THE LOST GEOMETRICAL PARTS OF THE *ISTIKMĀL* OF YŪSUF AL-MU'TAMAN IBN HŪD (11th CENTURY) IN THE REDACTION OF IBN SARTĀQ (14th CENTURY): An Analytical Table of Contents

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1. *Introduction*

In 1991 I published in this journal a table of contents of the extant geometrical parts of the *Kitāb al-Istikmāl* (Book of Perfection) of al-Mu'taman ibn Hūd, who died in 1085.[1] The author was the king of the Islamic kingdom of Saragossa in Spain from 1081 until his death, and at the same time an important mathematician in the medieval Andalusian tradition. The *Istikmāl* was believed to be lost until the discovery of four fragments in anonymous Arabic manuscripts around 1984. The work is basically an intelligent compilation from many Arabic sources and Arabic translations of Greek sources. Al-Mu'taman simplified and revised these sources, classified the material, and added some discoveries of his own. The *Istikmāl* was to consist of two "genera", but it seems that al-Mu'taman only finished the first "genus" before his accession to the throne in 1081, and never had time to write the second "genus". Even so, the first genus of the *Istikmāl* is one of the longest medieval mathematical works that is extant today.

The text of the first genus of the *Istikmāl* has come down to us in incomplete form. Figure 1[2] displays the manuscript sources and the subdivision of the first genus into "species", "species of species", and "sections" (the white and shaded rectangles), according to al-Mu'taman's system of classification. Al-Mu'taman's introduction to the first species on arithmetic is missing, but the rest of the first species is extant in two manuscripts **C** = Cairo, Dār al-Kutub, Muṣṭafā Fāḍil Riyāḍa 40m, 1b-37b, and **D** = Damascus, Ẓāhiriyya, ʿāmm 5648. A detailed survey of the first species has been published by Djebbar[3]. Large parts of the species 2-5 on geometry are extant in manuscript **L** = Leiden, University Library 123/1, and in manuscript **K** = Copenhagen, Royal Library, Or. 82, which also includes part of species 1. Manuscript **L** is a fragment, and many leaves of manuscript **K** are missing

[1] Jan P. Hogendijk, "The Geometrical Parts of the *Istikmāl* of Yūsuf al-Mu'taman ibn Hūd (11th Century): An Analytical Table of Contents", *Archives internationales d'histoire des sciences*, *41* (1991), 207-281.

[2] The figure is taken from *op. cit.* (footnote 1), 212.

[3] A. Djebbar, "Les livres arithmétiques des *Éléments* d'Euclide dans le traité d'al-Mu'taman du XIe siècle", *Llull*, *22* (1999), 589-653.

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and the remainder is bound in incorrect order. The extant leaves of **K** can be divided into twelve continuous groups which I have labeled **a**, **b**, ... **l**, see Figure 1. Only two of the gaps can be filled by the manuscript **L**, and the remaining nine gaps amount to an estimated 25 percent of the text.

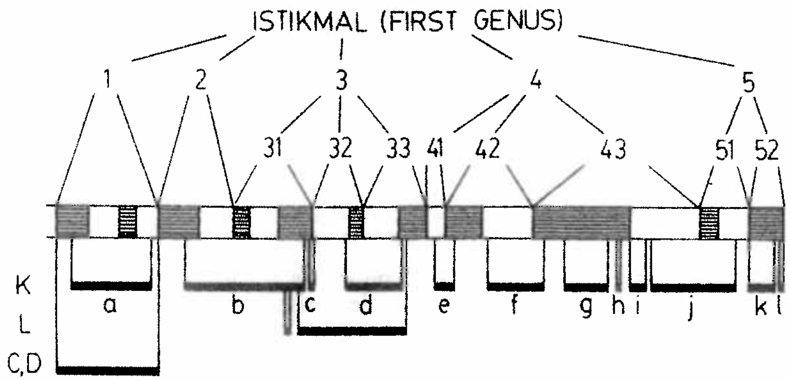

Fig. 1

The nine gaps can now to some extent be filled thanks to the discovery of a complete text of another work entitled *al-Ikmāl al-Riyāḍī* (The Mathematical Achievement). Two manuscripts of this work were identified by Mr. Ihsan Fazioğlu in the 1990s. The work is a revision of the *Istikmāl* by Ibn Sartāq al-Marāghī (ca. 1315).[4] The *Ikmāl* (Achievement) is not the same work as the *Istikmāl* (Perfection), because Ibn Sartāq changed the wording and some of the proofs. As far as we can judge from the extant parts of the *Istikmāl*, Ibn Sartāq used the same numbering for the chapters and, with very few exceptions, also for the individual propositions.

The purpose of the present paper is to fill the gaps in the table of contents of the *Istikmāl* by means of the *Ikmāl*. As in my earlier paper, I will only discuss the geometrical part of the *Istikmāl*, that is to say, the species 2-5 of the first genus of the work[5]. I use the same conventions as in my earlier paper, which will be abbreviated as GP ("Geometrical Parts"). Thus a notation such as 312.18 refers to proposition 18 of section 2 of species 1 (or class 1) of species 3 of the first genus of the *Istikmāl* or *Ikmāl*[6]. For the *Ikmāl* I have used the manuscript Cairo, University Library

[4] See A. Djebbar, "La rédaction de *L'istikmāl* d'al-Mu'taman (XIe s.) par Ibn Sartāq, un mathématicien des XIIIe-XIVe siècles", *Historia Mathematica*, *24* (1997), 185-192.

[5] The *Ikmāl* includes a version of al-Mu'taman's general introduction to the *Istikmāl* and a table of contents of the projected "second genus". See the paper of Djebbar in footnote 4.

[6] Instead of al-Mu'taman's "species of species" (*nawᶜ al-nawᶜ*), Ibn Sartāq uses the slightly different term "class of species" (*ṣinf al-nawᶜ*).

23209, which will be indicated by the symbol Q. The following abbreviations will be used for fundamental Greek texts, which were available in medieval Arabic translation:

El.	Euclid, *Elements*,
Th.	Theodosius, *Spherics*,
Men.	Menelaus, *Spherics*,
Conics	Apollonius, *Conics*,
SC	Archimedes, *On the Sphere and Cylinder.*

References to editions and translations of these works can be found in the bibliography. I use the numbering of propositions in the Greek editions, except in the case of Menelaus' *Spherics*, where the numbering is as in Krause's edition of the version of Abū Naṣr ibn ᶜIrāq. As in GP, I often avoid long and complicated descriptions of propositions by a simple reference to the source from which the proposition (theorem or construction) was taken. Such a reference means that the theorem is the same as in the source, or that the same thing is constructed, but it does not always imply that the constructions or the proofs are the same in detail.

The new information on the gaps in species 2-5 does not lead to a new view of the *Istikmāl* as a whole, and most of my conjectures about these gaps in my earlier paper GP turn out to be correct[7]. One or two new sources of the *Istikmāl* can also be traced. Propositions 431.4-5 below contain material which is either al-Mu'taman's own work, or based on an unidentified source on cylindrical sections. Proposition 512.16 below is based on a lost Arabic text on regular polyhedra, related to Book XIV of the *Revision of the Elements* by Muḥyī al-Dīn al-Maghribī (died 1283).

Ibn Sartāq's own contributions in the *Ikmāl* are extensive and interesting, but fall outside the scope of this paper. In Appendix 1 of this paper I present the Arabic text with English translation of a brief proposition in the *Istikmāl* and the corresponding proposition in the *Ikmāl*. The reader can see that Ibn Sartāq changed the wording quite drastically but made no essential changes in the mathematical proof. The proposition in question is the so-called "theorem of Ceva", which has been attributed to the 17^{th} century Italian mathematician Giovanni Ceva[8] until recently.

GP 224. Gap between manuscripts C-D and fragment b of manuscript K

This gap concerns the beginning of species 2 until 21.26.

The title of the second species is, according to the *Ikmāl*: On lines, plane figures and angles, not in combination (*fī l-khuṭūṭ wa l-suṭūḥ wa l-zawāyā min ghayr iḍāfatⁱⁿ*).

[7] My erroneous conjectures in GP are corrected in the present paper.

[8] On Giovanni Ceva see the article by Herbert Oettel in C.G. Gillispie, *Dictionary of Scientific Biography* (New York, 1970-1986), III, 181-183.

The species begins with unnumbered definitions of the following concepts (Q 25b:22-26a:12): a. boundary; b. figure; c. point; d. line; e. surface; f. solid; g. straight line; h. round line; i. plane surface; j. round surface; k. parallel lines; l. meeting lines; m. angle; n. right, obtuse, acute angle; o. circle; p. radius of a circle; q. diameter, chord; r. segment *(qiṭ'a)*; s. arc; t. angle in a segment of a circle; u. sector of a circle *(qaṭṭā')*; v. similar arcs of a circle; w. triangle: equilateral, isosceles, scalene, right-angled, obtuse-angled, acute-angled; x. quadrilaterals: square, rectangle, rhombus, rhomboid, trapezium; y. polygons, such as the pentagon.

Postulates *(muṣādarāt)* (Q 26a:12-16):

a. to draw a circle with any centre and distance; b. to draw a straight line between any two points; c. to extend a straight line to where we wish; d. two straight lines cannot contain an area; e. if part of a straight line concides with part of another straight line, it does not diverge from it in another part; f. a straight line cannot be joined to two straight lines in such a way that it is the rectilinear extension of both; g. all right angles are equal; h. things which coincide are equal; i. "it is impossible that the greater [Q: lesser] of two quantities of the same kind exceeds the lesser [Q: greater] by other than a finite number of times a quantity of the same kind; j. the common part of two lines is a point; k. the common part of two planes is a line; l. the common part of two solids is a plane.

The title of Section 21 is, according to the *Ikmāl*: On figures with straight sides *(fī l-ashkāl dhawāt al-aḍlā' al-mustaqīma)*.

Propositions:

21.1 (Q 26a:20-26b:14). a. If there are two pairs of intersecting straight lines, and one angle in the first pair is equal to one angle in the second pair, then all angles in the first pair are equal to all angles in the second pair; b. El. I:4, c. El. I:15.

21.2 (Q 26b:14-27a:8). El. I:5-6.

21.3 (Q 27a:8-19). El. I:8.

21.4 (Q 27a:19-27b:10). El. I:1,10,9,11,12.

21.5 (Q 27b:10-14). El. I:16.

21.6 (Q 27b:15-25). El. I:26.

21.7 (Q 27b:25-28a:4). El. I:18-19.

21.8 (Q 28a:4-12). El. I:20,21.

21.9 (Q 28a:12-22). El. I:24,25.

21.10 (Q 28a:22-28b:3). El. I:14.

21.11 (Q 28b:3-7). El. I:2.

21.12 (Q 28b:7-17). El. 1:23. Cor: El. I:22.

21.13 (Q 28b:17-29a:2). El. I:27. Corollary: El. 1:31.

21.14 (Q 29a:2-24). A "proof" of the parallel postulate of Euclid. The gist of this proof is as follows: (Figure 2) Consider a line *HGT* intersecting a second line *DGBA* at point G. Through point *B* draw line *EBZ* in such a way that points *Z* and *T* are on the same side of line *AD* and $\angle ZBG + \angle BGT < 180°$. Theorem: Lines *BZ* and *GT* intersect on the side of points Z and *T*.

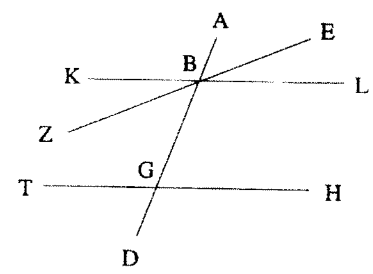

Fig. 2

"Proof": We draw through point *B* a straight line *KBL* such that point *K* is on the same side of line *AD* as points *Z*, *T* and such that $\angle KBG = \angle TGD$. Assume that line *KB* moves from *B* to *G* in such a way that the angle *KBG* remains constant. (Thus in any position *K'B'G* of line *KB*, we have $\angle K'B'G = \angle KBG = \angle TGD$.) At the beginning of this motion, *KB* intersects *ZB* at *B*. Thereafter the point of intersection of *ZB* and *K'B'* moves on *BZ* in the direction of *Z*.

The final position of the motion is *K"B"* = *GT*. Suppose that in this final position, *K"B"* and *BZ* do not meet. In the initial position, *KB* and *BZ* do meet at point *B*. Thus we have two straight lines which are extended indefinitely on one side, and these two lines meet in the beginning of the motion and do not meet in the end. According to the text, this can only happen if the two straight lines coincide somewhere in the course of their motion, at least in part[9]. This is impossible, so *GT* and *BZ* must have a point of intersection, q.e.d. According to a remark at the end of the proof, the proposition was "edited" *(muḥarrar)* by Ibn Sartāq. In the absence of further evidence we can assume that the basic idea of the proof was due to al-Mu'taman.

21.15 (Q 29a:24-29b:4). Corollary (introduced by 'and here it has become clear'): El. I:29,30. Correction to GP 225: 21.15 has no relation to El. II:6 (the "21.15" in the margin of manuscript **K**, on which I based this conjecture, was my misreading of the poorly legible "21.25").

21.16 (Q 29b:4-8). El. I:32.

21.17 (Q:29b:8-30a:5). A conglomerate of El. I:33-40.

21.18 (Q 30a:5-17). El. I:42, and El. I:43, two propositions which have no direct relationship to one another.

21.19 (Q 30a:17-30b:1). To construct a square with a given side (El. I:46). If points *E*, *Z*, *H*, *T* are chosen on the sides *AG*, *GD*, *DB*, *BA* of a square in such a way that *AE* = *GZ* = *DH* = *BT*, then *EZHT* is also a square (not in the *Elements*, but used in 21.22).

[9] This assumption does not follow from the other axioms and postulates in the *Elements*, so here we have the flaw in the "proof".

21.20 (Q 30b:1-7). El. II:1. Corollaries: El. II:2,3.

21.21 (Q 30b:7-10). El. II:4. The text says that the proof is "as before, in numbers" (treated in Species 1). Corollaries: a. the square of a segment is four times the square of its half; b. El. II:7.

21.22 (Q 30b:10-25). The theorem of Pythagoras El. I:47, and its converse El. I:48. The proof by 21.19, second part, is unlike Euclid's proof in the *Elements*.

21.23 (Q 30b:25-31a:5). El. II:12.

21.24 (Q 31a:6-11). El. II:13.

21.25 (Q 31a:11-15). El. 11:5,6. The text says that the proofs are 'as in numbers'.

21.26 (Q 31a:16-20). Construction of a square equal to a given triangle (El. II:14). The second half of this construction is the beginning of fragment **b** in manuscript **K** of the *Istikmāl*.

GP 253. Gap between manuscript L and manuscript K, fragment e.

This gap affects the end of section 332 and the beginning of section 41. The text of section 332 breaks off in Ms. **L** in the middle of a sentence in the proof of 332.3 (Q 87b:21-88a:7).

332.4 (Q 88a:7-19). If a line is tangent to two circles which are known in position and magnitude, the line itself is also known. This is Ibn al-Haytham, *On Known Things* II:24-25.[10]

Propositions 332.5-332.7 are theorems on tangent circles. Ibn al-Haytham proves these theorems for the case where the circles are internally tangent. The *Ikmāl* includes the case of external tangency.

332.5 (Q 88a:19-88b:5). Ibn al-Haytham, *On Known Things*, I:18.

332.6 (Q 88b:5-89a:3). Ibn al-Haytham, *On Known Things*, I:19. Corollary: Ibn al-Haytham, *On Known Things*, I:20.

332.7 (Q 89a:3-14). Ibn al-Haytham, *On Known Things*, I:21.

Species 4.

The title is in the *Ikmāl* (Q 89a:20-22): "The fourth species of the first genus of the two genera of the mathematical sciences, on the principles (*mabādi'*) of the species of solids <and> of the sections occurring in them. Three classes (*aṣnāf*):[11] the first on rectilinear (figures), the second on spheres and the sections occurring in them, the third on cylinders and cones and their sections."

[10] For Ibn al-Haytham, *On Known Things*, see the summary in Sédillot, Louis-Amélie "Notice du traité des connues géométriques de Hassan ben Haithem", *Journal Asiatique*, 2ème Série *13* (1834), 435-458, reprinted in F. Sezgin (ed.), *Islamic Mathematics and Astronomy*, vol. 57, Frankfurt 1998, and the Arabic edition with French translation in Rashed, Roshdi, *Les Mathématiques infinitésimales*, vol. 4. (London: Al-Furqān Foundation, 2002), 444-582.

[11] Al-Mu'taman used the term "species", see footnote 6 above.

In the *Ikmāl*, "Class" 41 begins with definitions of the following concepts (Q 89a:22-89b:3): a. solid figure; b. solid angle; c. line perpendicular to a plane; d. plane perpendicular to another plane; e. plane inclined to another plane; f. parallel planes; g. cube; h. prism (*manshūr*).

Propositions:

41.1 (Q 89b:4-6). El. XI:1.

41.2 (Q 89b:6-9). El. XI:2.

41.3 (Q 89b:9-12). El. XI:3.

41.4 (Q 89b:12-23). El. XI:4.

41.5 (Q 89b:23-90a:3). El. XI:5.

41.6 (Q 90a:3-7). El. XI:6.

41.7 (Q 90a:8-10). El. XI:7.

41.8 (Q 90a:11-15). El. XI:8.

41.9 (Q 90a:15-19). El. XI:9.

41.10 (Q 90a:19-23). El. XI:10.

41.11 (Q 90a:24-90b:3). El. XI:11.

41.12 (Q 90b:3-5). El. XI:12. The text of this proposition is missing in the manuscripts of the *Istikmāl*, but the figure is the beginning of fragment **e** of manuscript **K**.

GP 256. Gap in manuscript K between fragments e and f

This gap affects the end of section 421 and the beginning of section 422.

The text of 421.4 is extant in fragment **e** of manuscript **K**, but the figure is missing.

421.5 (Q 94b:16-95a:8). Th. II:11,12.

421.6 (Q 95a:8-95b:2). Th. II:15,14.

421.7 (Q 95b:2-10). Th. II:10.

421.8 (Q 95b:11-96a:13). Th. II:13.

421.9 (Q 96a:13-96b:2). Th. II:16.

421.10 (Q 96b:2-24). Th. II:17.

421.11 (Q 96b:24-97a:17). Th. II:19,20.

421.12 (Q 97a:17-97b:12). Th. II:21.

421.13 (Q 97b:12-98b:3). Th. II:22.

421.14 (Q 98b:4-22). Th. III:4.

I have not taken account of small variations such as the following. In Th. II:11,12 it is proved that two arcs DE, AB are equal if and only if two segments HB, TE are equal. The *Ikmāl* also proves that $AB > DE$ is equivalent to $HB > TE$.

Correction to GP 256: The marginal reference in ms. **K** to 421.19 (?), which supposedly included Th. III:3, is incorrect. Correction to GP 254: the number of propositions in section 421 is 14, not "at least 19".

Section 422. In the *Ikmāl,* the title of section 422 is (Q 99a:1-2): "On the arcs of great circles in the sphere and their sines, according to ratio and composition (*taʿlīf*) ."

Then the following concepts are defined (Q 99a:2-9): a. spherical triangle; b. spherical angle; c. the magnitude of a spherical angle; d. the sine of an arc; e. the complement of an arc.

422.1 (Q 99a:10-17). Men. I:5: in any spherical triangle, the sum of any two sides is greater than the remaining side.

422.2 (Q 99a:17-99b:23). Congruence of two spherical triangles, Men. I:4,8. Corollary: construction of a great circle passing through a given point and making a given spherical angle with a given great circle through the point.

422.3. This proposition is extant in manuscript **K**, fragment **f**, except the first few words.

4. GP 261. Gap in manuscript K between fragments f, g.

This gap affects section 413. The text in fragment **f** breaks off in the middle of the proof of 413.3

431.4 (Q 119a:7-124a:21). Fundamental property of the ellipse, as a plane section of the cone (*Conics* I:13, 21) and as a plane section of a cylinder. This is followed by a long discussion of cases: in a right cone, the ordinates are perpendicular to the original diameter and the latus rectum is less than the latus transversum; in an oblique cone this need not be the case, etc. At the end of the proposition (Q 123b:9-124a:21) there are three easy lemmas on triangles and circles.

431.5 (Q 124a:22-127b:14). Consider a given cylinder intersected by a given plane in an ellipse. We want to know if a given cone contains a section congruent to this ellipse. This proposition is either based on an unidentified source or al-Mu'taman's own contribution. According to the preface to the Arabic translation of the *Conics*, al-Ḥasan ibn Mūsā proved that a given cylindrical section is contained in some cone (cf. Toomer, p. 626-627).

431.6 (Q 127b:14-128a:6). Fundamental property of the parabola, *Conics* I:11,20.

431.7 (Q 128a:6-129b:19). Fundamental property of the hyperbola and the "opposite sections", *Conics* I:12,14,21. This is followed by a short discussion along the lines of the second part of 431.4.

431.8 (Q 129b:19-130b:15). Conjugate and second diameters, *Conics* I:15,16.

431.9 (Q 130b:15-132a:12). Number of points of intersection of a straight line and a conic section in various cases, *Conics* I:18-19, 22-27, 31.

431.10 (Q 132a:12-134a:1). Construction of a tangent to a conic section at a given point, and proof that this tangent is unique, *Conics* I:17, 32-36. The proof by Apollonius in *Conics* I:34 for an ellipse and a hyperbola is shortened in Q 132b:9 by a reference to an earlier proposition, which turns out to be 312.17. The shortened proof in the *Ikmāl* (which is probably identical to the lost proof in the *Istikmāl*) resembles the historical analysis by Zeuthen,[12] who argues that Apollonius used a proposition just like proposition 312.17 in the *Istikmāl.* This remarkable agreement suggest that proposition 312.17 in the *Istikmāl* is of Greek origin.

431.11 is almost completely extant in fragment **g** of manuscript **K**.

5. GP 264. Gaps in manuscript K between fragments g, h, i.

Fragment **g** of manuscript **K** ends in the middle of proposition 431.25 (Q 152b:10-153a:21).

431.26 (Q 153a:21-154b:11). Construction of a conic section of given type such that the diameter makes a given angle with the corresponding ordinates. This is the same thing as constructing a diameter which makes a given angle with the tangent through the vertex (*Conics* II:51,53).

431.27 (Q 154b:12-157b:22, most figures on f. 158a). Construction of a conic section from certain data, as in *Conics* I:52-60. The opposite sections and conjugate hyperbolas are also mentioned. The text ends with the construction of a hyperbola through a given point and having given asymptotes (*Conics* II:4). Fragment **h** of manuscript **K** is part of this proposition (corresponding to **Q** 156a:17 - 157a:20). This is the final proposition of section 431.

Fragment **i** begins with section 432.

GP 266. Gap between fragments i and j in manuscript K.

Fragment **i** ends with 432.5.

432.6 (Q 164a:1-11). Preliminary theorem on tangents to a hyperbola or ellipse, *Conics* VII:4.

432.7 (Q 164a:11-166b:4). This proposition begins with a proof that the sum of squares of the axes of an ellipse is equal to the sum of the squares of any two conjugate diameters (*Conics* VII:12). The rest of this proposition is in fragment **j** of manuscript **K**.

GP 273-274. Gap between fragments j and k in manuscript K.

Fragment **j** ends in the middle of the proof of 512.13.

512.14 (Q 202a:6-202b:25). Construction of a regular icosahedron in a sphere and

[12] Hieronymus G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum* (Copenhagen, 1886; reprint Hildesheim, 1966), 82-83.

proof that if the diameter of the sphere is rational, the side is the irrational straight line called "minor", as in El. XIII:16.

512.15 (Q 202b:25-203b:12). The square of the cube and the triangular face of the octahedron inscribed in the same sphere have equal circumscribed circles (this theorem is also found in al-Maghribī's *Revision of the Elements,* XV:3). If an icosahedron and a dodecahedron are inscribed in the same sphere, the triangular and pentagonal face have equal circumscribed circles (found in *Elements*, Book XIV, written by Hypsicles; also in al-Maghribi, *Revision of the Elements*, XV:5).

512.16 (Q 203b:12-204b:24) We want to find the ratios of the perpendiculars drawn from the centre of a sphere to the faces of the five regular solids inscribed in that sphere. The same problem is treated in al-Maghribī's *Revision of the Elements*, XV:33. One of the figures in the proof is the mirror image of the figure in al-Maghribī's *Revision of the Elements* XV:25, and the proof in 512.16 uses essentially the same ideas as al-Maghribi in his propositions XV:7, 24. Assuming that this proof also occurred in the *Istikmāl,* al-Mu'taman (died 1085) probbaly used the same unidentified source as al-Maghribi (died 1283).

This theorem is followed by the theorem that there cannot be more than five regular solids, and a description of two semi-regular solids, one truncated cube or octahedron (with 14 faces) and one truncated icosahedron or dodecahedron. The text refers to the *Revision of the Elements* by Naṣīr al-Dīn al-Ṭūsī (died 1274). For chronological reasons, this reference must have been added by Ibn Sartāq.

8. GP 276. Gap between fragments k and 1 in manuscript K.

Fragment **k** breaks off in the middle of the proof of 52.12.

52.13 (Q 223a:15-224a:15). Construction of a sphere with surface area equal to a given circle or with volume equal to a given cone or cylinder. Construction of a cone or cylinder similar to a given (right) cone or cylinder, and either with surface area equal to that of a given circle, or such that the volume is equal to the volume of a given sphere, cone or cylinder. The proposition includes Archimedes, SC II:1.

52.14 (Q 224a:15-225a:10). Construction of a segment of a sphere similar to a given segment of a sphere and such that (a) the surface area is equal to the surface area of another given segment, or (b) the volume is equal to the volume of another given segment. (a) is Archimedes, SC II:6, (b) is based on 511.1. Fragment **1** is the end of part (b) of this proposition. This is the final proposition of (the first genus of) the *Istikmāl* and the *Ikmāl*.

Appendix 1

The "Theorem of Ceva" in the *Istikmāl* and the *Ikmāl*

The following editions and translations of the last proposition in section 2 of species 1 of species 3 in the *Istikmāl* (prop. 18) and section 2 of class 1 of species 3 in the *Ikmāl* (prop. 17) [13] will give the reader a feeling for the differences between the two works. This proposition is about a theorem which is named after the Italian mathematician Giovanni Ceva, who published it in 1678 in his work *De lineis rectis se invicem secantibus statica constructio.* No medieval Arabic proofs of the theorem of Ceva have been published hitherto. I suspect that al-Mu'taman took the "theorem of Ceva" together with the preceding theorems 312.16 (on the projective invariance of general cross-ratios) and 312.17 from an Arabic translation of an ancient Greek work which has not otherwise come down to us[14].

The "theorem of Ceva" is found in the margin of f. 43a of manuscript **K** of the *Istikmāl* [15]. In **K**, the names of points in geometrical figures are always written out in full, as in *alif bā'*. For sake of brevity I have written in my Arabic edition only the letters *AB,* not the names *alif bā',* but I have indicated the letters by slashes, as is usual in medieval Arabic geometrical texts. The corresponding passage in the *Ikmāl* is in manuscript Q, f. 51b:4-10. My own explanatory additions to the translations appear in parentheses.

[13] Ibn Sartāq omitted proposition 312.9 in the *Istikmāl* to the effect that two figures which are similar to a third figure are themselves similar. This explains the difference between the proposition numbering in the rest of the section. Section 312 is one of the very few exceptions to the rule that the proposition numberings in the two works are equal.

[14] See proposition 431.10 above.

[15] The fact that the theorem also occurs in the *Ikmāl* shows that the marginal addition in **K** is genuine.

Istikmāl, proposition 312.18. Arabic Text.

يح كل مثلث يخرج من كل واحدة من زواياه خط يقطع الضلع المقابل بها وتلقى الخطوط داخل المثلث على نقطة واحدة، فإن نسبة أحد قسمي أضلاع المثلث إلى الآخر مثنّاة بنسبة القسم الذي يلي التالي إلى القسم الآخر من ذلك الضلع كنسبة قسمي الضلع الباقي من المثلث أحدهما إلى الآخر من جهة ابتدال النسبة، وعكس ذلك.

مثال ذلك: مثلث ابج أخرج من نقطة ا خط اد يلقى ضلع بج على نقطة د ، وأخرج من نقطة ب خط بح يلقى خط اد على نقطة ه وضلع اج على نقطة ح ووصلت نقطتا جه حتى يلقى خط جه ضلع اب على نقطة ز . فأقول إن نسبة بز إلى زا مثنّاة بنسبة اح إلى حج كنسبة بد إلى دج .

برهان ذلك إن نسبة بز إلى زا كنسبة به إلى هح مثنّاة بنسبة حج إلى جا ونسبة اح إلى حج مثنّاة بنسبة حج إلى جا هي كنسبة حا إلى اج . فنسبة بز إلى زا مثنّاة بنسبة اح إلى حج كنسبة به إلى هح مثنّاة بنسبة حا إلى اج التي هي كنسبة بد إلى دج فنسبة بز إلى زا مثنّاة بنسبة اح إلى حج هي كنسبة بد إلى دج وذلك ما أردنا أن نبيّن.

وهنالك تبيّن أنه إذا كانت نسبة بز إلى زا مثنّاة بنسبة اح إلى حج كنسبة بد إلى دج ووصل بح و جز و اه تمر بنقطة د ، لأنه إن لم تمر بها انقسم خط بج على نقطتين مختلفتين بنسبة واحدة من جهة واحدة ذلك خلف. فخط اه إذا أخرج مرّ بنقطة د وذلك ما أردنا أن نبيّن. هذا الشكل ...

Istikmāl, proposition 312.18. Translation.

18. (Figure 3)[16] (In) every triangle in which from each of its angles a line issues to intersect the opposite side, such that the three lines meet inside the triangle at one point, the ratio of one of the parts of a side of the triangle to the other (part), doubled with the ratio of the part adjacent to the second term (of the first ratio) to the other part of that side is as the ratio of the two parts of the remaining side of the triangle, if the ratio is inverted.[17] And the converse of this.

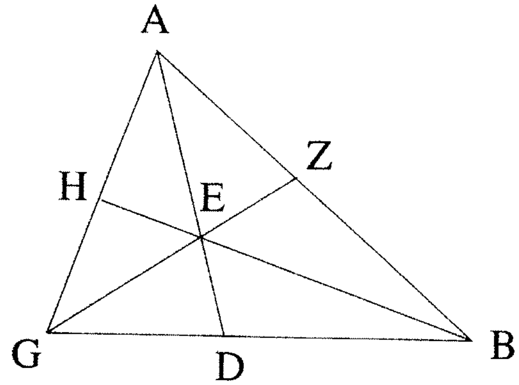

Fig. 3

Example of this: (In) triangle *ABG* from point *A* line *AD* has been drawn to meet side *BG* at point *D*, and from point *B* line *BH* has been drawn to meet line *AD* at point *E* and side *AG* at point *H*, and points *G*, *E* are joined, until line *GE* meets side *AB* at point *Z*. I say that the ratio of *BZ* to *ZA* doubled with the ratio of *AH* to *HG* is as the ratio of *BD* to *DG*.

Proof of this: The ratio of *BZ* to *ZA* is as the ratio of *BE* to *EH* doubled with the ratio of *HG* to *GA*.[18] But the ratio of *AH* to *HG* doubled with the ratio of *HG* to *GA* is as the ratio of *HA* to *AG*. Thus the ratio of *BZ* to *ZA* doubled with the ratio of *AH* to *HG* is as the ratio of *BE* to *EH* doubled with the ratio of *HA* to *AG*, which (ratio) is as the ratio of *BD* to *DG*.[19]

16 Only a very small trace of the figure is visible in manuscript **K**, on the top margin of f. 43a.

17 In the notation of Figure 2, $(BZ : ZA) \cdot (AH : HG) = DB : GD$. The text considers the segments in the order *BZ*, *ZA*, *AH*, *HG*, *GD*, *DB*. Segment *AH* is adjacent to the second term *ZA* of the ratio *BZ* : *ZA*. The text means that one should not take the ratio *GD* : *DB* but rather the inverted ratio *BD* : *DG*.

18 Here the text uses the theorem of Menelaus for a plane transversal figure.

19 Again Menelaus' theorem is used.

Thus the ratio of *BZ* to *ZA* doubled with the ratio of *AH* to *HG* is as the ratio of *BD* to *DG*. That is what we wanted to demonstrate.

Here it has become clear that if the ratio of *BZ* to *ZA* doubled with the ratio of *AH* to *HG* is as the ratio of *BD* to *DG*, and *BH, GZ* and *AE* are joined, it (*AE*) passes through point *D*. For if it would not pass through it, line *BG* would be divided at two different points in the same ratio on the same side (i.e., internally); this is absurd. Thus if line *AE* is extended, it passes through point *D*. That is what we wanted to demonstrate. This proposition is ... [20]

Ikmāl, proposition 312.17. Arabic text.

يز إذا خرج من زوايا مثلّث إلى أضلاعه ثلاثة خطوط ملتقية على نقطة واحدة داخل المثلث فنسبة قسمي أحد أضلاعه مؤلفة من نسبتي أقسام الضلعين الباقيين على أن يؤخذ مقدّم إحدى البسيطتين ما يقطع مقدّم المؤلفة وتالي البسيطة الأخرى ما يقطع تاليها.

فليخرج من زوايا ابج من مثلّث ابج إلى أضلاع بج جا اب خطوط اهد بهح جهز ملتقية على ه داخل المثلّث. فلأن نسبة بد دج مؤلفة من نسبتي به هح حا اج ونسبة حا اج مؤلفة من نسبتي اح حج حج جا بجعل حج وسطًا فنسبة بد دج مؤلفة من نسب به هح اح حج حج جا.

وإذ نسبة بز زا مؤلفة من نسبتي به هح حج جا فنأخذ بدل نسبتي به هح حج جا نسبة بز زا فيكون نسبة بد دج مؤلفة من نسبتي بز زا اح حج وذلك ما أردناه.

تمّ الفصل الثاني والحمد لله ربّ العالمين.

[20] I have not been able to read the final one or two words.

Ikmāl, proposition 312.17. Translation.

17. (Figure 3) If from the angles of a triangle three lines issue to its sides, which (lines) meet at one point inside the triangle, then the ratio of the two parts of one of its sides is composed of the two ratios of the parts of the two remaining sides, if one takes as the first term in one of the two simple ratios the (part) which intersects the first term in the composed (ratio), and as the second term in the other simple ratio the (part) which intersects the second term of it (the composed ratio).[21]

Thus let from the angles *A, B, G* of triangle *ABG* to the sides *BG, GA, AB* lines *AED, BEH, GEZ* issue, which (lines) meet at *E* inside the triangle. Then, since the ratio *BD, DG*[22] is composed of the two ratios *BE, EH, HA, AG* and the ratio *HA, AG* is composed of the two ratios *AH, HG, HG, GA* by making *HG* the mean, thus the ratio *BD, DG* is composed of the ratios *BE, EH, AH, HG, HG, GA*.

Since the ratio *BZ, ZA* is composed of the two ratios *BE, EH, HG, GA*, we take instead of the two ratios *BE, EH, HG, GA* the ratio *BZ, ZA*. Then the ratio *BD, DG* is composed of the two ratios *BZ, ZA, AH, HG*. That is what we wanted.[23]

End of the second section. Praise be to God, Lord of the Worlds.

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[21] Here Ibn Sartāq means the following. We choose any ratio between the two parts of the sides as "composed ratio"; suppose we choose $BD : DG$ as in Fig. 3. Then $BD : DG = (p : q) \cdot (r : s)$ for two simple ratios $(p : q)$ and $(r : s)$, where $p : q$ and $r : s$ are ratios between the parts of the other two sides *AB* and *AG*. Then *p* is the only of the four parts which "intersects" *BD*, so $p = BZ$, hence $q = ZA$; and similarly *s* is the only part which "intersects" *GD*, so $s = HG$, hence $r = AH$. We can also choose as our "composed ratio" any of the five other ratios between parts of the sides of the triangle. In this way we obtain six equivalent ways to state the theorem.

[22] Ibn Sartāq writes ratios in a shorthand way. He omits the preposition *ilā* "to", which is normally used in Arabic geometrical texts in expressions such as "the ratio of BD to DG".

[23] Ibn Sartāq omits the converse of the theorem.

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