscripts). These were restored by Rosenfeld and Akhmedov in [Al-Bīrūnī *1976a*, 62] to $DK \cdot KL + BK^2 = BD^2$ and $\frac{1}{2}(DK + KL) = DX$. There is no doubt that this is what Al-Bīrūnī intended, not only because he was a good mathematician, but also because the two statements were consequences of his favourite theorems on arcs in a circle.

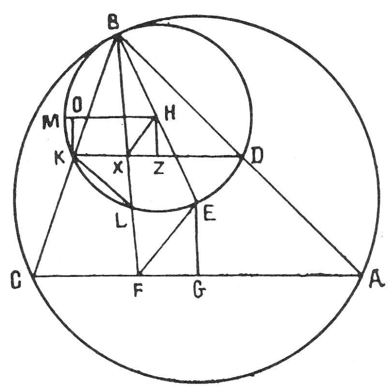

Figure 2

These two examples show that in editing and translating an Arabic mathematical or astronomical manuscript, one has to check the mathematics. While this is true for mathematical texts in general, there is a special difficulty for al-Bīrūnī, with the consequence that his works are vastly more difficult to translate than those of other Arabic mathematicians and astronomers. Al-Bīrūnī had mastered Arabic (which was not his native language) to a much greater extent than many of his colleague-scientists. He was a poet, who liked to play with words, as is evident from the titles of some of his works. In the following three examples he used rhyme:

1. Ifrād al-Maqāl

fī amr aẓ-ẓilāl

"Excess of treatise in the matter of shadows" (translated in [al-Bīrūnī *1987*], English translation and commentary by Prof. E.S. Kennedy in [Al-Bīrūnī *1976*])

2. Kitāb al-tafhīm

li-awā'il ṣinā^{c}at al-tanjīm

"Book of the instruction in the elements of the art of astrology"

(translated in [al-Bīrūnī *1975*], English translation in [al-Bīrūnī *1934*])

3. Maqāla fī 't-taḥlīl

wa-t-taqṭī^c li-t-ta^{c}dīl

"Treatise on the analysis and the subdivision of the equation"

(Russian translation in [al-Bīrūnī *1987*]; no English translation exists but the work is analyzed in [Kennedy and Muruwwa *1958*]).

The interpretation of these titles is not trivial; for example the "equation" in the third title is the solar equation, that is the difference between the mean and true position of the sun, and the treatise has no relation to algebra.

As has been mentioned above, the translation by Bulgakov and Rosenfeld [*1973*] and Rosenfeld and Akhmedov [*1976a*] is the only modern translation of the entire Qānūn al-Mas^{c}ūdī, which consists of 13 'Books'. Only the relatively easy Book III on trigonometry was translated into German by Carl Schoy in 1927. As a further example of the linguistic difficulties of translating al-Bīrūnī, it is interesting to compare the beginning of Chapter 5 of Book III of the Qānūn al Mas^{c}ūdī in the two translations.

In Schoy's translation the chapter begins as follows [*1927*, 31]:

> "On the ratio between the diameter and the circumference of the circle.
>
> This is easy to express in a (small) number. And verily, only the one of those who wanted to (establish it), really changed (?) anything in this matter, but these changes only concerned the procedure and the improvement (of it?), as also the division of the circumference, in which the people of this science differ from one another. The circumferences have 360 parts . . . And the circumference of a circle has a ratio to the diameter, and one should make the ratio between the number of the circumference and that of the diameter, and this gives the approximate knowledge of that ratio."

(my translation from the German; words in parentheses and question marks occur in Schoy's translation).

Bulgakov-Rosenfeld translate as follows: [Al-Bīrūnī *1973*, 217, cf. *1954*, 303]

> "On the ratio between the diameter and the circumference of the circle.
>
> Although unity is related to the objects of counting, if unity is considered in the [totality of essences] possessing matter, then it does not appear truly according to its own essence. But it is [taken] as a convention and in the way of common agreement, just like the parts into which the circumference of the circle is divided; of which the people of this art have agreed that they are 360 ... The circumference of the circle has some ratio to its diameter, and the number of the circumference has also a ratio to the number of the diameter, although this (ratio) is irrational."

(my translation from the Russian).

It is hard to see that the two translations are translations of the same text. The Bulgakov-Rosenfeld translation is the correct one. Schoy did not realize that the beginning of this text is really a philosophical discussion of the number one. The example shows how easily one can be deceived by the difficult language of al-Bīrūnī.

The Russian translation of the Qānūn al-Mas^{c}ūdī contains thousands of footnotes, including many corrections to the Arabic text in the Hyderabad edition [*1954*]. These corrections are indispensable for any future editor or translator of the text. The Russian translation is interesting for another reason, too: as Professor Rosenfeld once remarked to me, it is the first work on astrology printed in the Soviet Union after the revolution.

Rosenfeld and his collaborators have done an immense amount of labour to make medieval Arabic texts available in Russian translations. This constitutes a great service to the field.

5 Interpretations

Rosenfeld has always been interested in relationships between medieval Arabic mathematical texts and modern mathematical ideas. A good example is the *Treatise on the sections of the cylinder and on their surface area* by

Thābit ibn Qurra (836-901). This is no. 18 of the the 35 texts published in [Sabit 1984]; the text was translated by L. Karpova and B.A. Rosenfeld. Rosenfeld indicated a relationship between this text and a type of geometrical transformation called "equi-affine". An affine transformation of the (real) plane is a transformation which maps straight lines on straight lines. An affine transformaton is always the product of a bijective linear transformation and a translation. In coordinates, the most general affine transformation is given by $\phi(x, y) = (x', y')$ with $x' = ax + by + c, y' = dx + ey + f$ for constant a, b, c, d, e, f such that the determinant $(ae - bd)$ is non-zero. If a figure undergoes an affine transformation, its area is multiplied by a factor $|ae - bd|$. The transformation is called equi-affine if $ae - bd = \pm 1$. An equi-affine transformation preserves the area of a figure, while it may distort the shape.

I now turn to theorem 14 in part 2 of Thābit's *Treatise on the sections of the cylinder and their surface area* [Sabit *1984*, 206]. Thābit first shows that the areas of a circle with diameter a and an ellipse with major axis a and minor axis b are in the ratio $a : b$. The idea is as follows in modernized notation. Thābit inscribes a regular polygon $A_n = P_1 \dots P_{2^n}$ with 2^n angles in the circle and he consider the related regular polygon $B_n = Q_1 \dots Q_{2^n}$ in the ellipse, see Figure 3 (drawn for $n = 3$). Clearly $A_n : B_n = a : b$. He then shows that for every area ϵ, one can find n such that the difference between either A_n and the circle or B_n and the ellipse is less than ϵ. Thābit then proves in a rigorous way that the areas of the circle and the ellipse are also in the ratio $a : b$.

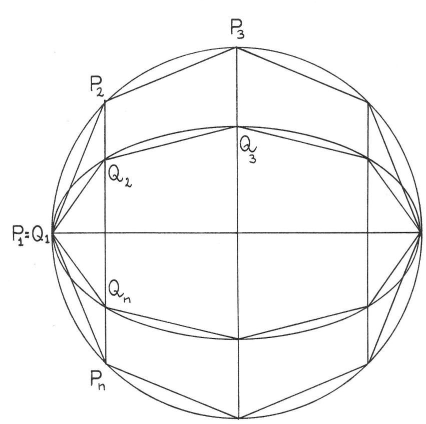

Figure 3

Somehow, Thābit seems to have preferred an equality over a ratio $a : b$. He therefore introduced a new circle whose diameter is the mean proportional $\sqrt{ab}$ between the major and the minor axes of the ellipse. The ratio between the old circle with diameter a and this new circle is $a : b$, so the whole ellipse is equal to the new circle. Thābit then shows that a segment of the ellipse cut off by a straight line perpendicular to an axis is also equal to a certain segment of this new circle (theorems 15, 16), and he then goes on to prove in theorem 17 that an arbitrary segment of the ellipse is equal to another segment of the circle, which he defines in a complicated way, as we will presently see.

In order to understand Thābit's theorem 17, it will be useful to introduce Rosenfeld's equi-affine transformation. Consider the affine transformation ϕ which maps the centre of the ellipse to the centre of the new circle and the major and minor axes of the ellipse on two perpendicular diameters of the new circle. This transformation is clearly equi-affine (as can be seen from the fact that the areas of the ellipse and the circle are equal). Using this transformation, Thābit's theorem 17 (and also his earlier theorems) can be expressed as follows: Every segment of the ellipse is equal to its image under the equi-affine transformation ϕ. Here we need one line to express a theorem for which Thābit needs the following complicated and vague expression [Sabit *1984*, 210, cf. Rosenfeld *1988*, 131, Karpova and Rosenfeld *1974*, 69]:

> "Every segment of an ellipse is equal to a segment of a circle of the same area (as the ellipse) such that if we drop two perpendiculars from the endpoints of its base to the diameter of the circle, then each of them is to the diameter as the corresponding perpendicular, dropped from an endpoint of the base of the segment of the ellipse to one of its axes, is to the other axis; provided that the segments are both smaller or both greater [than half the ellipse or the circle], the position of the center of the ellipse relative to the perpendiculars is the same as the position of the centre of the circle relative to its perpendiculars, and the position of the feet of the perpendiculars of the ellipse on its axis is the same as the position of the feet of the perpendiculars of the circle on its diameter."

This is not very clear, and Thābit continues with an even longer, but somewhat clearer “setting-out”[1] with eight figures (Figure 4, copied from [Sabit *1984*, 211])

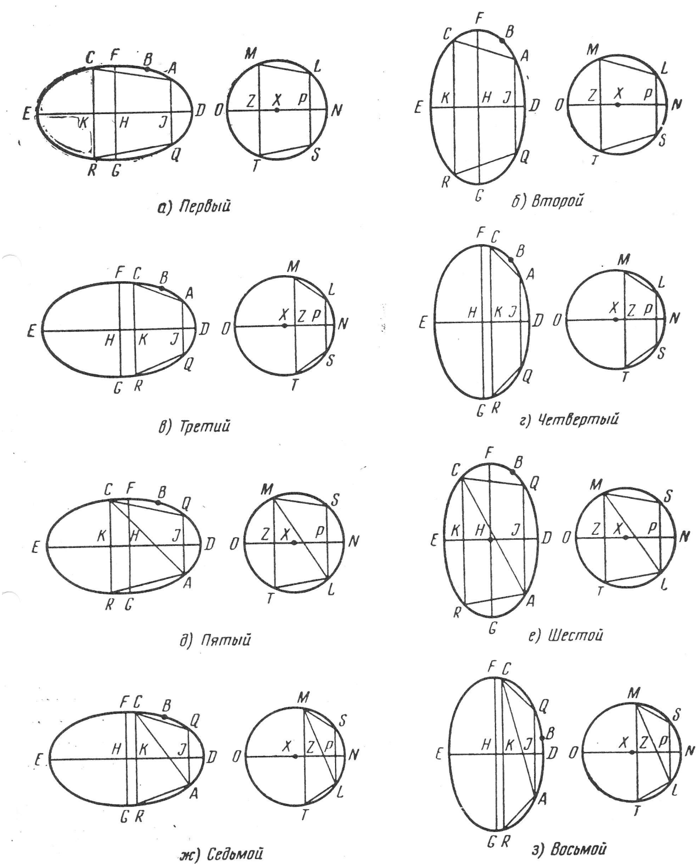

Figure 4

> “Let ABC be a segment of ellipse $ABCD$, with base AC. Let it first be less than half of the ellipse, of which the centre is H, and ED the major axis, GF the minor axis in the first, third, fifth and seventh figure, and let ED be the minor axis and GF the major axis in the second, fourth, sixth and eighth figure. We draw from points A and C perpendiculars AJ and CK onto axis ED. Let circle LMN with center X be equal to ellipse $ABCD$ and let segment LM be less than half of it. We draw from points

[1] Arabic: *mithāl*

> L and M perpendiculars LP and MZ to the diameter KO of the circle. Let the ratio of AJ to FG be equal to the ratio of [LP to NO, and let the ratio of CK to FG be equal to the ratio of][2] MZ to NO, and let both perpendiculars AJ and CK fall on one side of axis ED and both perpendiculars LP and MZ also fall on one side of diameter NO, as in the first four figures, or let both perpendiculars AJ and CK fall on different sides of axis ED and both perpendiculars LP and MZ also fall on different sides of diameter ON, as in the last four figures. Let H be between perpendiculars AJ and CK and X between perpendiculars LP and MZ as in the first, second, fifth and sixth figure, or let H be outside AJ and CK and X on one side of LP and MZ, as in the third, fourth, seventh and eighth figure. Then I say that the segments ABC and LM are equal."

The details of the proof can be found in [Sabit *1984*, 211-212]. The basic idea is that the segment can be obtained by taking half the ellipse and cutting off two segments by straight lines perpendicular to the axis (which segments have been compared to a segment of the circle in theorems 15 and 16) and one quadrilateral (which can be shown to be equal to the corresponding quadrilateral of the circle by theorems from Euclid's *Elements*).

The fact that theorem 17 can be rendered simply in terms of an equi-affine transformation leads to the question, to what extent was the idea of an equi-affine transformation present in Thābit's mind? Of course he did not have the modern ideas of coordinates and of a transformation as a mapping of points. On the other hand, he may have had, at least intuitively, the idea of a transformation as a special relation between entire figures. As Professor Rosenfeld pointed out to me, Thābit's intuition may have been related to Euler's idea of affinity as a relationship between figures (curves). Euler's notion of affinity reflects the use of the terms "figures of the same kind" and "figures of the same species" by Newton and Clairaut [Rosenfeld *1988*, 143-144]. Of course, Thābit's werk was not transmitted to medieval and renaissance Europe, so Newton, Clairaut and Euler could not have been influenced by it.

Rosenfeld also discussed other types of geometrical transformations (general affine transformations, inversions) in relation to ancient and medieval

[2] Passage restored by Karpova and Rosenfeld.

texts, such as the *Quadrature of the Parabola* by Thābit's grandson Ibrāhīm ibn Sinān. Details may be found in [Rosenfeld *1988*, Chapter 3] and in the articles in the bibliography.

6 Prehistory of Non-Euclidean geometry

One of Rosenfeld's favourite topics in the history of mathematics is the discovery of non-Euclidean geometry. This discovery completely changed the general idea of what mathematics is about. Before non-Euclidean geometry was discovered, it was believed that there was only one kind of mathematics, which is closely related to "reality". After the discovery, mathematicians gradually realized that different mathematical theories are possible, for example Euclidean and non-Euclidean geometry. These theories contradict each other, and we cannot tell which one is "real". Mathematics now becomes a science of structures which are invented by the human mind. Rosenfeld observed in the foreword to [*1988*] that we are now entering a new period, in which the computer plays a fundamental role in the development of mathematics. He suggests calling the period between 1870 and the advent of the computer the "era of non-Euclidean mathematics."

The discovery of non-Euclidean geometry has a long and complicated prehistory in antiquity and the middle ages. In Euclidean geometry, there exists for each straight line ℓ and each point P not on ℓ precisely one straight line ℓ' in the plane through P and ℓ such that ℓ' contains P and does not meet ℓ. In order to prove this fact, Euclid assumed in his *Elements* the so-called parallel postulate, to the effect that if two straight lines are cut by a straight transversal in such a way that the two interior angles on the same side of the transversal are less than two right angles, the two lines meet on that side (of the transversal) on which the angles are less than two right angles. For 2000 years mathematicians in various cultures (Greek, Islamic, Hebrew, Latin, etc.) tried to prove this postulate or to replace it by a postulate which they believed to be simpler. In the early 19th century, Lobachevsky, Bolyai and Gauss discovered that it is possible to base a geometry on the negation of the parallel postulate. In this geometry (often called Lobachevskian geometry) there are for a given point P and a given straight line ℓ, infinitely many straight lines ℓ' in the plane through P and ℓ such that ℓ' contains P and does not meet ℓ.

The parallel postulate was investigated by many Arabic geometers, and the complicated history of their "proofs" and alternatives to the postulate has been described by Rosenfeld in [*1983, 1988*]. One Arabic "proof" of the postulate was transmitted to 17th-century Europe, because it appeared in an Arabic version of Euclid's Elements, attributed to "Khwāja Naṣīr al-Dīn al-Ṭūsī" (1201-1274), and printed in Arabic in 1594 in Rome [Euclid, *1594*]. Thanks to the researches of various historians, including Rosenfeld, we now know that the author cannot be al-Ṭūsī, and I will therefore call him pseudo-Ṭūsī. One of the reasons that the author cannot be Naṣīr al-Dīn al-Ṭūsī is the fact that the treatise was written in 1294, twenty years after Naṣīr al-Dīn al-Ṭūsī's death. The proof of the parallel postulate in [Euclid *1594*] was quite influential in Europe. It was translated into Latin by Pococke and published by the English mathematician John Wallis in 1697, who criticized it and then went on to give his own "proof". The crucial part of the proof of pseudo-Ṭūsī concerns the existence of a rectangle. More precisely (Figure 5), if two equal perpendiculars AC and BD are erected on a straight line AB, pseudo-Ṭūsī states (and thinks he can prove) that the angles at C and D are right angles.If this is granted, it is not difficult to prove $CD{=}AB$, and then the parallel postulate can be proved on the basis of the axiom of Archimedes-Eudoxos, see [Bonola *1955*, 11].

Rosenfeld's analysis (rendered in [*1988*, 80-85] and in other articles) shows that pseudo-Ṭūsī's "proof" is primitive, but related to a much more subtle proof by the real Naṣīr al-Dīn al-Ṭūsī in his edition of Euclid's *Elements*. This is an instance of the general rule that the transmission of Arabic science to medieval and renaissance Europe was very incomplete.

In this proof, the real al-Ṭūsī also begins with a line AB with two equal perpendiculars AC and BD (Figure 5). He draws line CD and proves that angles C and D must be equal in the following way. Draw the diagonals AD and BC. Then triangles CAB and DBA are congruent ($AB = BA, AC = BD, \angle A = \angle B$), so $AD{=}BC$. Thus triangles CDA and DCB are congruent ($DC = CD, CA = DB, DA = CB$), hence $\angle C = \angle D$. (The same proof was given by the the 12th-century mathematician-poet al-Khayyām).

The real al-Ṭūsī then continues as follows: Angles C and D can be right, acute or obtuse. Assume that they are acute. Now it is possible to drop a perpendicular BE onto CD; then because $\angle D$ is acute, point E falls between D and C. In a triangle, the greater side subtends the greater angle, and the exterior angle is greater than any non-adjacent interior angle (this is proved by Euclid without using the parallel postulate, and these theorems are valid in Lobachevskian geometry.)

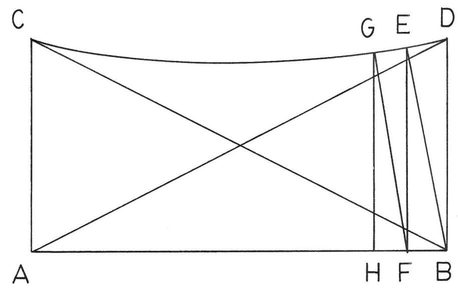

Figure 5

Therefore BE is less than BD, and angle ABE is acute. We now drop a perpendicular EF onto AB. Then point F falls between B and A, angle FEC is acute, and EF is less than EB. In this way we get a series of continuously decreasing perpendiculars DB, BE, EF, FG, GH and so on. Therefore DC converges toward BA. Since $\angle C$ is acute, we can prove in the same way that CD converges toward AB. Therefore two straight lines converge in the same direction into which they diverge, which is a contradiction in al-Ṭūsī's opinion.

In the same way, al-Ṭūsī derives a "contradiction" if the angles C and D are obtuse. He concludes that the angles are right.

This proof is subtle, and at first sight it is difficult to see what is wrong with it. The problem is, as Rosenfeld remarks, that al-Ṭūsī has not proved that the series of points B, F, H ... converges toward A. In Lobachevskian geometry the series of points $B, F, H \ldots$ converges toward the midpoint of AB [Rosenfeld *1988*, 76; Rosenfeld-Yuschkevitch *1983*, 85]. The real al-Ṭūsī may well be excused for not having seen this gap in his "proof".

The Arabic text printed in Rome in [Euclid *1594*] was evidently written by someone who had adapted the proof by the real al-Ṭūsī in an incompetent way. Rosenfeld suggests that the author of the proof in [Euclid *1594*] may have been a student of al-Ṭūsī, with less insight in mathematics than al-Ṭūsī himself. He suggests as a possible candidate al-Ṭūsī's son, who was called Ṣadr al-Dīn ibn khwāja Naṣīr al-Dīn.

Of course this is only a very tiny bit of the history of the parallel postulate in the Islamic East. More information on the history of the parallel postulate in Islamic civilization (and its continuation in Europe, leading to the discovery of non-Euclidean geometry in the early 19th century) can be found in Rosenfeld's book [*1988*] on the history of non-Euclidean geometry.

This paper is not an exhaustive survey of Rosenfeld's work on the history of mathematics in Islamic civilization. I have rather attempted to give an inside view on some of his contributions, and show why his work deserves admiration.

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