1. INTRODUCTION

One of Professor van der Waerden's main contributions to the history of mathematics was his work on the development of algebraic thought. His book "Geometry and Algebra in Ancient Civilizations" (Springer, 1983) provides a comprehensive overview of the mathematical ideas of ancient civilizations, including the Babylonians, Egyptians, and Greeks.

2. THE EARLY BABYLONIAN MATHEMATICS

Around 1800 B.C, the Babylonians were already using algebraic methods to solve problems. They were particularly interested in the Pythagorean theorem and its applications.

3. THE DEVELOPMENT OF EGYPTIANS

The Egyptians were also skilled in mathematics, particularly in the area of astronomy. They used their knowledge of the stars to create calendars and predict the seasons.

4. THE GEOMETRIC IDEAS OF THE GREEKS

The Greeks made significant contributions to mathematics, particularly in geometry. They were the first to develop a system of deductive reasoning, which is still used today in the field of mathematics.

5. TRACES OF EGYPTIANS IN ANCIENT CIVILIZATIONS

The Egyptians' influence on mathematics can be seen in the development of algebraic thought in Europe. Their use of the decimal system and the concept of zero have had a lasting impact on the way we think about numbers.

6. THE DEVELOPMENT OF ALGEBRA IN EUROPE

The development of algebra in Europe is closely tied to the work of mathematicians such as Pierre de Fermat and René Descartes. Their work laid the foundation for modern algebra.

7. THE DEVELOPMENT OF GEOMETRY IN EUROPE

The development of geometry in Europe is closely tied to the work of mathematicians such as Euclid and Archimedes. Their work laid the foundation for modern geometry.

Let us hope that in the near future all his research papers, which contain so many new and diverse results, will be brought together in a Collected Works of B.L. van der Waerden.
After 500 B.C. the Babylonians developed simple and not so simple arithmetical schemes for the prediction of celestial phenomena. Babylonian clay tablets containing astronomical computations have been excavated since the middle of the last century, and part of the complicated mathematical methodology was clarified by the pioneering work of three Jesuit scholars in Cernian around 1880. In 1955, O. Neugebauer published all available astronomical clay tablets in his Astronomical Cuneiform Texts (ACT).

Vander Waerden's main contribution to the understanding of Babylonian lunar theory will be presented here on the basis of clay tablet No. 1 published in ACT (vol. 3). Figure 1 displays part of Neugebauer's transcription of the front of the tablet. The tablet consists of columns of numbers, which are intermediary stages in the computation of the times of full moon and lunar eclipses. Like most other tablets, the tablet is severely damaged, and the numbers and symbols in square brackets are Neugebauer's restorations.  

![Figure 1](image-url)
In the seventh column we find values of a function C, which indicates the length of the synodic month. _G(X)_ is the time interval from the full moon in month _X_ -- 1 to the full moon in month _X_. The synodic month is always 29 days plus a fraction, and the Babylonians only recorded the fraction in "time" month #X -- 1 to the full moon in month #X. The synodic month is always 29 days plus a fraction, and the Babylonians only recorded the fraction in "time".

In the second position we find values of a function C, which indicates the length of the synodic month.
Jan P. Hogendijk difference, because the minima of \( \Omega \) come (1—) synodic months later than the maxima of \( \omega \).

Because the Babylonian time-degree corresponds to 4 modern minutes (units of time), the last sexagesimal in the numerator such as C2—4,46;42,57,46 indicates of a second. It is obvious that the Babylonians could not measure such small time-intervals. The functions \( \omega \) and \( C \) therefore belonged to a mathematical model, the motivation of which was unclear in 1955, when Neugebauer published his Astronomical Cuneiform Texts.

Around that time a clue was found by the famous Assyriologist A. Sachs, Neugebauer's colleague in the History of Mathematics Department at Brown University in Providence (11.1.). On a tablet for the computation of \( \omega \), Sachs was able to read the sentence "17,46,40 is the addition or subtraction for 18 years." We have seen the number 17,46,40 = before, in the computation of \( G \), from \( \omega \) near the extrema. Sachs observed that "18 years" is the Babylonian name for a period of 223 synodic months, which is nowadays called a Saros period. The Saros period is important in (ancient and modern) lunar theory because it is very nearly an integer multiple of the periods of other lunar phenomena.

Lunar and solar eclipses often occur with intervals of 1 Saros, so the Babylonians probably discovered the Saros period by analyzing the eclipse observations that had been made since the eight century B.C.

Sachs' work was continued by Neugebauer[5]. Neugebauer showed that \( \omega(X+223)—\omega(X)=\pm 17,46,40 \), so the passage which Sachs deciphered makes sense. Thus we can write

\[
\omega(X+223)—\omega(X)=\pm 4.
\]

Neugebauer noticed that \( \omega \) and \( C \) have the same period; hence also

\[
\phi_\omega(X+223)—\phi_\omega(X)=\sin(—25;48,38,31,6,40)\approx 4,
\]

he concluded

\[
\phi_\omega(X+223)—\phi_\omega(X)=\pm 4
\]

and he stated that "\( \omega \) measures time in comparing the length of lunations one Saros apart"[5,p.8]. Thus the unit of measurement of \( \omega \) is the time-degree. \( \omega \), however, the precise meaning of \( \omega \) and the connection between \( \omega \) and \( \phi_\omega \) were still unclear to Neugebauer.

Now Vander Waerden enters the scene. In the first edition of his book Science Awakening (1966)[8], pp. 148—153, Vander Waerden explained the phenomenon of \( \omega \) in terms of the so-called modern sexagesimal numerical system. The sexagesimal number is given by

\[
(x_1, x_2, x_3, \ldots, x_n) = x_1 \times 60^n + x_2 \times 60^{n-1} + \cdots + x_n \times 60^0.
\]

The number 17,46,40 is the decimal equivalent of the Babylonian sexagesimal number 17,46,40. The Babylonians used a base-60 number system, which was preferred for its divisibility by 2, 3, 4, 5, and 6. This is why many of the numbers in ancient texts are represented as sexagesimal fractions. For example, 17,46,40 is one-tenth of 1,746,400, which is 1,746,400/60 = 29,100.

In 1968 A. Aaboe[1] found a tablet which shows the system of modern sexagesimal numbers (units of time) used in ancient times. The tablet contains dates of the first appearance in the sky of a certain planet, which were recorded in the Babylonian calendar. The Babylonians used a sexagesimal numeral system, which allowed them to express fractions with great precision.
From the Indian point of view, reality is more complicated than the preceding model, which assumes that the motion of the planets is purely uniform circular motion. The Indian astronomers developed more complex models to account for the observed motions of the planets. The first step in this model is to define the epicycle, which is a small circle on which the planet moves uniformly. The center of this small circle is the apogee, which is the point where the planet is furthest from the Earth. The radius of the epicycle is called thedeferent, and it is used to account for the variation in the planet's velocity as it moves along its orbit.

In order to compute the position of the planet at a given moment, the Indian astronomers first found the angle $\alpha = \angle ZAE$, called the "centre angle," as a linear function of the time. Then, they computed the correction $\delta$ by elementary trigonometry, $\delta = LSE/A$. This gives the apparent position of the planet with respect to the apogee. The velocity of the other planets seen from the Earth depends to some extent on their positions relative to the Sun, but to a much larger extent on their positions with respect to the Earth. To account for the second effect, the Indian astronomers used a second type of epicycle, the sphaera-epicycle (Figure 4), which is a larger epicycle with a center $C$ and radius $r_0$. The center $C$ rotates uniformly on a deferent circle with radius $r$ and center the Earth $E$. For the superior planets, the planet $P$ moves uniformly on the sphaera-epicycle, such that the relation between $P$ and $G$ is also explained in Neugebauer's Almanac [6, vol. I, pp. 505-511], but I find his explanation less clear than that given by van der Waerden.

The Indian astronomers knew that the apogee $A$ is variable, but the variation is so slow that the effect is only noticeable after centuries. The term is derived from Sanskrit, "kendra" in Greek, suggesting a transmission from Greece to India. The Indians invented the sine function.

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Vander Waerden's analysis of the procedure described in the Khandakhadyaka of Brahmagupta (seventh century). I use the notation $u$ for the nanda—correction (ZMEA in the model without sighra-epicycle, Figure 3, with parameters $A$ and $r$ adjusted to the planet) and $a$ for the sighra—correction (TPEC in the simple model without nanda-epicycle, Figure 4).

1. Find angles $c$ and $a$ (as linear functions of time).

2. Compute the following quantities:

   \[ \sqrt{K^2 - K(3K - 1)} \]

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\[ + u \]

1. a

\[ + u \]

This procedure does not make the position of the planet.

Nevertheless, one might entertain the notion of the nanda-epicycle. In the nanda-epicycle, the center of the planet is fixed at the center of the epicycle, and the planet moves uniformly in a circle of radius $r$ centered at this point. The nanda-epicycle is used to model the motion of the planet in the sky, and it is based on the assumption that the planet moves uniformly in circles, with the center of the epicycle moving uniformly in a circle.

According to Vander Waerden, the Indian astronomers knew that the planetary positions were computed using an approximating function of position in the sky, which was a geometric model of the planet's motion. This approximating function was used to determine the position of the planet at any given time, and it was based on the assumption that the planet moves uniformly in circles, with the center of the epicycle moving uniformly in a circle.
In this case, an analysis of the common pattern of these motions is possible: the motion of the moon around the earth is not the same as the motion of the planets. This is because the moon's orbit is an ellipse, while the planets' orbits are somewhat more complex. The moon's elliptical orbit is due to the gravitational force of the Earth. The planets, on the other hand, follow a different set of forces, including the Sun's gravitational pull. This results in more complex orbits, which are not perfectly elliptical.

The significance of this analysis is that it provided a precise way to predict the positions of the planets, which was crucial for navigation and astrology. The Ptolemaic system of astrology was based on this understanding, and it remained the dominant system until the 16th century, whenCopernicus and other astronomers began to develop more accurate models of the solar system.

Eratosthenes' method was a significant improvement over previous models, but it still had limitations. It was not able to accurately predict the movement of the outer planets, such as Saturn and Jupiter. However, it was a major step forward in the development of astronomical models.

The understanding of the solar system and the planets' movements was crucial for the development of modern astronomy. It allowed astronomers to make more accurate predictions about the movements of the planets, which in turn allowed for more accurate navigation and timekeeping. The development of these models also laid the groundwork for the development of more advanced models, such as Kepler's laws of planetary motion.

Appendix

The reader who has read this far may wonder how accurate the ancient models are, or whether there are more accurate models. Is worth mentioning that there are various models of the two models to read. They are expressed in different times and places, and may not always be consistent with each other. The models are also based on different assumptions about the nature of the universe, and may not always be consistent with our modern understanding of it. The models described here are not exhaustive, and there may be other models that are more accurate or more useful in certain contexts.
I know of no detailed investigation of the overall accuracy of Babylonian (or Greek) lunar theories, but it seems that the Babylonian theories provided very reasonable predictions of lunar eclipses and first sightings of the lunar crescent. We have to bear in mind that the Babylonians possessed neither accurate clocks nor instruments for measuring the longitude and latitude of the moon at a given instant. It is historically interesting to ask how the complicated Babylonian theory of 4 and G could have been derived from observations. This question has been elucidated in a recent paper by Mrs. L. B. ACK—BRIN (2), see also J. P. B. VINTON (3).

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