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AL-KĀSHĪ'S FUNDAMENTAL THEOREM

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Abstract: In the first section of al-Risāla al-muhītīyya (1424), al-Kāshī proved the following seminal theorem: If on a semicircle with the diameter $AB = 2r$ and the center E we consider an arbitrary arc AG, and we call the middle of the arc GB the point D, and we draw the chord AD, then $r(2r + \overline{AG}) = \overline{AD}^2$. This theorem is the foundation of al-Kāshī's arguments and calculations in al-Risāla al-muhītīyya, so we call this theorem "al-Kāshī's fundamental theorem." For proof, he used four propositions from Euclid's Elements. In this article, we present, for the first time, an English translation of al-Kāshī's fundamental theorem, using the original manuscript of al-Risāla al-muhītīyya, handwritten by al-Kāshī himself. Also, we comment on the German translation and commentary by P. Luckey, the Persian summary by A. Qurbānī, as well as Russian translations and commentaries by B. Rosenfeld, A. Youschkevitch, and M. Chelebi.

AMS Subject Classification: 01A30
Key Words: al-Kāshī, al-Kāshānī, Kāshānī, Euclid's Elements, Lambert identity, pi, al-Risāla al-muhītīyya

1. Introduction

One of the foremost mathematical achievements of Ghiyāth al-Dīn Jamshīd Mas'ūd al-Kāshī (also known as Kāshī, al-Kāshānī, or Kāshānī)

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is al-Risāla al-muhītīyya (The Treatise on the Circumference), which was written longhand in Arabic by al-Kāshī himself (see the Note), and was completed in July 1424 (Sha'bān 827 A.H.L.) [2]. This original manuscript of al-Risāla al-muhītīyya was donated by Nāder Shāh (King Nāder) in 1145 A.H.S. to the library of the Āstāne Qudse Razawi, Mashhad, Iran [1]. In this masterpiece of computational techniques, al-Kāshī tested his extraordinary computational talents to produce fascinating results by means of elementary operations. These multi-step and tedious calculations required a high degree of precision, determination, dedication, and persistence: characteristics which describe al-Kāshī himself.

In the first section of al-Risāla al-muhītīyya, al-Kāshī determines the chord of an arc which is the sum of an arc of a known chord and half of its supplement to the half circle. Although this was the foundation and the main ingredient of all of his calculations in this treatise, he did not state it as a theorem. He stated and proved the following seminal theorem which we call al-Kāshī's fundamental theorem: If on a semicircle with the diameter $AB = 2r$ and the center E we consider an arbitrary arc AG, and we call the middle of the arc GB the point D, and we draw the chord AD, then $r(2r + \overline{AG}) = \overline{AD}^2$.

Al-Kāshī's proof of his fundamental theorem is based on four different propositions from Euclid's Elements [3] that he called principles. However, he never mentioned Euclid or Euclid's Elements in his proof: an indication that Euclid's Elements had been well-known and widely used for geometrical proofs at that time. In the proof he used Proposition Thirty-One, but erroneously he referred to it as the Thirtieth Principle instead of the Thirty-First Principle.

Our goal in this article is to use the original manuscript of al-Risāla al-muhītīyya and present, for the first time, an English translation of al-Kāshī's fundamental theorem.

2. Note

At the conclusion of al-Risāla al-muhītīyya [1], al-Kāshī writes, "katabahoo muallefohoo", which means "written by its [the] author". Also, several sources confirm that [1] is indeed the authentic copy of al-Risāla al-muhītīyya, including Abu'l-Qāsim Qurbānī [6, p. 130, 1989 edition] and the staff at the central library of the Āstāne Qudse Razawi. Moreover, on the second page of [1], the authenticity of this copy has been

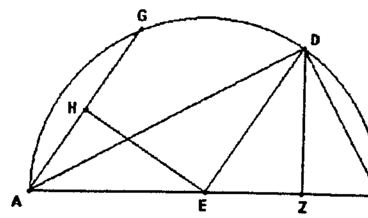

Figure 1:

certified by Bahā al-Dīn al-Āmelī, who had this copy of al-Risāla al-muhītīyya in his possession for some time.

3. Al-Kāshī's Fundamental Theorem

In this section we present a translation of the first section of al-Risāla al-muhītīyya, which we call al-Kāshī's fundamental theorem. We insert numbers in the Arabic manuscript to indicate the beginning of a sentence in the English translation. The number at the beginning of each sentence in the English translation corresponds to the same number in the Arabic manuscript. Extra words or phrases in brackets have been inserted by this author for the purpose of clarification of a word or sentence. Section one of al-Risāla al-muhītīyya starts at 1 in Figure 2 and ends at 17 in Figure 2. Al-Kāshī used red ink made of crushed berries to write the section headings as well as to draw figures. In Figure 2 the section heading (Section 1) and the drawing of his proof is not clear, for this ink did not copy well in black and white. Here now is the translation.

[1]The First Section. [2]On determining the chord of an arc which is the sum of an arc of known chord and half of its supplement to the half circle. [3]I say that the surface [the area, the rectangle, the product] of the sum of the whole diameter and the chord of any arc less than half of the circumference by half of the diameter is equal to the square of the chord of the arc that is equal to the sum of the first arc and half of its supplement to the half of the circle .

[4]For illustration we draw on the line [segment] AB the half circle AGB over the center E and we make arbitrarily, the chord AG and we halve BG, its supplement [the supplement of the arc AG] to the half circle, at the point D and we connect AD (Figure 1). [5]We claim that

the surface of half of the diameter by the sum of AB and [the chord] AG is equal to the square of [the chord] AD.

[6]For proof we connect BD, then the angle ADB will be a right angle according to the Thirtieth Principle [Proposition 31] of the Third [Book of Elements]. [7]Then, we draw out of the point D a perpendicular DZ to the line [segment] AB. [8]Next, it makes [we have] two similar triangles $DBZ$and DAZ that are similar to the triangle ADB according to the Eighth Principle [Proposition 8] of the Sixth [Book of Elements]. [9]Then, the ratio of the diameter AB to [the chord] AD will be the same as the ratio of [the chord] AD to AZ, and according to the Nineteenth Principle [Proposition 19] of the Seventh [Book of Elements], the surface of the diameter AB by AZ is equal to the square of [the chord] AD. [10]Then, we draw out of the point E a perpendicular EH to [the chord] AG, to make the point H at the middle of [the chord] AG, according to the Third Principle [Proposition 3] of the Third [Book of Elements]. [11]And we connect ED. [12]And since the inscribed angle BAG is equal to half of the arc GB, which is equal to the angle BED, then the two angles are equal. [13]As a result, the two triangles AHE and EZD are equal, because angles H and Z are right [angles], angles at E and A are equal, and two sides AE and ED are equal, that makes the side EZ equal to the side AH, which is half of [the chord] AG. [14]And then, the surface of AZ, I mean the sum of half of the diameter and EZ, actually half of [the chord] AG, by the diameter is equal to the square of [the chord] AD. [15]And since the surface of one of the two lines [line segments] by half of the other is equal to the surface of half of the first by the whole of the other, this makes the surface of the sum of the diameter and double of EZ, I mean the sum of the diameter and [the chord] AG, by half of the diameter equal to the square of [the chord] AD, and this is what we wanted [to prove].

[16]If [the chord] AG were known in the parts [that is, if the length of the chord AG were known] that makes half of the diameter sixty, we add to it the diameter, and we raise the sum by a [sexagesimal] place, the result will be the square of [the chord] AD.

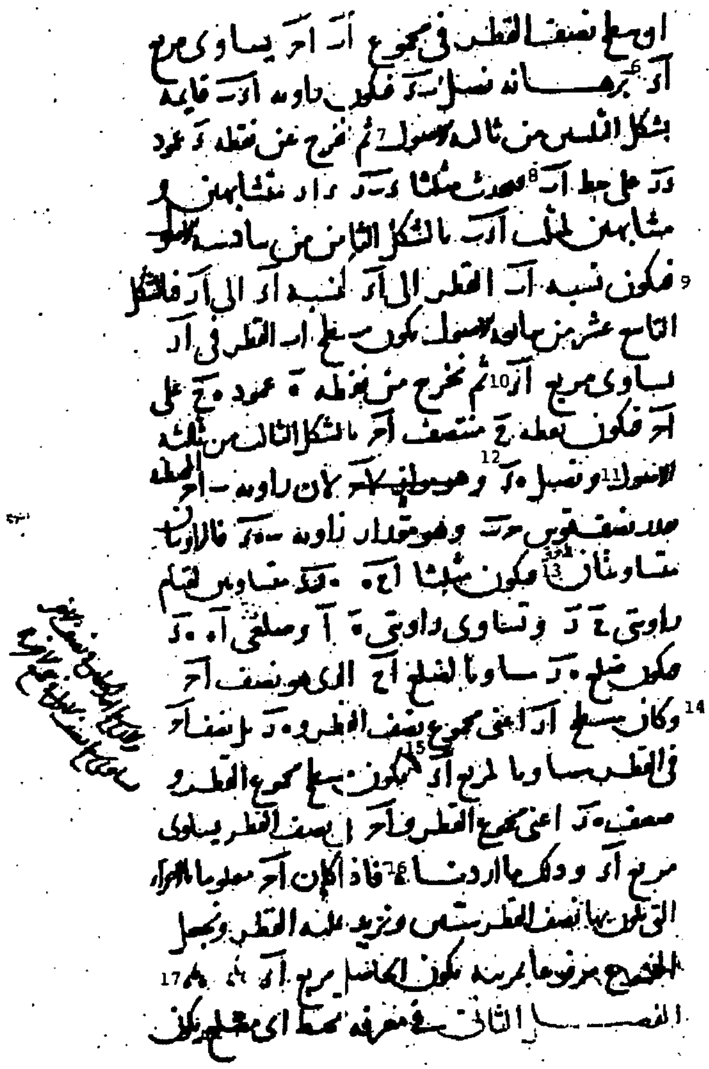

Figure 2: Continuation

4. Comment

To verify the last paragraph of the proof let us assume that $\overline{AG} = 30$. Then $r(2r + \overline{AG}) = 60(120 + 30) = 60(150) = 9000 = (2, 30, 0)_{60}$ and $150 = (2, 30)_{60}$, which affirms al-Kāshī's statement.

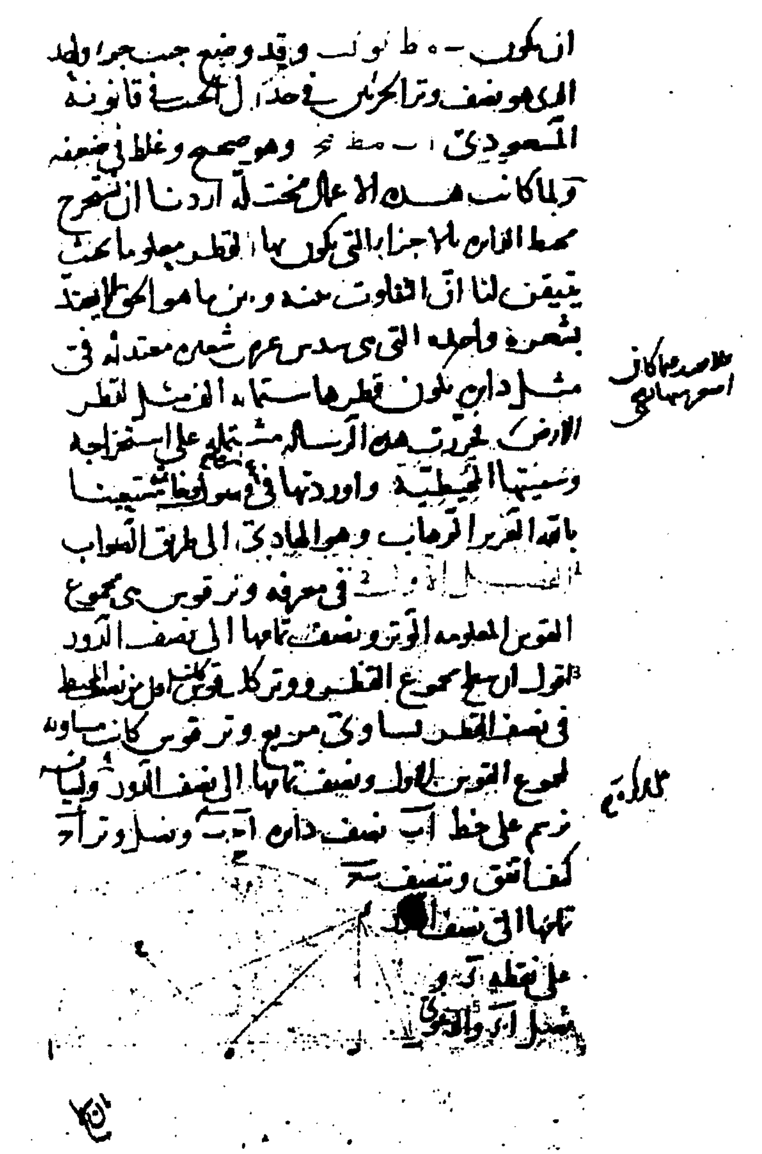

Figure 2: First section of al-Risāla al-muḥīṭīyya (1424). Courtesy of the Astane Qudse Razawi, Mashhad, Iran

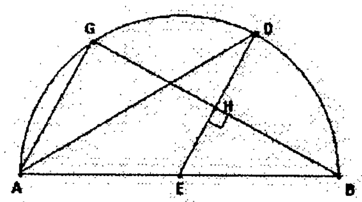

Figure 3:

5. Another Approach

Although al-Kāshī's proof is short and simple, it certainly is not unique. Here we present another proof of al-Kāshī's Fundamental Theorem using modern notations.

Recall that we want to prove that if on a semicircle with the diameter $AB = 2r$ and the center E we consider an arbitrary arc AG, and we call the middle of the arc GB the point D, and we draw the chord AD, then $r(2r + \overline{AG}) = \overline{AD}^2$.

For proof we connect GB and ED, and we let H be the point where GB and ED intersect at right angles (Figure 3). Now, from the right triangle BHD we have

$$\overline{DB}^2 = \overline{BH}^2 + (r - \overline{EH})^2. \tag{1}$$

Similarly, from the right triangle ADB we obtain

$$\overline{DB}^2 = 4r^2 - \overline{AD}^2. \tag{2}$$

Next, from (1) and (2), we deduce that

$$\overline{BH}^2 + (r - \overline{EH})^2 = 4r^2 - \overline{AD}^2. \tag{3}$$

Also, since two right triangles AGB and EHB are similar triangles,

$$\overline{BH} = \frac{\overline{BG}}{2} \text{ and } \overline{EH} = \frac{\overline{AG}}{2}.$$

Thus (3) can be rewritten as

$$\left(\frac{\overline{BG}}{2}\right)^2 + \left(r - \frac{\overline{AG}}{2}\right)^2 = 4r^2 - \overline{AD}^2. \tag{4}$$

Moreover, from the right triangle AGB we get

$$\overline{BG}^2 = 4r^2 - \overline{AG}^2. \tag{5}$$

Finally, putting the value of $\overline{BG}^2$ from (5) into (4) and simplifying we achieve the desired equality.

6. Some Immediate Applications of al-Kāshī's Fundamental Theorem

It takes a brilliant mind to come up with such a concept: simple, yet with remarkable applications. This theorem certainly gave al-Kāshī a distinct advantage over his predecessors by having found the most accurate value of π at that time. From his fundamental theorem, al-Kāshī deduced that in a unit circle if the arc AG is equal to 2ϕ radians, then $\overline{AD} = 2\sin\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$, and from his fundamental theorem al-Kāshī obtained the identity

$$\sin\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \sqrt{\frac{1+\sin\phi}{2}} \quad (\text{where } \phi < \frac{\pi}{2}),$$

which, as the trigonometric form of his fundamental theorem, is now known as the Lambert identity. It is interesting to note that this identity appeared in Johann Heinrich Lambert's work in 1770, while al-Kāshī proved it before 1424 (827 A.H.L.), some 346 years earlier than Lambert. Al-Kāshī applied his fundamental theorem to calculate successively the value of the chord c_n of the arc $\alpha_n^o = 180^o - \frac{360^o}{3(2^n)}$ (where $n \geq 0$). From here he found the lengths of the sides of inscribed and circumscribed regular polygons each with $3(2^n)$ sides ($n \geq 1$), in a given circle. Then he determined the number of sides of the inscribed regular polygon in a circle whose radius is six hundred times the radius of the Earth in a such a way that the difference between the circumference of the circle and the perimeter of the inscribed regular polygon in this circle will become less than the width of a horse's hair. Al-Kāshī continued to use his fundamental theorem to calculate the value of π, correct to 16 decimal places, using inscribed and circumscribed polygons, each with $3(2^{28}) = 805{,}306{,}368$ sides. In an article by Glen Van Brummelen's, al-Kāshī's argument here has recently been paraphrased and explained [9]. The best approximation of π correct to six decimal places obtained before al-Kāshī was by the Chinese scientist Tsu Ch'ung-chih, around 480 A. D. While al-Kāshī's approximation of π seems ordinary to us, knowing over 1.24 trillion digits of π (Yasumasa Kanada, September 2002, in [8]), this approximation of π, in 1424, was a remarkable achievement for that time, and it far surpassed all approximations of π by all previous mathematicians throughout the world, including Archimedes, Ptolemy, and Muhammad al-Khwarizmi (Archimedes' approximation in 250 B.C. was $\pi = 3.14$, Ptolemy's result in circa 150 A.D. was $\pi = 3.14166$, and al-Khwarizmi's approximation of π around 800 A.D. was 3.1416). It took European scientists 186 years to improve on al-Kāshī's approximation of π. François Viète, 155 years later in 1579 obtained a value for π, correct only to 9 decimal places. Adriaan Van Roomen, in 1593, calculated the value of π correct to 15 decimal places and finally, Ludolf van Ceulen obtained the value of π correct to 35 decimal places in 1610.

7. Comments on German, Persian, and Russian Manuscripts

Paul L. Luckey was the first to write an impressive and comprehensive German translation of the al-Risāla al-muhītiyya, which of course included al-Kāshī's fundamental theorem, in 1949, by using the manuscript number 576 of the Istanbul Military Museum. Unfortunately, he did not live long enough to see the fruits of his labor in published form. P. Luckey's German translation with an eloquent and detailed commentary along with the Arabic text was published in Berlin, posthumously in 1953 [4]. Like al-Kāshī himself, P. Luckey in the translation of al-Risāla al-muhītiyya, did not present the first section as a theorem either.

The typed Arabic manuscript of the first section of al-Risāla al-muhītiyya that was used by P. Luckey and was published along with his translation contains a typographical error and a deletion of a word. In the third line of the second paragraph the correct word is "qā'ima", which means "perpendicular", and not "qā'ina", which does not have any mathematical meaning. In the fourth line of the second paragraph, the word "mushābehīn", which means "similar", is missing. However, neither one caused a problem with the translation of the proof. In the first case, Luckey's translation of "qā'ina" is that of "qā'ima", which is correct. In the second case, the missing word did have an effect on his translation, but even then, mathematically his translation is correct. His translation with the missing word, using my notations, reads "... we have two triangles $DBZ$and DAZ that are similar to the triangle ADB ...", when indeed it should read, "... we have two similar triangles DBZ and DAZ that are similar to the triangle ADB ...". But, clearly, if two triangles are similar to a third triangle, they themselves are similar as well.

A. Qurbānī in Kāshānī Nāmeh [5], a book written in Persian about al-Kāshī and his achievements in mathematics and astronomy, stated the result of the first section of al-Risāla al-muhītīyya as al-Kāshī's fundamental theorem, without presenting al-Kāshī's proof. However, he did emphasize the importance of this theorem and presented some applications.

Also, Boris A. Rosenfeld and Adolf P. Youschkevitch, wrote a Russian account of al-Risāla al-muhītīyya with commentaries in 1954 [7]. According to A. Qurbānī and Professor B. A. Rosenfeld himself, the typed manuscript that was available to Rosenfeld and Youschkevitch contained some errors, which carried over into the Russian translation. However, the corrected version of [7] along with extended commentaries was published in 1956 in a book by B. A. Rosenfeld [6]. It is interesting to note that this book [6] contains an exposition by Mirim Chelebi who was the grandson of both Qādī zāde al-Rūmī and Āli al-Qushji.

Acknowledgments

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