Secrets of Ancient MATH

Vedic Mathematics and the Calculations of Guru Tīrthajī*

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Introduction

This article was written as a reaction to several publications in the Netherlands on methods of mental calculation, which have been presented as "Vedic mathematics." The adjective "Vedic" suggests that these methods date back to the ancient Vedic period before 500 BCE.[1] In this article I will first give two examples of genuine Vedic mathematics. After a brief intermezzo on the decimal position system and decimal fractions, the so-called "Vedic" methods of mental calculation will be identified as the inventions of a twentieth-century Guru. More information on the subject can be found in [3].

The March 2003 issue of π *in the Sky* also includes an article on Vedic Mathematics. It was written by Jeganathan Sriskandarajah from Madison Area Technical College.

The Vedas

The Vedas are ancient religious and philosophical scriptures that originated in India in the Vedic period. These texts were composed in an Indo-European language that was the predecessor of Sanskrit. Initially the Vedas were transmitted orally, but later on they were recorded in writing. The word Veda has the same root as the Dutch word "weten," the German "wissen" (meaning: to know), and the English "witness," "unwitting" and "to wit." The Vedic texts refer to rituals and fire-altars. Rules for the precise construction of these altars are given in the *Śulba-sūtra's* (rules for the measuring cord), which also belong to the authentic Vedic literature. Some of these sutras deal with the geometric knowledge that is necessary for the construction of brick altars of different shapes and sizes.

Examples of Vedic Mathematics

The following four sutras are taken from the Baudhāyanaśulbasūtra, which was composed before 500 BCE. The text and translation are adapted from the edition of Sen and Bag, which is based on old Indian manuscripts in Benares, Munich, London and Ujjain. In this work, the sutras are stated without further explanation and without proof. I will not provide any commentary either, so as to leave the reader the pleasure of pondering the meaning of these sutras. (See [5, pp. 18, 19, 78–80, 150–154, 164–169]; the numbering from [5] is used here.)

(1.9) samacaturaśrasyākṣṇayārajjurdvistāvatīṃ bhūmiṃ karoti

The diagonal of a square produces double the area.

(1.12) dīrghacaturaśrasyākṣṇayārajjuḥ pārśvamānī tiryaṅmānī yatpṛthagbhūte kurutastadubhayaṃ karoti

The areas produced separately by the length and the breadth of a rectangle together equal the area produced by the diagonal.

(1.13) tāsāṃ trikacatuskayordvādaśikapañcikayoḥ pañcadaśikāṣṭikayoḥ saptikacaturviṃśikayor dvādaśikapañcatriṃśikayoḥ pañcadaśikaṣaṭtriṃśikayorityetāsūpalabdhiḥ

This is observed in rectangles having sides three and four, twelve and five, fifteen and eight, seven and twenty-four, twelve and thirty-five, fifteen and thirty-six.

(2.12) pramāṇaṃ tṛtīyena vardhayettacca caturthenātmacatustriṃśonena saviśeṣaḥ

The measure is to be increased by its third and this again by its own fourth less the thirty-fourth part [of the latter (Ed.)]; this is the diagonal of a square.

The last sutra has been interpreted as a close approximation to $\sqrt{2}$. Historians generally assume that this approximation was discovered not by accident but by some clever mathematical method.

Historic Intermezzo: Discovery of the Decimal System

In the Vedic sutras, numbers are expressed in words. Indo-European roots of number words can be recognized in the preceding quotations: tri = three, catur = quattuor in Latin (cf. quadrilateral), pañca = pente in Greek (cf. pentagon), dva = two, etc.

From the third century BCE onwards, that is to say, after the Vedic period, number symbols were used in North India for the numbers one through nine. These symbols are the predecessors of the modern $1, 2, 3, \ldots, 9$. Other symbols were used for ten, twenty..., hundred, two hundred..., and a symbol for zero had not yet been invented.

The oldest known symbols for "zero" occur in astronomical computations in the sexagesimal place-value system, on Babylonian clay tablets written in last three centuries BCE. The zero was used as a place holder, as in "29 degrees, 0 minutes, and 43 seconds." The sexagesimal system, including the zero, was adapted by the Greek astronomers in the second century BCE, and are still to be found today in angles (degrees, minutes, and seconds) and units of time (hours, minutes and seconds). From 300 BCE onwards, astronomical methods from Babylon and ancient Greece, including the sexagesimal system, were transmitted to India as well. Probably around 300 CE (six centuries later), the Indian astronomers developed the modern decimal place-value system for integers. This system was transmitted into Arabic in the eighth and early ninth century, but it never became popular in the medieval Islamic world. In the eleventh and twelfth centuries, the Indian number symbols were introduced in Christian Europe. The final victory of the system began in the Renaissance as a result of the increase of trade and commercial transactions. Decimal fractions were introduced by a few medieval Islamic mathematicians in order to write the end results of computations, but they were hardly used in the computations themselves (which were usually done in the sexagesimal

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[1] The notation BCE stands for Before Common Era and CE stands for Common Era. The BCE/CE system is synonymous with the much more common BC/AD system (Ed.).

system). In Europe, decimal fractions were independently discovered by the Flemish mathematician Simon Stevin, who published his discovery in 1585 in Leiden in Holland. Decimal fractions probably arrived in India in the seventeenth century, when French astronomical tables were studied by astronomers in and around Delhi. During the English colonial regime, decimal fractions were of course taught in schools.

The "Vedic" Mathematics of Guru Tirthajī

The "Vedic" methods of mental calculations in the decimal system are all based on the book *Vedic Mathematics* by Jagadguru (world guru) Swami (monk) Śrī (reverend) Bhāratī Kṛṣṇa Tīrthajī Maharaja, which appeared in 1965 and which has been reprinted many times [4]. This Guru was born in 1884 near Madras as Venkatraman Shastri. He studied English, mathematics, Sanskrit, and other subjects at various schools and colleges in India. In 1911 he quit his teaching job in order to devote himself completely to study and meditation in an ashram near Mysore. In 1919 he was initiated as a sannyasi (monk) and received the name Bhāratī Kṛṣṇa Tīrtha. Shortly afterwards, he became the leader of ashrams. He travelled extensively through India, and, at the invitation of the Self-Realization Fellowship of Yogananda, also to the United States in 1958. Bhāratī Kṛṣṇa Tīrtha died in 1960 and his main work *Vedic Mathematics* appeared posthumously.

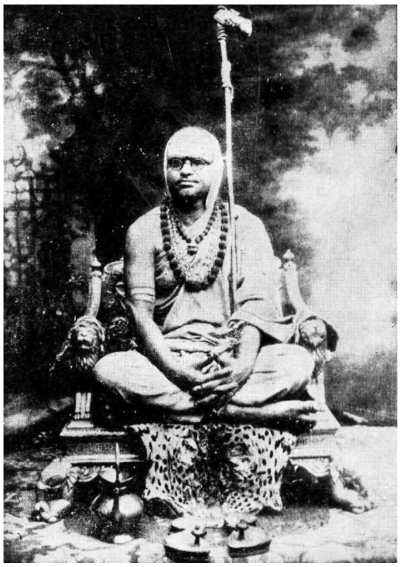


Figure 1: Guru Tīrthajī

The book contains sixteen brief sutras that can be used for mental calculations in the decimal place-value system. An example is the sutra *ekādhikena purvena*, meaning: *by one more than the previous one*. The Guru explains that this sutra can for example be used in the mental computation of the period of a recurring decimal fraction such as $1/19 = 0.\overline{052631578947368421}$, as follows:

The last decimal of the period is 1. We now apply the sutra *by one more than the previous one* to the number 19. The *previous one* is the penultimate decimal in the decimal representation of 19, so *one more than the previous one* is 2. We multiply the last decimal of the period by 2 and get the penultimate decimal of the period, and so on. Thus we find the final decimals $\overline{\ldots 8421}$. We now compute $8 \times 2 = 16$, write the 6 before the 8, and remember the 1. We add this 1 to the next product 6×2 to obtain $\overline{\ldots 368421}$, and so on.[2]

The same method can be used for 1/29, but now we have to take 3 as *one more than the previous one*.

The word "Vedic" in the title of the book suggests that these calculations are authentic Vedic mathematics. The question now arises how the Vedic mathematicians were able to write the recurrent decimal fraction of 1/19, while decimal fractions were unknown in India before the seventeenth century. We will first investigate the origin of the sixteen sutras. We cite the Guru himself [4, pp. xxxviii–xxix]:

"And the contemptuous or, at best, patronizing attitude adopted by some so-called orientalists, indologists, antiquarians, research-scholars etc. who condemned, or light heartedly, nay irresponsibly, frivolously and flippantly dismissed, several abstruse-looking and recondite parts of the Vedas as 'sheer nonsense' or as 'infant-humanity's prattle,' and so on, ...further confirmed and strengthened our resolute determination to unravel the too-long hidden mysteries of philosophy and science contained in ancient India's Vedic lore, with the consequence that, after eight years of concentrated contemplation in forest-solitude, we were at long last able to recover the long lost keys which alone could unlock the portals thereof.

"And we were agreeably astonished and intensely gratified to find that exceedingly tough mathematical problems (which the mathematically most advanced present day Western scientific world had spent huge lots of time, energy and money on and which even now it solves with the utmost difficulty and after vast labour involving large numbers of difficult, tedious and cumbersome 'steps' of working) can be easily and readily solved with the help of these ultra-easy Vedic Sūtras *(or mathematical aphorisms) contained in the* Pariśīsta *(the Appendix-portion) of the* Atharvaveda *in a few simple steps and by methods which can be conscientiously described as mere 'mental arithmetic.' "*

The Guru adds [4, p. xl]:

"The Sūtras *(aphorisms) apply to and cover each and every part of each and every chapter of each and every branch of mathematics (including arithmetic, algebra, geometry—plane and solid, trigonometry—plane and spherical, conics—geometrical and analytical, astronomy, calculus—differential and integral etc.). In fact, there is no part of mathematics, pure or applied, which is beyond their jurisdiction."*

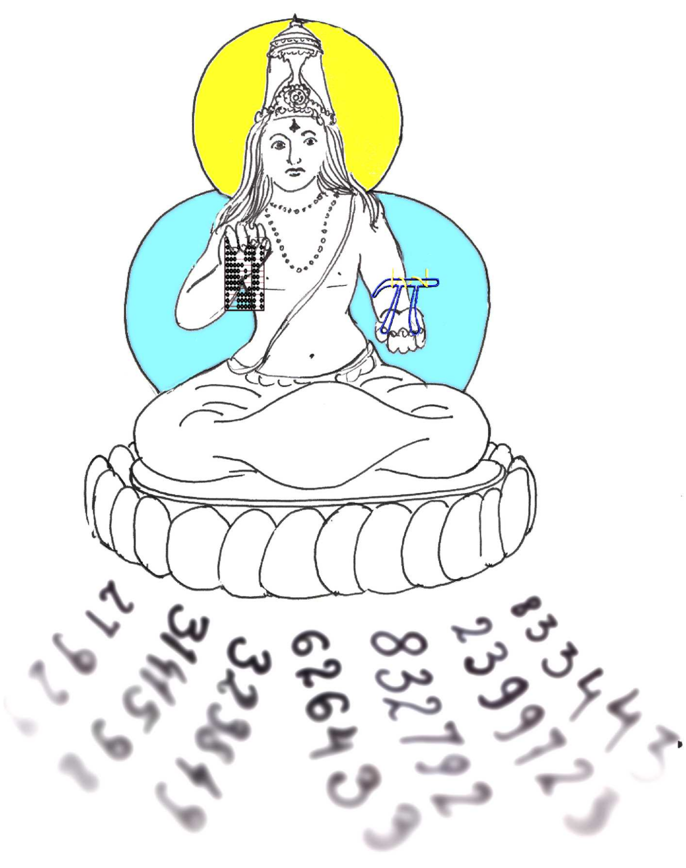
Concerning the applicability of the sixteen sutras to all mathematics, we can consult the Foreword to *Vedic Mathematics* written by Swami Pratyagātmānanda Saraswati. This Swami states that one of the sixteen sutras reads *Calanakalana*, which can be translated as *Becoming*. The Guru himself translates the sutra in question as "differential calculus" [4, p. 186]. Using this "translation" the sutra indeed promises applicability to a large area in mathematics; but the sutra is of no help in differentiating or integrating a given function such as $f(x) = 1/\sin x$.

Sceptics have tried to locate the sutras in the extant *Pariśīsta*'s (appendices) of the *Atharva-Veda*, one of the four Vedas. However, the sutras have never been found in authentic texts of the Vedic period. It turns out that the Guru had "seen" the sutras by himself, just as the authentic Vedas were, according to tradition, "seen" by the great Rishi's or seers of ancient India. The Guru told his devotees that he had "reconstructed" his sixteen sutras from the *Atharva-Veda* in the eight years in which he lived in the forest and spent his time on contemplation and ascetic practices. The book *Vedic Mathematics* is introduced by a General Editor's Note [4, p. vii], in which the following is stated about the sixteen sutras: "[the] style of language also points to their discovery by Śrī Swāmijī (the Guru) himself."

[2] The Guru does not give a proof of why the method works. Here is a simple number-theoretic proof: let n_k be the k-th decimal in the decimal expansion of 1/19 (thus $n_1 = 0, n_2 = 5, n_3 = 2\ldots$). Then for $k = 1, 2, \ldots$ we have $10^k = 19(10^{k-1}n_1 + 10^{k-2}n_2 + \ldots + 10n_{k-1} + n_k) + m_k$ where m_k is an integer such that $0 < m_k < 19$; thus $m_1 = 10, m_2 = 5, m_3 = 12\ldots$. Because $19n_k + m_k$ must be a multiple of 10, which implies that $-n_k + m_k$ must be a multiple of 10, we see that n_k is equal to the last digit of m_k, which is the remainder of 10^k after division by 19. Because $10^{18} \equiv 1 \mod 19$, we have $n_{18} = m_{18} = 1$. Since $2 \cdot 10^{k+1} = 20 \cdot 10^k \equiv 10^k \mod 19$, we have $2m_{k+1} \equiv m_k \mod 19$, hence $n_{17} = m_{17} = 2$, $n_{16} = m_{16} = 4, n_{15} = m_{15} = 8$, $m_{14} = 16$, and $n_{14} = 6$. Then $m_{13} \equiv 2 \cdot 16 \equiv 2 \cdot (10 + 6) \equiv 20 + 2 \cdot 6 \equiv 1 + 2 \cdot 6 \mod 19$. Thus the Guru correctly adds the penultimate digit 1 of the number 16 to two times the last digit 6. The same argument works for all other $m_i \geq 10$.

The Guru finishes his book on *Vedic Mathematics* with a (supposedly ancient) Indian verse that is a hymn to Krishna or Shankara. The Guru states that the verse also contains the value of $\pi/10$ expressed in what he calls an *Alphabetical Code-Language* [4, pp. 360]. We finish this article with a brief analysis of this verse, derived from [1]. In the verse, the decimals of π are encoded according to the Indian Kaṭapayādī system. This system was used by Indian astronomers in order to encode complicated numbers (mathematical and astronomical constants) into words and verses that were easy to remember. The system was developed in Kerala in South-India, probably in the ninth century CE. This dating already presents a problem in connection with the supposed "Vedic" origin of the verse. In order to explain an even more serious problem with the verse, we first give some details about the Kaṭapayādī system and then present the verse itself.

The Kaṭapayādī system works as follows. To each digit between 0 and 9, a small group of consonants of the Sanskrit alphabet is assigned (see the list below). This Kaṭapayādī system can be compared to, but is more complicated than the modern telephone keypad system 1 = ABC, 2 = DEF, and so on. To encode a digit between 0 and 9, we choose any Sanskrit consonant from the group belonging to this digit. For the digit 1, for example, we can choose among the four possibilities k, ṭ, p, and y (this is why the system is called Kaṭapayādī). Of course, each consonant can encode at most one digit. In order to dispel all illusions about the Guru's verse for π, it is necessary to list the possible encodings in detail (abbreviations such as ḍh represent a single Sanskrit consonant):



1 = k, ṭ, p, y;   2 = kh, th, ph, r;   3 = g, ḍ, b, l;

4 = gh, ḍh, bh, v;   5 = ṅ, ṇ, m, ś;   6 = c, t, ṣ;

7 = ch, th, s;   8 = j, d, h;   9 = jh, dh;   0 = ñ, n.

To make words and verses, the following rules are added. If two successive consonants (such as 'vr') are placed in a word only the last consonant r counts, and the v is ignored. A double consonant such as jj counts only once. Vowels and consonants that do not occur in the list do not have a numerical meaning, so they can be inserted anywhere without changing the numerical value of the word or verse. Thus the system was very flexible, and it was not difficult to compose words or verses for complicated (series of) numbers. The digits were encoded in reverse order.

A fifteenth century example from Kerala in South-India is the verse:

vidvāṃs
tunnabalaḥ
kavīśanicayaḥ
sarvārthaśīlasthiro
nirviddhāṅganarendraru
[2, p. 53].

The verse means:

The wise ruler whose army has been struck down gathers together the best of advisers and remains firm in his conduct in all matters; then he shatters the king whose army has not been destroyed [2, pp. 57–58].

This verse consists of five Sanskrit words, and the letters that carry numerical values are vv=44, tnbl=6033, kvśncy =145061, sv(th)śl(th)r=7475372, nv(dh)gnrrr=04930222. Because the digits were encoded in reverse order, the five words encode the sexagesimal numbers

$44/60^3$,
$33/60^2 + 06/60^3$,
$16/60 + 05/60^2 + 41/60^3$,
$273/60 + 57/60^2 + 47/60^3$,
$2220/60 + 39/60^2 + 40/60^3$,

which astronomers wanted to memorize because they occur in an expression used for computation of the Indian sine, equivalent to a modern Taylor series. These numbers are written in terms of the first three powers of 1/60, and they are rounded values of (in modern terms) $\frac{90}{11!}(\frac{\pi}{2})^{10}$, $\frac{90}{9!}(\frac{\pi}{2})^{8}$, $\frac{90}{7!}(\frac{\pi}{2})^{6}$, $\frac{90}{5!}(\frac{\pi}{2})^{4}$, and $\frac{90}{3!}(\frac{\pi}{2})^{2}$, respectively.

Now we know enough about the authentic Kaṭapayādī system to identify the origin of the Guru's verse about $\pi/10$. Here is the verse: (it should be noted that the abbreviation ṛ represents a vowel in Sanskrit):

gopī bhāgya madhuvrata
śṛṅgiśo dadhi sandhiga
khala jīvita khātāva
gala hālā rasandhara.

According to the Guru, decoding the verse produces the following number:

31415 92653 58979 32384 62643 38327 92.

In this number we recognize the first 31 decimals of π (the 32-th decimal of π is 5).

In the authentic Kaṭapayādī system, the decimals are encoded in reverse order. So according to the authentic system, the verse is decoded as

29723 83346 26483 23979 85356 29514 13.

We conclude that the verse is not medieval, and certainly not Vedic. In all likelihood, the Guru is the author of the verse.

Conclusion

There is nothing intrinsically wrong with easy methods of mental calculations and mnemonic verses for π. However, it was a miscalculation on the part of the Guru to present his work as ancient Vedic lore. Many experts in India know that the relations between the Guru's methods and the Vedas are faked. In 1991 the supposed "Vedic" methods of mental calculation were introduced in schools in some cities, perhaps in the context of the political program of saffronisation, which emphasizes Hindu religious elements in society (named after the saffron garments of Hindu Swamis). After many protests, the "Vedic" methods were omitted from the programs, only to be reintroduced a few years later. In 2001, a group of intellectuals in India published a statement against the introduction of the Guru's "Vedic" mathematics in primary schools in India.

Of course, there are plenty of real highlights in the ancient and medieval mathematical tradition of India. Examples are the real Vedic sutras that we have quoted in the beginning of this paper; the decimal place-value system for integers; the concept of sine; the cyclic method for finding integer solutions x, y of the "equation of Pell" in the form $px^2 + 1 = y^2$ (for p a given integer); approximation methods for the sine and arctangents equivalent to modern Taylor series expansions; and so on. Compared to these genuine contributions, the Guru's mental calculation are of very little interest. In the same way, the Indian philosophical tradition has a very high intrinsic value, which does not need to be "proved" by the so-called applications invented by Guru Tīrthajī.

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[6] Interesting web site on Vedic ritual: http://www.jyotistoma.nl.

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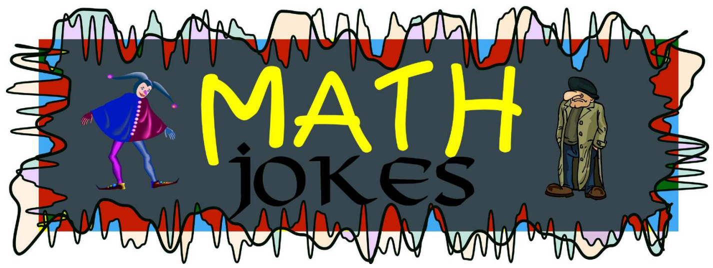

Q: "How does one insult a mathematician?"
A: "Your brain is smaller than any $\epsilon > 0$!"

Two mathematicians are studying a convergent series.
The first one says: "Do you realize that the series converges even when all the terms are made positive?"
The second one asks: "Are you sure?"
"Absolutely!"

Q: What does a mathematician present to his fiancée when he plans to propose to her?
A: A polynomial ring!

Q: Why don't mathematicians ever spend time at the beach?
A: Because they don't need the sun to get a tan, they have sin and cos.

Q: Why is it that derivatives can never finish telling a story?
A: Because they are always going off on tangents.