Europe between 3000 and 2500 B.C. It should have left traces not only in the early Babylonian and Greek mathematics, but also with the Indians and Chinese. It should have been characterized in particular by an extraordinary knowledge about Pythagorean triples and by the theorem of Pythagoras on right triangles. An exciting exposition of these theses can be found in the book *Geometry and Algebra in Ancient Civilizations* (Springer, 1983) which is devoted to the algebra of the ancient times. A continuation entitled *A History of Algebra* (Springer, 1985) concerns the history of algebra from the Arabs until about 1930. Finally, van der Waerden's book *Sources of Quantum Mechanics* (North-Holland, 1967) is also of a historical nature, containing a rich source of material on the history of quantum mechanics.

Let us hope that in the near future all his research papers, which contain so many new and diverse results, will be brought together in a *Collected Works of B.L. van der Waerden.*

B.L. van der Waerden's Detective Work in Ancient and Medieval Mathematical Astronomy

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1. Introduction

One of Professor Van der Waerden's favourite research interests is the history of astronomy in Antiquity and the Middle Ages, a rather neglected subject closely related to the history of mathematics. In Antiquity and the Middle Ages, astronomy was the only field in which sophisticated mathematics was used, and a large part of mathematics was inspired by possible astronomical applications.

By mathematical and logical analysis of ancient and medieval astronomical sources, Van der Waerden has been able to discover many mathematical and historical connections that had not been noticed before. He has written a large number of papers on the following themes:

1. Babylonian astronomy, from the earliest Venus observations in the 16th century B.C. to the mathematical methodology that was used between 500 B.C. – A.D. 0. We will discuss an example below.
2. The development of astrology and its influence on astronomy.
3. The Pythagoreans (6th – 4th B.C.).
4. The astronomy of Ptolemy (A.D. 150), especially his *Handy Tables*.
5. Traces of Egyptian astronomy in the last few centuries B.C., and its relations with Babylonian astronomy.
6. Astronomy in pre-Islamic Persia (A.D. 250–650).
7. Indian astronomy between A.D. 400 and A.D. 1850 (at that time, Tamil astronomers were still using methods from ancient and medieval astronomy).

In the style of Van der Waerden, I will now present two concrete examples as illustrations of his methodology. These examples concern his discovery of the meaning of the function Φ in Babylonian lunar theory of the last centuries B.C, and his analysis of a medieval Indian method for computing planetary positions.

2. The Saros in Babylonian astronomy

Around 750 B.C., Babylonian scribes began to make systematic observations of celestial phenomena, which they recorded on clay tablets. These clay tablets were stored in archives, so that the observations could be used and analysed by later astronomers.[1] After 500 B.C. the Babylonians developed simple and not so simple arithmetical schemes for the prediction of celestial phenomena. Babylonian clay tablets containing astronomical computations have been excavated since the middle of the last century, and part of the complicated mathematical methodology was clarified by the pioneering work of three Jesuit scholars in Germany around 1880. In 1955, O. NEUGEBAUER [4] published all available astronomical clay-tablets in his *Astronomical Cuneiform Texts (ACT)*.

Van der Waerden's main contribution to the understanding of Babylonian lunar theory will be presented here on the basis of clay tablet No. 1 published in *ACT* (vol. 3). Figure 1 displays part of Neugebauer's transcription of the front of the tablet. The tablet consists of columns of numbers, which are intermediary stages in the computation of the times of full moon and lunar eclipses. Like most other tablets, the tablet is severely damaged, and the numbers and symbols in square brackets [ ] are Neugebauer's restorations.[2]

month		Φ		F	G
2,4	bar	1,59,35,33,20	...	1[1,26]	[4,55,39,15,33,20]
	gu_4	2,2,21,28,53,20	...	12, [8]	4, [33,55,33,20]
	sig	2,5,7,24,26,40	...	12,50	4, [8,6,54,48,53,20]
	šu	2,7,53,20	...	[13,32]	[3,42,18,16,17,46,40]
	[i]zi	2,10,39,[15],33,20	...	[14,14]	[3,16,29,37,46,40]
	[kin]	2,13,25,11,6,[40]	...	[14,56]	[2,50,46,40]
	[du_6]	[2],16,11,6,[40]	...	[15,38]	[2,40]
	[apin]	[2,1]5,12,35,33,[20]	...	[15,34]	[2,4]0
	[gan]	[2,12],2[6,4]0	...	14,52	2,41,46,4[0]
	[ab]	[2,9,40,44,26,40]	...	14,10	3,2,33,[5,11,6,40]
	[ziz]	[2,6,54,4]8,53,20	...	13,28	3,28,21,[43,42,13,20]
	[še]	[2,4,]8,53,20	...	12,46	3,54,10,[22,13,20]

FIGURE 1.

The Babylonian astronomers wrote the number $10p + q$ (for p, q integers, $0 \leq p \leq 5, 0 \leq q \leq 9$) as p left angle-brackets $<$, immediately followed by q thin wedges. Larger numbers and fractions were represented sexagesimally, and the different sexagesimals were separated by narrow spaces. Thus the number $293\frac{1}{2} = 4 \cdot 60 + 53 + 30 \cdot 60^{-1}$ was written as follows: four thin wedges, a narrow space, five angle-brackets plus three thin wedges, a narrow space, three angle-brackets. I will transcribe this sequence of wedges and angle-brackets according to Neugebauer's convention, thus: 4,53;30. The Babylonians had no special symbol to separate the integer from the fractional part of the number,

[1] Some of the Babylonian records were transmitted to Greece; Ptolemy (150 A.D.) used Babylonian observations of lunar eclipses that occurred in 721–720 B.C.

[2] The missing numbers can be restored as soon as one knows the mathematical structure of the columns. Sometimes it can be proved in this way that various fragments, which may even be in different museums, once belonged to the same tablet.

and therefore the reader should bear in mind that the same sequence can also be transcribed as 4;53,30 or 4,53,30. The correct interpretation can only be determined from the astronomical context. The Babylonian astronomers had a special symbol for zero, which was used between two non-zero sexagesimals, but never at the beginning or the end of a number.

The tablet in Figure 1 is divided (from left to right) into columns, which I have indicated by the modern names of the functions tabulated in them (month, Φ, F, G). For our purpose, only the first, second, sixth and seventh columns are important. I will first discuss what was known about these columns in 1955, the year in which the *ACT* was published.

The first column begins with a year number 2,4 (=124) in the Seleucid Era. A Seleucid year consists of 12 or 13 months, and a month is the period between two first sightings of the lunar crescent. In every 19-year cycle, 7 years of 13 months were intercalated, according to a certain scheme, between 12 years of 12 months, with the result that the beginning of the year was always close to the beginning of spring. Year no. 1 in the Seleucid Era began in April -310 (= April 311 B.C.), so the tablet is for the period between the spring of -187 and the spring of -186. The symbols "bar", "gu_4" etc. are the names of the successive months of the year. These names are indicated as I, II, ... XII in Figure 2.

The second column displays values of a function which is called Φ in the modern literature (following a convention of Neugebauer). The mathematical structure of Φ has been known since 1880. Φ is a zigzag-function; a zigzag-function is a periodic and piecewise linear function that oscillates between its maximum M(= 2,17,4,48,53,20) and minimum m(= 1,57,47,57,46,40) (see Figure 2 for a graph corresponding to Figure 1, but note that no such graphs are to be found on Babylonian clay tablets). The numbers are rendered in sexagesimal notation; therefore $M = 2 \cdot 60^5 + 17 \cdot 60^4 + 4 \cdot 60^3 + 48 \cdot 60^2 + 53 \cdot 60^1 + 20$ in units of the last sexagesimal. Because we do not know what this number means, we cannot say where the semicolon should be placed. In most cases, the difference between successive values $\Phi(X)$ and $\Phi(X+1)$ is $+d$ or $-d$, where $d = 2, 45, 55, 33, 20$ is a constant. Φ increases in this way until $\Phi(X) + d > M$, then $\Phi(X+1) = M - (\Phi(X) + d - M)$. Then Φ decreases, until $\Phi(X) - d < m$, then $\Phi(X+1) = m + (m - \Phi(X) + d)$. Now Φ starts to increase again, and so on.

The period of Φ is $\frac{2(M-m)}{d} = \frac{6247}{448} \approx 13.94$ months. Thus if the table of full moons is continued, the same values of Φ recur after an interval of 448 periods = 6247 months $\approx$ 505 years. Because Babylonian mathematical astronomy did not extend beyond the period between 500 B.C. and the year 0, and because in almost all tablets the same function Φ was used, we can use a single value of Φ to determine the date of a full moon tablet. This is of interest because most of the tablets we have are so badly damaged that all the year numbers and the names of the months are missing (tablet No. 1 is really an exception).

Kugler, the Jesuit scholar who deciphered the first astronomical tablets around 1880, thought that Φ was related to the apparent diameter of the moon. Neugebauer called Φ "monthly variation", because Φ is mathematically related

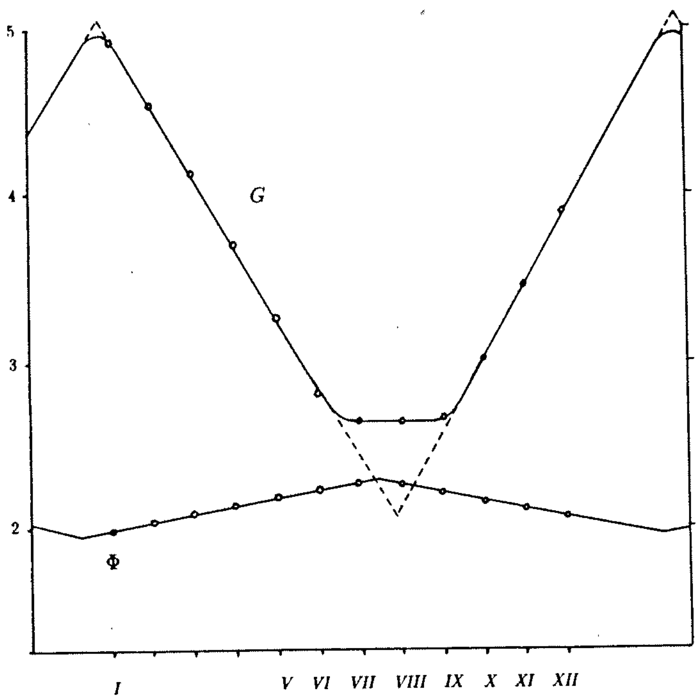

FIGURE 2.

to two other functions F and G, which are tabulated in the sixth and seventh column of the table (see Figure 1, in which the third, fourth and fifth columns, dealing with the position of the sun, the length of daylight and the lunar latitude have been omitted). To avoid confusion, the maximum, minimum and constant difference of Φ will be indicated from now on as M_Φ, m_Φ and d_Φ.

The sixth column displays values of a function F, the lunar daily motion in degrees. As a matter of fact, F is an abbreviated form of a function F^* which appears on other tablets and which is a zigzag-function with maximum M_{F^*} = 15;56,54,22,30, minimum m_{F^*} = 11;4,4,41,15, and constant difference d_{F^*} = 0;42,0,0,0. The unit of measurement is the degree (note that the Babylonian astronomers introduced the division of the circle into 360 degrees). Experience tells us that Φ and the unabbreviated function F^* are exactly in phase. The abbreviated function F on tablet No. 1 has the same constant difference, but since the maximum and minimum are rounded to $M_F = 15;57$, $m_F = 11;4$, the period of F is slightly different from that of F^* and Φ.

In the seventh column we find values of a function G, which indicates the length of the synodic month. $G(X)$ is the time interval from the full moon in month $X - 1$ to the full moon in month X. The synodic month is always 29 days plus a fraction, and the Babylonians only recorded the fraction in "time-degrees". One "time-degree" is the 360th part of one day, that is to say 4 modern minutes. In reality the length of the synodic month depends on the motion of the moon and the sun, and at a later stage a simple correction is made to account for the variable solar velocity. We are not concerned with this correction here.

The Babylonians found the values of G from Φ according to a complicated scheme, which can be restored on the basis of several clay tablets. These tablets are "procedure-texts", which give in cookbook-form the recipe for certain computations, without any motivation. I quote a short extract from one of these tablets (*ACT* vol. 1, pp. 203-4) with my own additions in square brackets:

> Opposite [Φ=] 2,13,20, decreasing, you shall put [G=] 2,40.
>
> ⋮
>
> Opposite [Φ=] 1,58,54,48,53,20, increasing, you shall put [G =] 4,56,35,33,20
> for everything up to [Φ=] 1,59,12,35,33,20, increasing, you shall put [G=] 4,56,35,33,20
> anything beyond [Φ=] 1,59,12,35,33,20, increasing to [Φ=] 1,59,30,22,13,20 increasing, multiply by 2 and subtract it from [G=] 4,56,35,33,20, you shall write it down
> etc. etc.

Mathematically, the relationship between G and Φ can be defined as follows (compare Figure 2, but note that no such figure is found on the clay tablets). Put G_1=2,53;20 and $G_2 = 4,46;42,57,46$.

If $\Phi = 2,10,40,0,0,0$ decreasing, or $\Phi = 2,13,8,8,53,2$ increasing, $G = G_1$. If $\Phi = 1,58,31,6,40,0$ decreasing, or $\Phi = 2,0,59,15,33,2$ increasing, $G = G_2$. For Φ decreasing from $2,10,40,0,0,0$ to $1,58,31,6,40,0$, and for Φ increasing from 2,0,59,15,33,20 to 2,13,8,8,53,20, G is a linear function of Φ.

The function G is rounded near the maximum and minimum. Between Φ = $1,58,31,6,40,0$ (decreasing) and Φ = 2,0,59,15,33,20 (increasing), and also between $\Phi = 2,13,8,8,53,20$ (increasing) and $2,10,40,0,0,0$ (decreasing), G is a piecewise linear function of Φ according to a complicated scheme, in which the domain is divided into more than 30 intervals, most of which have length equal to 17,46,40. I will not give the details here but refer to the table in VAN DER WAERDEN, [9] *Science Awakening* II, p. 224. The effect is that the maximum of G is M_G=4,56;35,33,20 and that the minimum of G is a nice number, namely m_G = 2,40. Note that $G = m_G$ over rather a long stretch (see Figure 2).

The function G can be considered as a modification of a zigzag-function $\hat{G}$ (dotted lines in Figure 2) with maximum $M_{\hat{G}} = 5,4;57,2,13,20$, minimum $m_{\hat{G}} = 2,4;59,45,11,6,40$, and constant difference[3] $d_{\hat{G}} = 25;48,38,31,6,40 = \frac{28}{3} \cdot d_\Phi$. We have $G = \hat{G}$ except near the maximum and the minimum. $\hat{G}$ and Φ have the same period of $\frac{6247}{448}$ synodic months and rather a curious phase

[3] In the formula relating $d_{\hat{G}}$ and d_Φ, the semicolon in $d_{\hat{G}}$ has to been omitted, since we do not yet know the meaning of Φ.

difference, because the minima of $\hat{G}$ come $\frac{1}{2} \cdot (1 - \frac{3}{28})$ synodic months later than the maxima of Φ.

Because the Babylonian time-degree corresponds to 4 modern minutes (units of time), the last sexagesimal in a number such as $G_2 = 4,46;42,57,46$ indicates $\frac{46}{900}$ of a second. It is obvious that the Babylonians could not measure such small time-intervals. The functions Φ and G therefore belonged to a mathematical model, the motivation of which was unclear in 1955, when Neugebauer published his *Astronomical Cuneiform Texts*.

Around that time a clue was found by the famous assyriologist A. Sachs, Neugebauer's colleague in the History of Mathematics Department at Brown University in Providence (R.I.). On a tablet for the computation of Φ, Sachs was able to read the sentence

"17,46,40 is the addition or subtraction for 18 years".

We have seen the number 17,46,40 $= \frac{3}{28} d_\Phi$ before, in the computation of G from Φ near the extrema. Sachs observed that "18 years" is the Babylonian name for a period of 223 synodic months, which is nowadays called a Saros period. The Saros period is important in (ancient and modern) lunar theory because it is very nearly an integer multiple of the periods of two other lunar phenomena, namely the anomalistic month (the mean period of lunar velocity) and the draconitic month (the mean period of lunar latitude). Lunar and solar eclipses often occur with intervals of 1 Saros, so the Babylonians probably discovered the Saros period by analysing the eclipse observations that had been made since the eight century B.C.

Sachs' work was continued by NEUGEBAUER [5]. Neugebauer showed that $\Phi(X+223) - \Phi(X) = \pm 17,46,40$, so the passage which Sachs deciphered makes sense. Thus we can write

$$\Phi(X + 223) - \Phi(X) = \pm\frac{3}{28} d_\Phi.$$

Neugebauer noticed that Φ and G have the same period; hence also

$$\hat{G}(X + 223) - \hat{G}(X) = \pm\frac{3}{28} d_{\hat{G}}.$$

Since $d_{\hat{G}} = 25;48,38,31,6,40 = \frac{28}{3} \cdot d_\Phi$, he concluded

$$\hat{G}(X + 223) - \hat{G}(X) = \pm d_\Phi$$

and he stated that "Φ measures time in comparing the length of lunations one Saros apart" [5, p. 8]. Thus the unit of measurement of Φ is the time-degree.[4] However, the precise meaning of Φ and the connections between Φ and G were still unclear to Neugebauer.

Now Van der Waerden enters the scene. In the first edition of his book *Science Awakening* (1966) [8], pp. 148-153, Van der Waerden explained the

[4] or: 60 time-degrees, called a "large hour".

mystery of Φ and the connections between Φ and G on the basis of the following idea. If we delete the circumflexes in the last formula of Neugebauer, we obtain $G(X+223) - G(X) = \pm d_\Phi$, a relation which will be valid most of the time (we are not yet concerned with details here). We have

$$G(X + 223) - G(X) = (G(X + 1) +$$
$$+ \ldots + G(X + 223)) - (G(X) + \ldots + G(X + 222)).$$

The quantity $(G(X + 1) + \ldots + G(X + 223))$ is the length of the Saros period which begins with the full moon in month X. This period is always 6585 days plus a fraction, and Van der Waerden then conjectured that Φ is exactly this fraction, measured in time-degrees. (Note that also in the case of G, the length of the synodic month, the Babylonian scribes omitted the constant number of days). Thus Van der Waerden conjectured that

$$G(X + 1) + G(X + 2) + \ldots + G(X + 223) = 6585 \text{ days } + \Phi(X) \text{ time-degrees.}$$

Using this definition of Φ, Van der Waerden could explain the phase difference between G and Φ. Since 16 periods of Φ (and of G) equal $223\frac{3}{28}$ mean synodic months, we have[5] $G(X + 223) = G(X - \frac{3}{28})$. Therefore
$\Phi(X) - \Phi(X - 1) = G(X + 223) - G(X) = G(X - \frac{3}{28}) - G(X)$.

Thus if the minimum of Φ is at $-\frac{1}{2}$, the maximum of $\hat{G}$ is at $-\frac{1}{2} \cdot \frac{3}{28}$, so the phase difference is $\frac{1}{2} \cdot (1 - \frac{3}{28})$.

Van der Waerden realised that the relation

$$G(X - \frac{3}{28}) - G(X) = \Phi(X) - \Phi(X - 1) \quad (*)$$

can be used to compute G from Φ as soon as all values of Φ and one initial value $G(X_0)$ are known. It turned out that this computation produced a zigzag-function rounded at the extrema (like G), but not exactly equal to the function G on the clay tablets. Van der Waerden also observed that the function G on the tablets is produced exactly if we replace Φ in (*) by a truncated function $\bar{\Phi}$, defined as follows:

If $\Phi \geq 2;13,20$ then $\bar{\Phi} = 2;13,20$.
If $\Phi \leq 1;58,31,46,40$ then $\bar{\Phi} = 1;58,31,46,40$.
If $1;58,31,46,40 \leq \Phi \leq 2;13,20$ then $\bar{\Phi} = \Phi$.

Thus Van der Waerden's interpretation led to the assumption that such a truncated function $\bar{\Phi}$ must have been used in Babylonian astronomy.

In 1968 A. AABOE [1] actually found a truncated $\bar{\Phi}$ on a cuneiform tablet, as predicted by Van der Waerden. We now know that the truncation at 2;13,20 was so important for the Babylonians that they actually named Φ after this

[5] The function G only makes astronomical sense for integer arguments. For fractional arguments such as $X - \frac{3}{28}$, G can be defined by means of the graph of the zigzag-function in Figure 2. Of course it is not clear whether the Babylonians defined G for fractional arguments; they have left us with numbers without motivations.

number. They would say "the 2;13,20 of this month is 1;59,48, ...", meaning: Φ of this month is 1;59,48, ... In the second edition of *Science Awakening* [9, pp. 226-229], Van der Waerden included the evidence found by Aaboe, but he omitted most of the reasoning which he had used to unravel the meaning of Φ.[6]

We now turn to another example of Van der Waerden's research, concerning a medieval Indian method for computing planetary positions, which appeared in the Khaṇḍakhādhyaka of Brahmagupta (7th century) and in a variety of other Sanskrit astronomical textbooks of the same period. First we have to explain some basic principles of geocentric motion and the corresponding technical terminology. From a geocentric point of view, the sun moves around the earth E in a period of one year, with a slightly varying velocity depending on its position in the ecliptic. The motion is slowest in the late spring, when the sun in the sign Gemini, and is fastest in the late autumn, when the sun is in the sign Sagittarius. The medieval Indian astronomers explained the solar motion as follows (Figure 3). They considered a point M moving uniformly in the plane of the ecliptic on a circle with centre the earth E, and they assumed that the sun S moved uniformly on a very small circle with centre M. The rotations of M and S are in opposite directions but with the same period (a year), so MS is always parallel to a line from E to a fixed point A, which is located in the sign Gemini in the ecliptic. Point A is the apogee; in other words S is at the greatest distance from the earth when M is at A.[7]. The large circle is called the deferent, and the Indian astronomers put its radius R equal to some integer (for example 150). They called the small circle a *manda*-epicycle, and they expressed its radius r_μ in the same units as R. Numerical values for the ratio $r_\mu : R$ and the position of A in the ecliptic were determined from observations of the sun or derived from earlier Indian or Greek sources.

In order to compute the position of the sun S at a given moment, the astronomers first found the angle $\kappa = \angle MEA$, called the "centre",[8] as a linear function of time. Then the astronomers computed by elementary trigonometry[9] the *manda correction* $\mu(\kappa) = \angle SEM$. Then $\kappa - \mu(\kappa)$ gives the apparent position of S with respect to the apogee A.

The velocity of the other planets as seen from the earth depends to some extent on their positions in the ecliptic, but to a much larger extent on their positions with respect to the sun. To account for the second effect, the Indian astronomers used a second type of epicycle, the *sighra-epicycle* (Figure 4), which is a larger epicycle with centre C and radius r_σ. The centre C rotates uniformly on a deferent circle with radius R and centre the earth E. For the superior planets, the planet P moves uniformly on the *sighra*-epicycle such

[6]The relation between Φ and G is also explained in Neugebauer's *HAMA* [6, vol. 1, pp. 505-511], but I find his explanation less clear than that given by Van der Waerden.

[7]The Indian astronomers knew that the apogee A is variable, but the variation is so slow that the effect is only noticeable after centuries.

[8]The term is *kentron* in Greek, "kendra" in Sanskrit, suggesting a transmission from Greece to India

[9]The Indian astronomers invented the sine-function.

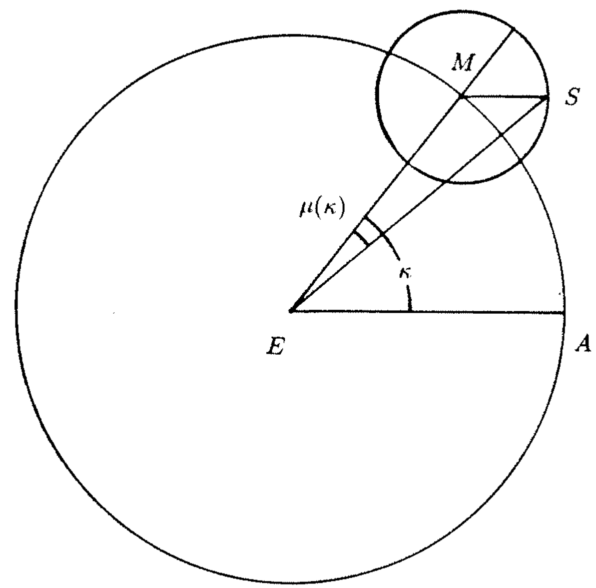

FIGURE 3.

that line PC is always parallel to EM, which is the line joining the Earth and the centre of the *manda-epicycle* of the sun.[10] In the case of Jupiter, for instance, P revolves on the *sighra*-epicycle in approximately 400 days,[11] the period between two superior conjunctions, and C revolves on the deferent in about twelve years, which, from the modern point of view, is the period of Jupiter's revolution around the sun.[12]. The radius r_σ of the *sighra*-epicycle was again expressed in the same units as the radius R of the deferent circle. The parameters of the model were derived from observations or copied from earlier sources.

Let EC intersect the epicycle at Q. The Indian astronomers found the *anomaly* $\alpha = \angle QCP$ as a linear function of time. From α, r_σ and the constant R they found by elementary trigonometry the so-called *sigra-correction* $\angle PEC = \sigma(\alpha)$.

From the Indian point of view, reality is more complicated than the preceding

[10]Thus point P on the *sighra*-epicycle does not revolve with the same angular velocity as point C on the deferent; the *manda*-epicycle revolves with the same angular velocity as the deferent, but in the opposite direction.

[11]The revolution is measured with respect to line EC.

[12]To make the connection with a heliocentric model, complete the parallelogram $ECQZ$ and suppose that the sun is at Z.

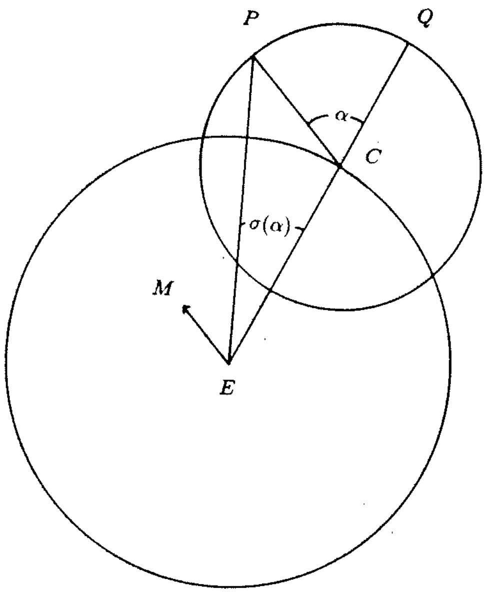

FIGURE 4.

model suggests. The Indian astronomers knew that the velocity of the planet also depends on its position in the zodiac, and they therefore assumed that each planet had a *sighra*-epicycle as well as a *manda*-epicycle like the sun (with a different fixed point A and a different r_μ for each planet). The next problem was how to combine the effects of the *manda*- and *sighra*-epicycles. The Indian astronomers prescribed various procedures for the computation, in the style of a cookbook, but they did not answer the question of how the *manda*- and *sighra*-epicycles fit together in one geometrical model.

I will now discuss Van der Waerden's analysis [7] of the procedure described in the Khaṇḍakhādyaka of Brahmagupta (seventh century). I use the notation μ for the *manda*-correction ($\angle MEA$ in the model without *sighra*-epicycle, Figure 3, with parameters A and r_μ adjusted to the planet) and σ for the *sighra*-correction ($\angle PEC$ in the simple model without *manda*-epicycle, Figure 4).

1. Find angles κ and α (as linear functions of time).
2. Compute the following quantities:
   $\kappa_1 = \kappa + \frac{1}{2}\sigma(\alpha)$
   $\kappa_2 = \kappa - \frac{1}{2}\mu(\kappa_1)$
   $\kappa_3 = \kappa - \mu(\kappa_2)$

   $\kappa_4 = \kappa_3 + \sigma(\alpha - \mu(\kappa_2))$.

Then κ_4 is assumed to be the arc between A and the position of the planet as seen from the Earth.

This procedure does not seem to make sense from a geometrical point of view. Of course it is not necessary to assume that the Indian astromomers explained the planetary motions according to what we would call a consistent model, involving a point P moving on an epicycle, the centre of which moves on another epicycle, etc. In Indian natural philosophy, the deviations of the planets from uniform motion were attributed to the deities *Sighrocca* "Conjunction" and *Mandocca* "Apogee", who were sitting in the zodiac, and pulling the planet in one way or the other, with their right or left hands, by means of cords of wind. Nevertheless, one may ask what the rationale of the procedure was. Van der Waerden came up with the following explanation.

First he showed by mathematical analysis that the κ_4 of the Indian astronomers is very nearly equal to the $\angle PEA'$ in the so-called equant model (Figure 5) in the *Almagest* of the Greek astronomer Ptolemy (which was written around A.D. 150). In this model, the planet P moves uniformly on an epicycle with centre C' and radius r', and C' moves on a deferent circle with radius R and centre F, which does not coincide with the earth E. Line EF extended intersects the ecliptic at A', which we can assume to be a fixed point. We now introduce the *equant point* G as the point on line EF such that $EF = FG$. We let GC intersect the epicycle at Q' and we put $\alpha' = \angle Q'C'P$ and $\kappa' = \angle C'GA'$. Ptolemy assumes that the motion is such that α' and κ' are linear functions of time. (Thus the motion of C is uniform with respect to the equant point G).

Using Taylor expansions Van der Waerden showed that $\kappa_4 \approx \angle PEA'$ if in Ptolemy's model we take $r' = r_\sigma, \cancel{e} = 2r_\mu, A' = A, \alpha' = \alpha, \kappa' = \kappa$. The EG approximation is very good if the quantities e^3, e^2r' and er'^2 can be ignored with respect to R^3. This is the case for all planets except Mars. e = EF = FG

According to Van der Waerden, the mathematical agreement between the Indian texts and the *Almagest* cannot be a coincidence. If we assume that the Indian model produces good approximations to the real positions, we can explain the coincidence, because Ptolemy's equant model can also be used to predict planetary positions accurately. However, Van der Waerden prefers the hypothesis that the mathematical resemblance between the equant model and the Indian procedure is the result of a historical connection. All modern historians believe that the Indian astronomers around A.D. 500 did not know Ptolemy's *Almagest*, because there is no trace of Ptolemaic trigonometry and Ptolemaic spherical astronomy in India. However, it is well known that Greek astronomical works written before Ptolemy (or at least, not in the Ptolemaic tradition) were transmitted to India. Van der Waerden now draws the following conclusions about the history of the equant model:

1. The equant model must have been invented in Greece before Ptolemy. (This is rather a revolutionary conclusion.)
2. The "Indian" procedure for the computation must have been invented in Greece, as part of an approximate computation of planetary positions in an equant model.

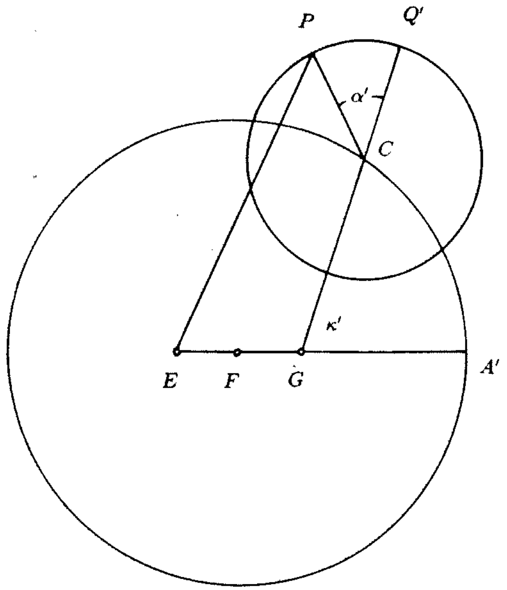

FIGURE 5.

3. The inventor must have been a great mathematician, probably Apollonius of Perga (ca. 200 B.C.), who worked with epicycles and knew trigonometry.
4. The approximate computation (but not the geometric model) were transmitted from Greece to India, probably via (pre-Islamic) Iran. Van der Waerden found traces of a similar (but simpler) method in the astronomical handbook of al-Khwārizmī (ca. A.D. 830), who used pre-Islamic Iranian sources.

There seems to have been little reaction to Van der Waerden's explanation in the modern literature. As far as I know, his conclusions regarding Apollonius of Perga as the possible author of the approximation method have not been accepted by other historians. However, his analysis is an interesting piece of work, which could lead to further research. It would be interesting to make a numerical comparison between the planetary positions predicted in the following three ways: by means of Ptolemy's equant model and by means of the Indian procedures (with historically attested parameters, and in a relevant time-interval, say, between 200 B.C. and A.D. 800), and according to modern recomputations. Thus one can try to find out whether the mathematical coincidence is the result of a historical connection or of a close correspondence

of the two models to reality. It is worth remembering that there are various similar Indian methods for predicting the planetary positions, and that Van der Waerden's analysis does not cover all variations in these methods, such as oscillations in the sizes of the epicycles during the revolutions of the planets, etc. To understand these methods, one has to familiarise oneself with the curious Sanskrit terminology and face the fact that many fundamental sources have not been edited or are only available in Sanskrit. Whatever the conclusions of future research may be, Van der Waerden is the first to have shown that the Indian method is sensible from a mathematical point of view.

In this case and in many other cases, Van der Waerden has used his own mathematical insight to elucidate the works of the ancient and medieval mathematicians and astronomers.

APPENDIX

The reader who has come this far may wonder how accurate the ancient astronomical theories are in comparison to modern ones. The following is an abstract from the non-technical exposition by O. NEUGEBAUER [6, vol. 3, pp. 1095–1108], to which the reader is referred for further details. According to Kepler's laws, the planets and the Earth move in ellipses with the sun at one of the foci. From a geocentric point of view, the sun therefore moves in an ellipse with the Earth at one of its foci. Because the eccentricity of the solar ellipse is small (about 0.0175), the solar motion is approximately uniform around the earth, and the maximal deviations of the sun from uniform motion are only 2 degrees of arc. It is therefore quite surprising that already the Babylonians were aware of the non-uniformity of the solar motion. If one assumes that the sun moves uniformly on a circle with centre outside the Earth, the error can be reduced to 1 minute of arc, provided that the centre of the circle is located suitably (at the second focus of the apparent elliptical orbit of the sun around the earth). The following argument shows that the heliocentric motion of a superior planet in a Keplerian ellipse can be approximated by Ptolemy's geocentric equant model. Suppose that the earth is E, the sun is Z and the planet is P and draw a parallelogram $EZPC$. Now we let P move around C in the same way as the sun Z moves around E, and we let C move around E in the same way as the planet P moves around Z. Because the apparent motion of the sun around the earth is approximately circular, we can assume that the motion of P around C is circular. The Ptolemaic motion of C around E (on a deferent circle with equant point) is a very good approximation of the motion of P around Z in a Keplerian ellipse. The distance of the planets is badly distorted in the Ptolemaic models, because Ptolemy locates the planets (with their epicycles and deferents) in concentric spheres with the earth as centre. The reader should bear in mind that planetary distances could not be measured by ancient methods, because the parallaxes of the planets (that is, their apparent displacements because the observer is not in the centre but on the surface of the earth) are too small to be observed with the naked eye.

The motion of the moon around the earth is very complex, because the lunar orbit is essentially a solution of the three-body problem of classical mechanics.

I know of no detailed investigation of the overall accuracy of Babylonian (or Greek) lunar theories, but it seems that the Babylonian theories provided very reasonable predictions of lunar eclipses and first sightings of the lunar crescent. We have to bear in mind that the Babylonians possessed neither accurate clocks nor instruments for measuring the longitude and latitude of the moon at a given instant. It is a historically interesting question how the complicated Babylonian theory of Φ and G could have been derived from observations. This question has been elucidated in a recent paper by Mrs. L. Brack-Bernsen [2], see also J.P. Britton [3].

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Van der Waerden's Foundations of Algebraic Geometry

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A description of Van der Waerden's foundations of algebraic geometry is preceded by a discussion of the classical Italian conceptions and followed by some remarks as to the usefulness of Van der Waerden's concepts at present.

This talk has naturally three parts, corresponding to three periods in time. The middle part is on Van der Waerden's set-up as given in a series of papers republished in 1983 [2] and in his *Einführung* [1] of 1939 (second edition 1973). This is roughly the period 1925-1975. In order to appreciate his work, we must go back to the situation from where he started: in the later part of the Italian algebraic geometry, when Severi was its dominant leader. So, the first part of the talk is about Severi's ideas; the period of Severi's work is about 1900–1950. For the third part, take the period 1950–2000. I cannot escape the somewhat tricky question: what is, or will be, the lasting significance of Van der Waerden's foundations? Do not expect a definite answer; I shall only make a few remarks.

1. Severi's algebraic geometry

The Italians' algebraic geometry was complex algebraic geometry, the study of algebraic varieties embedded in complex projective space. Their aim was to further generalize the theory of algebraic curves (abelian integrals, theorem of Riemann-Roch, Jacobian variety, classification,...) to surfaces and higher-dimensional varieties. They had obtained a rich collection of results, but some mathematicians complained about unclear definitions and incomplete proofs. Newcomers to the field became frustrated. They experienced difficulties similar to the ones we mathematicians have nowadays in following the arguments of mathematical physicists (see [5], where the Italian algebraic geometry is especially mentioned).

In 1949, Severi gave, in some talks, an explanation of the Italian way of thinking [9]. It was, what Severi called, synthetic, as opposed to analytic, and intuitive. And indeed, his writings are not overcrowded with formulae. One avoids calculations, imagines to be calculating, without doing it actually. Severi distinguishes *substantial* rigor from *formal* rigor. An example might be an inductive proof by showing a statement for $n = 1, 2, 3$, and say "etc." instead