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Al-Khwārizmī's Table of the "Sine of the Hours" and the Underlying Sine Table

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1. Introduction and Summary

In [1983, p. 10] D. King published a facsimile of the first half of a "table of the sine of the hours", which is appended to a treatise by Muḥammad ibn Mūsā al-Khwārizmī on the astrolabe in a Berlin manuscript. King tentatively attributed the table to al-Khwārizmī, and in the course of this paper several arguments will be given in support of al-Khwārizmī's authorship. An edited version of the entire table is to be found in Tables 1 and 2 in this paper, and details on the edition can be found in Section 2.

The "table of the sine of the hours" is concerned with the problem of determining the solar altitude at a given time from the solar altitude at noon. The time is measured in seasonal hours since sunrise. A seasonal hour is one-twelfth of the time interval between sunrise and sunset, so that the length of a seasonal hour varies with the season. The table contains for every degree of solar noon altitude H and for every seasonal hour t a number f(t, H) which can be used in the computation of the solar altitude. The manuscript does not explain the precise meaning of the numbers f(t, H) and there are no instructions for the use of the table. In Section 3 of this paper I will explain the mathematical structure of the table, following an unpublished paper by Girke and King [1988] on the approximate formula on which the table appears to be based.

The "table of the sine of the hours" is of unusual historical interest, because it is possible to exactly reconstruct the sine table which was used in its computation (Table 3). Instead of the modern sine function, the Islamic astronomers tabulated the function R sin(x) for fixed values of the "base" R. It turns out that the table of the sine of the hours is based on a sine table to base R = 150, a parameter which was common in Indian and early Islamic science. This sine table will be reconstructed in Section 4. The reconstructed sine table has not been found in any medieval source, and it is also new to the modern literature. It appears that the sines of the six multiples of 15° were taken from the Indian tradition, and that the sines of the other degrees were found by a curious non-linear interpolation scheme. The basic idea of this

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Table 1

Table of the sine of the hours for meridian altitudes from 25 to 90
(Jadwal jaib al-sā^{c}āt li-rtifā^{c} niṣf al-nahār min 25 ilā 90)

	hour 1	hour 2	hour 3	hour 4	hour 5	hour 6
25	16;29	15;13	13;06	10;09	6;20	2;06
26	17;06	15;47*	13;35	10;31	6;34	2;11
27	17;26	16;06	13;52	10;44	6;42	2;14
28	18;18*	16;54	14;33	11;16	7;02	2;21
29	18;54	17;27	15;01	11;38	7;16	2;25
30	19;30	18;00	15;30	12;00	7;30	2;30
31	20;04	18;31	15;57	12;20	7;43*	2;34
32	20;38	19;02	16;24	12;42	7;56	2;38
33	21;11	19;33	16;50	13;02	8;09	2;43
34	21;45	20;04	17;17	13;23	8;22	2;47
35	22;18	20;35	17;43	13;43	8;34	2;52
36	22;50	21;05	18;09	14;03	8;47	2;56
37	23;23	21;35	18;35	14;23	9;00	3;00
38	23;55	22;05	19;01	14;43	9;12	3;04
39	24;27*	22;34	19;26	15;02	9;24	3;08
40	24;59	23;03	19;51	15;22	9;36	3;12
41	25;30	23;32	20;16	15;42	9;48	3;16
42	26;01	24;01	20;41	16;01	10;00	3;20
43	26;32*	24;30	21;05	16;20	10;12	3;24
44	27;03	24;58	21;30*	16;39	10;24	3;28
45	27;33	25;26	21;54	16;57	10;36	3;32
46	28;00	25;51	22;15	17;14	10;46	3;35
47	28;27	26;15	22;37	17;30	10;56	3;39?
48	28;53	26;40	22;57	17;46	11;06	3;42
49	29;19	27;04	23;18	18;02	11;16	3;45
50	29;44	27;24	23;39	18;18	11;26	3;48
51	30;10	27;51	23;59	18;34	11;36	3;52
52	30;35*	28;14	24;19	18;48	11;46	3;55
53	31;00*	28;37	24;39	19;05	11;55	3;58*
54	31;25*	29;00	24;58	19;20	12;05	4;01
55	31;50	29;23	25;18	19;35	12;14	4;05
56	32;14	29;45	25;37	19;50	12;24	4;08
57	32;38*	30;07*	25;56	20;05	12;33	4;11

Apparatus (x means: illegible digit): [26,2] 15;17 [28,1] 19;18 [31,5] 7;34 [39,1] 24;x7 [43,1] 2x;32 [44,3] 21;40 [52,1] xx;35 [53,1] xx;00 [53,6] 3;18 [54,1] xx;25 [57,1] 32;58 [57,2] 30;54.

scheme is that the differences between successive sines decrease by one minute per degree. The results of this interpolation differ from the exact values by up to 40 minutes, which suggests that the sine table was computed early in the Islamic tradition.

By means of the reconstructed sine table one can recompute the table of the sine of the hours exactly. The recomputed values are in two sexagesimals, but the

Table 2

Table of the sine of the hours for meridian altitudes from 25 to 90.
(Jadwal jaib al-sā^{c}āt li-rtifā^{c} niṣf al-nahār min 25 ilā 90)

	hour 1	hour 2	hour 3	hour 4	hour 5	hour 6
58	33;01	30;29	26;15	20;19*	12;42*	4;14
59	33;25	30;50	26;33	20;33	12;51	4;17
60	33;47	31;12	26;52	20;48	13;00	4;20
61	34;05	31;28	27;06	20;59	13;06	4;22
62	34;22	31;44	27;19	21;09	13;13	4;24
63	34;39	31;59	27;33	21;20	13;20	4;26
64	34;57	32;14	27;46	21;30	13;26	4;29
65	35;12	32;30	27;59	21;40	13;32	4;31
66	35;28	32;44	28;12	21;50	13;39	4;33
67	35;44	32;59	28;24	22;00	13;45	4;35
68	36;00	33;14	28;37	22;09	13;51	4;37
69	36;15	33;28*	28;49	22;19	13;56	4;39
70	36;30	33;42	29;01*	22;28	14;02	4;41
71	36;45	33;56	29;13	22;37	14;08	4;43
72	37;00	34;09*	29;24	22;46	14;14	4;44
73	37;14	34;22	29;36	22;55	14;19	4;46
74	37;28	34;35	29;47	23;03	14;25	4;48
75	37;42	34;48	29;58	23;12	14;30	4;50
76	37;49	34;54	30;03	23;16	14;33	4;51
77	37;56	35;01	30;09	23;20	14;35	4;52
78	38;00	35;07	30;14	23;24	14;38	4;52
79	38;08	35;12	30;19	23;28	14;40	4;53
80	38;14	35;18	30;24	23;32	14;42	4;54
81	38;20	35;23	30;28	23;35	14;45	4;55
82	38;26	35;28	30;33	23;39	14;47	4;55
83	38;31	35;33	30;37	23;42	14;49	4;56
84	38;36	35;37	30;41	23;44	14;51	4;57
85	38;40	35;42	30;44	23;48	14;52	4;57
86	38;44	35;46	30;48	23;51	14;54	4;58
87	38;49	35;50	30;51	23;53*	14;56	4;58
88	38;53	35;53	30;54	23;56	14;57	4;59
89	38;56	35;57	30;57	23;58	14;59	4;59
90	39;00	36;00	31;00	24;00	15;00	5;00

Apparatus: [58,4] 20;15 [58,5] 12;52 [69,2] 33;23 [70,3] 29;10 [72,2] 33;09 [87,4] 23;58.

numbers in Tables 1 and 2 are given to one sexagesimal place. Therefore it is possible to see how al-Khwārizmī rounded his sexagesimal numbers in practice. See Section 4 for all details.

In this paper sexagesimal numbers will be transcribed using the following standard notation: a semicolon separates the integer from the fractional part, and

Table 3

Reconstructed sine table to base 150, used in the computation of the table for the sine of the hours. S(x) is the reconstructed sine of x, and diff. is the difference S(x+1)-S(x).

x	S(x)	diff.	x	S(x)	diff.	x	S(x)	diff.
0	0	163	30	75;00	131	60	130;00	67
1	2;43	162	31	77;11	130	61	131;07	66
2	5;25	161	32	79;21	129	62	132;13	65
3	8;06	160	33	81;30	128	63	133;18	64
4	10;46	159	34	83;38	127	64	134;22	63
5	13;25	158	35	85;45	126	65	135;25	62
6	16;03	157	36	87;51	125	66	136;27	61
7	18;40	156	37	89;56	124	67	137;28	60
8	21;16	155	38	92;00	123	68	138;28	59
9	23;51	154	39	94;03	122	69	139;27	58
10	26;25	153	40	96;05	121	70	140;25	57
11	28;58	152	41	98;06	120	71	141;22	56
12	31;30	151	42	100;06	119	72	142;18	55
13	34;01	150	43	102;05	118	73	143;13	54
14	36;31	149	44	104;03	117	74	144;07	53
15	39;00	151	45	106;00	103	75	145;00	27
16	41;31	150	46	107;43	102	76	145;27	26
17	44;01	149	47	109;25	101	77	145;53	25
18	46;30	148	48	111;06	100	78	146;18	24
19	48;58	147	49	112;46	99	79	146;42	23
20	51;25	146	50	114;25	98	80	147;05	22
21	53;51	145	51	116;03	97	81	147;27	21
22	56;16	144	52	117;40	96	82	147;48	20
23	58;40	143	53	119;16	95	83	148;08	19
24	61;03	142	54	120;51	94	84	148;27	18
25	63;25	141	55	122;25	93	85	148;45	17
26	65;46	140	56	123;58	92	86	149;02	16
27	68;06	139	57	125;30	91	87	149;18	15
28	70;25	138	58	127;01	90	88	149;33	14
29	72;43	137	59	128;31	89	89	149;47	13
						90	150;00	

sexagesimals are separated by commas. Thus 30;7,30 means $30+7/60+30/3600$.

2. Edition of the "Table of the Sine of the Hours"

Tables 1 and 2 above are my editions of the first and second half of the table of the sine of the hours in the manuscript Berlin, Deutsche Staatsbibliothek, Landberg 56 (Ahlwardt 5397) f. 95a–b. In the manuscript, the six entries in every row are almost always in the proportion 39 : 36 : 31 : 24 : 15 : 5, and the few exceptions to this rule can be explained as scribal errors, which have been corrected in Tables 1 and 2. All restored values are marked by an asterisk, and the entries in the manuscript have been noted in the apparatus at the bottom of the table. In the apparatus, a notation such as [26, 2] refers to the number in the row for H = 26 and the column for t = 2. The numbers in the manuscript are partly written in the alphabetical *abjad*-notation, as is usual in tables, and partly in the Hindu-Arabic notation, which is less common in this context. Most scribal errors can be explained as plausible misreadings in one

of these notations. Examples: in [26, 2] the *abjad*-number 17 (yā' – zāy) is a plausible mistreading of 47 (mīm—zāy); in [31, 5] the Hindu-Arabic "43" is a plausible error for "34".

In the second half of the table (Table 2), the scribe began the last three columns (t = 4, 5, 6) by putting the value for H = 57 in the first line (for H = 58), the value for H = 58 in the second line (H = 59), and so on. As a consequence, all numbers in the last three columns in the manuscript are displaced, and in each of these columns there are four numbers (for H = 87, 88, 89, 90) in the last three rows (for H = 88, 89, 90). I have restored the correct order of the entries in Table 2 without noticing all these changes of position in the apparatus.

In Section 3 it will be explained that the entries in Tables 1 and 2 are differences between Sines to base R = 150. I will conveniently ignore the fact that the 150 units of the base are called "minutes" in the table of sines in the *Canones Arzachelis* [Pedersen, pp. 150—152] and in the *Toledan Tables* [Toomer, p. 27], two 12th century Latin translations of Arabic originals; thus al-Khwārizmī may also have considered the 150 units as "minutes". If this interpretation is consistently carried through, one has to put R = 2;30 and move all sexagesimal semicolons in Tables 1 and 2 one position to the left, so that one would have to print 0;16,29 instead of 16;29, etc. However, I did not want to burden the table with so many zeros.

3. The Mathematical Structure of the "Table of the Sine of the Hours"

The following explanation is inspired by an unpublished paper by Girke and King [1988, Section 9.2] on an approximate formula which was widely used in the Islamic scientific tradition, and on which Tables 1 and 2 appear to be based. I first state this approximate formula in modern concepts and notation.

Let H be the solar altitude at noon, let t be the time in seasonal hours since sunrise, and let h be the solar altitude at t.

Then approximately

$$\sin(h) = \sin(H) \cdot \sin(15t)\,. \tag{1}$$

Formula (1) is in general incorrect, because h depends also on the geographical latitude. However, (1) is correct if the sun is at the equinoxes, and it is a good approximation for localities in the Middle East [Girke and King, Section 2.3].

Al-Khwārizmī used the sine to base 150. Writing Sin(x) = 150 sin(x), we have

$$\mathrm{Sin}(h) = \mathrm{Sin}(H) \cdot \mathrm{Sin}(15t)/150\,. \tag{2}$$

To show that al-Khwārizmī knew this formula, I quote a passage from the original version of his Zīj (astronomical handbook with tables). The passage does not occur in the extant Latin translation of the 10th century revision of the Zīj, but it has survived in the following form in the 11th century commentary by Ibn al-Muthannā on the original version of the Zīj. Ibn al-Muthannā says:

"Why did he (= al-Khwārizmī) say concerning the derivation of the (seasonal) hours of the day: Take the altitude of the sun whenever you wish and multiply its sine by 150 and keep it. Then take the noon altitude, and its sine is divided by what you kept. The result is the sine of the arc and convert it into an arc, considering each 15° to be one hour...." [Goldstein pp. 82, 207–208].

The passage that Ibn al-Muthannā quotes from al-Khwārizmī can be rendered in modern notation as $\mathrm{Sin}(15t) = 150\ \mathrm{Sin}(h)/\mathrm{Sin}(H)$, which is equivalent to (2).

We now return to Tables 1 and 2. These tables contain for every degree of solar noon altitude H ($25° \leq H \leq 90°$) and every seasonal hour t ($1 \leq t \leq 6$) an entry f(t, H). Let h(t, H) be the solar altitude at the beginning of the t-th seasonal hour on a day when the solar noon altitude is H. Then the f(t, H) are approximations of the following quantities:

$$\text{for } t = 1, \quad f(1, H) \approx \mathrm{Sin}(h(1, H)) \tag{3}$$

$$\text{for } 2 \leq t \leq 6, \quad f(t, H) \approx \mathrm{Sin}(h(t, H)) - \mathrm{Sin}(h(t-1, H)). \tag{4}$$

Tables 1 and 2 can be used to compute the sine to base 150 of the solar altitude at any given time, as in the following example. Suppose one wants to compute the sine of the solar altitude in the middle of the fifth seasonal hour on a day when the solar noon altitude is 26°. We read the first five numbers in the row H = 26° in Table 1, and we then compute $17;06 + 15;47 + 13;35 + 10;31 + (1/2) \cdot 6;34 = 60;16$, which is the required sine.

The entries f(t, H) were computed by means of the approximate formula (2), that is to say:

$$f(1, H) = \mathrm{Sin}(H) \cdot \mathrm{Sin}(15°)/150 \tag{5}$$

$$f(t, H) = \mathrm{Sin}(H) \cdot [\mathrm{Sin}(15t°) - \mathrm{Sin}(15t - 15°)]/150. \tag{6}$$

It is unclear why in Tables 1 and 2 differences between sines were tabulated and not the sines themselves. Then the sines for integer seasonal hours could have been read in the table and the user would not have had to make additions. For fractional seasonal hours linear interpolation could have been used. The efficiency of this procedure explains why Islamic astronomers did in general not construct tables such as Tables 1 and 2, in which only differences occur. This feature of Tables 1 and 2 supports their attribution to al-Khwārizmī, who composed a table for the astrological "casting of rays" in a similar inefficient way (see [Hogendijk 1989, 180—187, 191 no. 2]).

It is equally unclear to me why al-Khwārizmī wanted to tabulate the sine of the solar altitude at all. This quatity does not seem to be of any use by itself. It seems more useful to carry the computation one step further and tabulate the solar altitudes themselves. Thus the mathematical structure of the table for the sine of the hours is clear, but the motivation of the table is a mystery.

4. Reconstruction of the Underlying Sine Table

We will now reconstruct the sine table which al-Khwārizmī used in his computation of Tables 1 and 2 by means of formulas (5) and (6). It is conceivable that al-Khwārizmī used accurate values for Sin(H), but rounded values for Sin(15t), to save computational labour. We will therefore use the notation, $S_1(15t)$ and $S_2(H)$, for al-Khwārizmī's approximations of Sin(15t) and S(H). Sin

In this notation, (5) and (6) may be restated thus:

$$f(1, H) \approx S_1(15°) \cdot S_2(H)/150, \tag{7}$$

and for $2 \leq t \leq 6$

$$f(t, H) \approx [S_1(15t) - S_1(15t - 15)] \cdot S_2(H)/150. \tag{8}$$

It is our task to reconstruct the S_1 (15t) and $S_2(H)$.

The sign ≈ is used here to call attention to the fact that the f(t, H) are values rounded to one sexagesimal place of the quantities on the right side of the sign.

We first determine the S_1 (15t) by putting H = 90. Undoubtedly $S_2(90°) = 150$, so that for H = 90° the factor $S_2(H)/150$ can be omitted in (7) and (8).

We then read from the table $f(1, 90°) = 39;00$, so we can expect that $S_1(15°)$ is a number between 38;59 and 39;1. The exact value $\mathrm{Sin}(15°) = 38;49,22$ (correct to two sexagesimals) is not near either of these bounds. We therefore conclude that S_1 (15°) was rounded to integer degrees, thus $S_1(15°) = 39$. For similar reasons

$$S_1(30°) = 39 + 36 = 75,\ S_1(45°) = 106,\ S_1(60°) = 130,\ S_1(75°) = 145,$$

and of course $S_1(90°) = 150$.

Exactly the same sine table for multiples of 15° occurs in the *Brāhmasphuṭa-siddhānta*, a Sanskrit work written by Brahmagupta in 628 A. D. [Pingree, p. 579, Table V, 24]. The work was somehow transmitted into Arabic and it is known to have been used by al-Khwārizmī in composing his Zīj [GAS VI, pp. 118–120; Toomer, "al-Khwārizmī", pp. 360–361].

From (7) and (8) we now derive

$$f(1, H) : f(2, H) : \cdots : f(6, H) = f(1, 90°) : f(2, 90°) : \cdots : f(6, 90°)$$
$$= 39 : 36 : 31 : 24 : 15 : 5.$$

This is the relation that has been used to identify and correct scribal errors in the edition of Tables 1 and 2.

The numbers 39, 36, 31, 24, 15 and 5 also appear in a diagram in the *Toledan Tables* [Pedersen p. 150] and in a treatise on the sine quadrant [Girke and King 1988, Section 9.2]. In both cases, a quadrant of a circle of radius 150 is divided into six equal arcs of 15°, and the 39, 36, 31, 24, 15 and 5 are the lengths of the perpendicular projections of the equal arcs on one of the diameters that defines the quadrant.

In the following reconstruction of $S_2(H)$ a crucial role will be played by the sum of the entries in the row for a given solar noon altitude H. We therefore introduce for $25^\circ \le H \le 90^\circ$ the notation

$$\Sigma(H) = f(1, H) + f(2, H) + f(3, H) + f(4, H) + f(5, H) + f(6, H) . \qquad (9)$$

Substituting (7) and (8) in (9) we obtain

$$\Sigma(H) \approx S_2(H) ,$$

where the sign $\approx$ indicates that there may be a difference between these two quantities because of rounding errors made in the computation of the f(t, H). It is reasonable to assume that the average rounding error is at most 0;0,30, so that as a rule

$$|\Sigma(H) - S_2(H)| \le 6 \cdot 0;0,30 = 0;3 . \qquad (10)$$

We now determine S_2 for all multiples of 15°. The solar noon altitude $H = 15^\circ$ does not occur in Table 1, and clearly $S_2(30^\circ) = 75$ and $S_2(90^\circ) = 150$, so that the three cases $H = 45^\circ$, $H = 60^\circ$ and $H = 75^\circ$ remain.

From Tables 1 and 2 we derive

$$\Sigma(45^\circ) = 105;58, \ \Sigma(60^\circ) = 129;59 , \quad \text{and} \quad \Sigma(75^\circ) = 145;00 .$$

Modern recomputations show

$$\mathrm{Sin}(45^\circ) = 106;05, \ \mathrm{Sin}(60^\circ) = 129;54 , \quad \text{and} \quad \mathrm{Sin}(75^\circ) = 144;53 ,$$

correct to one sexagesimal place. It follows that

$$|\Sigma(H) - \mathrm{Sin}(H)| \ge 0;5 \quad \text{for} \quad H = 45^\circ, 60^\circ \quad \text{and} \quad 75^\circ .$$

Hence, by (10), $S_2(H)$ cannot have been obtained by rounding an accurately computed sine value.

On the other hand,

$$S_1(45^\circ) = 106, \ S_1(60^\circ) = 130 \quad \text{and} \quad S_1(75^\circ) = 145 ,$$

so that

$$|\Sigma(H) - S_1(H)| \le 0;2 \quad \text{for} \quad H = 45^\circ, 60^\circ, 75^\circ .$$

We conclude that $S_2(x) = S_1(x)$ in all cases where S_1 could be determined ($x = 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ$). Thus there is no need to assume that al-Khwārizmī used two different sine tables. We will therefore drop the indices and write S(x) for al-Khwārizmī's approximation to Sin(x).

In Table 4, $\Sigma(H)$ is compared with Sin(H), and the differences in minutes have been listed. For H in the neighbourhoods of multiples of 15° $\Sigma(H)$ is close to Sin(H), but for other H, $\Sigma(H)$ is often more than 20 minutes less than Sin(H). Because $|\Sigma(H) - S(H)|$ is as a rule less than 3 minutes, S(H) must behave in the same way as $\Sigma(H)$. It is therefore plausible that for H between integer multiples of 15° the values S(H) were derived not by crude rounding of an accurately computed value, but by some form of interpolation between the multiples of 15°.

Table 4

Sums of the rows in the "table of the sine of the hours" Σ(H), compared to the correct values of the sine (Sin(H)=150sin(H)) and the differences Σ(H)-Sin(H) in minutes.

H	Σ(H)	Sin(H)	Σ-Sin	H	Σ(H)	Sin(H)	Σ-Sin
25	63;23	63;25	-2	58	127;00	127;12	-12
26	65;44	65;45	-1	59	128;29	128;36	-7
27	67;04	68;07	-63	60	129;59	129;54	-5
28	70;24	70;25	-1	61	131;06	131;13	-7
29	72;41	72;43	-2	62	132;11	132;28	-17
30	75	75	0	63	133;17	133;39	-22
31	77;09	77;15	-6	64	134;22	134;49	-27
32	79;20	79;29	-9	65	135;24	135;58	-34
33	81;28	81;43	-15	66	136;26	137;03	-37
34	83;38	83;54	-16	67	137;27	138;06	-39
35	85;45	86;02	-17	68	138;28	139;06	-38
36	87;50	88;10	-20	69	139;26	140;02	-36
37	89;56	90;16	-20	70	140;24	140;57	-33
38	92;00	92;22	-22	71	141;22	141;51	-29
39	94;01	94;25	-24	72	142;17	142;41	-24
40	96;03	96;25	-22	73	143;12	143;28	-16
41	98;04	98;26	-22	74	144;06	144;11	-5
42	100;04	100;22	-18	75	145;00	144;53	+7
43	102;03	102;19	-16	76	145;26	145;34	-8
44	104;02	104;13	-11	77	145;53	146;09	-16
45	105;58	106;05	-7	78	146;15	146;43	-28
46	107;41	107;54	-13	79	146;40	147;16	-36
47	109;24	109;42	-18	80	147;04	147;43	-39
48	111;04	111;28	-24	81	147;26	148;09	-43
49	112;44	113;12	-28	82	147;48	148;32	-44
50	114;19	114;54	-35	83	148;08	148;54	-46
51	116;01	116;34	-33	84	148;26	149;12	-46
52	117;37	118;12	-35	85	148;43	149;27	-44
53	119;14	119;49	-35	86	149;01	149;38	-37
54	120;49	121;21	-32	87	149;17	149;49	-32
55	122;25	122;52	-27	88	149;32	149;56	-24
56	123;58	124;21	-23	89	149;46	150;00	-14
57	125;30	125;48	-18	90	150;00	150;00	0

The following example shows that linear interpolation cannot have been used. By means of linear interpolation between $S(75^\circ) = 145$ and $S(90^\circ) = 150$ one finds, for example, for $H = 78^\circ, 81^\circ, 84^\circ$ and 87° the values 146, 147, 148 and 149. The differences between these values and the corresponding values of $\Sigma(H)$ in Table 4 are much more than 3 minutes.

In order to establish what kind of interpolation was used, we consider the first differences $d(H) = \Sigma(H+1) - \Sigma(H)$. On the whole it can be expected that these differences behave in the same way as the (as yet unknown) first differences $S(H+1)-S(H)$, even though in individual cases $|d(H)-S(H+1)-S(H)|$ can be as large as 6 minutes.

The global behaviour of d(H) is as follows:

for $30^\circ \leq H \leq 44^\circ$, d(H) decreases from 129 to 116,
for $45^\circ \leq H \leq 59^\circ$, d(H) decreases from 103 to 89,
for $60^\circ \leq H \leq 74^\circ$, d(H) decreases from 67 to 54,
for $75^\circ \leq H \leq 89^\circ$, d(H) decreases from 26 to 14.

The behaviour of d(H) shows that al-Khwārizmī (or whoever else computed the sine table) used an interpolation scheme in which the first differences $S(H+1)-S(H)$ decrease by 1 minute with each degree of argument.

We therefore conclude that Table 3 was the table of sine values S(x) used by al-Khwārimzī. The values S(15°), S(30°), S(45°), S(60°), S(75°) and S(90°) were taken from Brahmagupta or another Indian source. In order to compute the values between, for example, S(60°) and S(75°), the differences between successive values have to be computed first, as follows. The average difference between two successive values is $(S(75^\circ)-S(60^\circ))/15$, that is 60 minutes. This average difference is the difference between the values S(x) for the two arguments $x=67^\circ$ and $x=68^\circ$ which are closest to the midpoint of the interval [60°, 75°]. The other differences can be found by successive addition or subtraction of 1 from the average difference, as can be seen in Table 3. Then the values themselves can be computed.

That the reconstructed Table 3 was really used in the computation of Tables 1 and 2 is proved by the fact that $|S(H)-\Sigma(H)|<0;3$ for 62 of the 66 solar noon altitudes H between 25° and 90°. We have $|S(H)-\Sigma(H)|=0;3$ only for $H=52$ and $H=78$, $|S(H)-\Sigma(H)|=0;6$ for $H=50$, and $|S(H)-\Sigma(H)|=1;1$ for $H=27$. The last exception can be explained by the assumption that al-Khwārizmī misread $S(27^\circ)=68;06$ as 67;06 while computing Table 1. Such a misreading is only likely if one works with Hindu-Arabic numbers, which is another argument for attributing Tables 1 and 2 to al-Khwārizmī.

Now that Table 3 has been reconstructed, we can recompute Tables 1 and 2 and compare the results with those of al-Khwārizmī. A few more uninteresting errors can thus be identified:

In [50, 2] ($H=50$, $t=2$) the manuscript has 27;24, probably as a result of scribal error, the correct value is 27;27,36.
In [52, 4] the ms. has 18;48, the correct value is 18;49,36.
In [60, 1] the ms. has 33;47, the correct value is 33;48,0.
In [64, 1] the ms. has 34;57, the correct value is 34;56,7.

These scribal and computational errors have not been corrected in Tables 1 and 2.

Because Tables 1 and 2 can be recomputed exactly, one can make a detailed comparison between the recomputed values (in two sexagesimal places) with the entries in Tables 1 and 2 (in one sexagesimal place) in order to determine the distribution of the rounding errors. The results are as follows.

If the second sexagesimal in the recomputed value is between 0 and 30, it is simply omitted in the manuscript, as was to be expected.

If the second sexagesimal is exactly 30, it is also omitted, and the first sexagesimal is never changed (there are 21 examples).

Even if the second sexagesimal is between 30 and 40, it is simply omitted. Thus in [89, 1] the recomputed value is 38;56,37 and the manuscript has 38;56. The only exception to this rule is [37, 5], where the recomputed value is 8;59,36, and the manuscript has 9;00.

If the third sexagesimal is 40 or more, al-Khwārizmī usually rounds the number by adding 1 to the first sexagesimal. However, in one-seventh of the cases, the first sexagesimal is left unchanged (example: in [31, 4] the recomputed value is 12;20,58, the manuscript has 12;20).

It is unlikely that the above-mentioned results represent a conscious policy on the part of al-Khwārizmī. He probably carried out his computation in one sexagesimal only, and he then made a reasonable guess about the rounding, without computing the second sexagesimal in detail. In any case, the rounding errors made by al-Khwārizmī in the computation of Tables 1 and 2 are not distributed uniformly on the interval [−0;0,30, +0;0,30], and the average rounding error is not as close to zero as one might expect. One may also note that al-Khwārizmī rounded a sexagesimal number of the form m;n,30 to the number m;n and never to m;n+1.

It is is well-known that al-Khwārizmī used a sine table to base 150 in the original version of his Zīj (compare the quotation from the 11th century commentator Ibn al-Muthannā in Section 3). This sine table is not extant, because it was replaced by a later editor of the Zīj by a more recent sine table to base 60. A manuscript of the Latin translation of the Zīj of al-Khwārizmī contains a sine table to base 150, computed to two sexagesimal places, and much more accurate than Table 3 [Neugebauer, pp. 189–190, the same table in Neugebauer-Schmidt, p. 226]. Although this table is attributed to "Arzachel" (the 10th century Andalusian astronomer al-Zarqāllu), one could be tempted to assume that this more accurate table was already in the Zīj of al-Khwārizmī. However, Benno van Dalen points out to me that almost all the second sexagesimals in the table in [Neugebauer, p. 190] end in 0, 2, 5, or 7, suggesting that this table was computed by multiplying a sine-table to base 60 by 2.5. This means that the table in [Neugebauer, pp. 189—190] was not the sine table in al-Khwārizmī's Zīj, and thus Table 3 emerges as another possible candidate. However, Table 3 should not be taken as "the" sine table used by al-Khwārizmī at all times and places. In fact, I have not been able to prove that Table 3 underlies any table attributed to al-Khwārizmī other than Tables 1 and 2. It is of course possible that such tables exist, and my research has not been exhaustive. The main interest of the reconstructed sine table is anyway not its possible relationship to al-Khwārizmī, but the fact that it probably dates back to an early stage of Islamic science, when accurate sine tables were not yet widespread.

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