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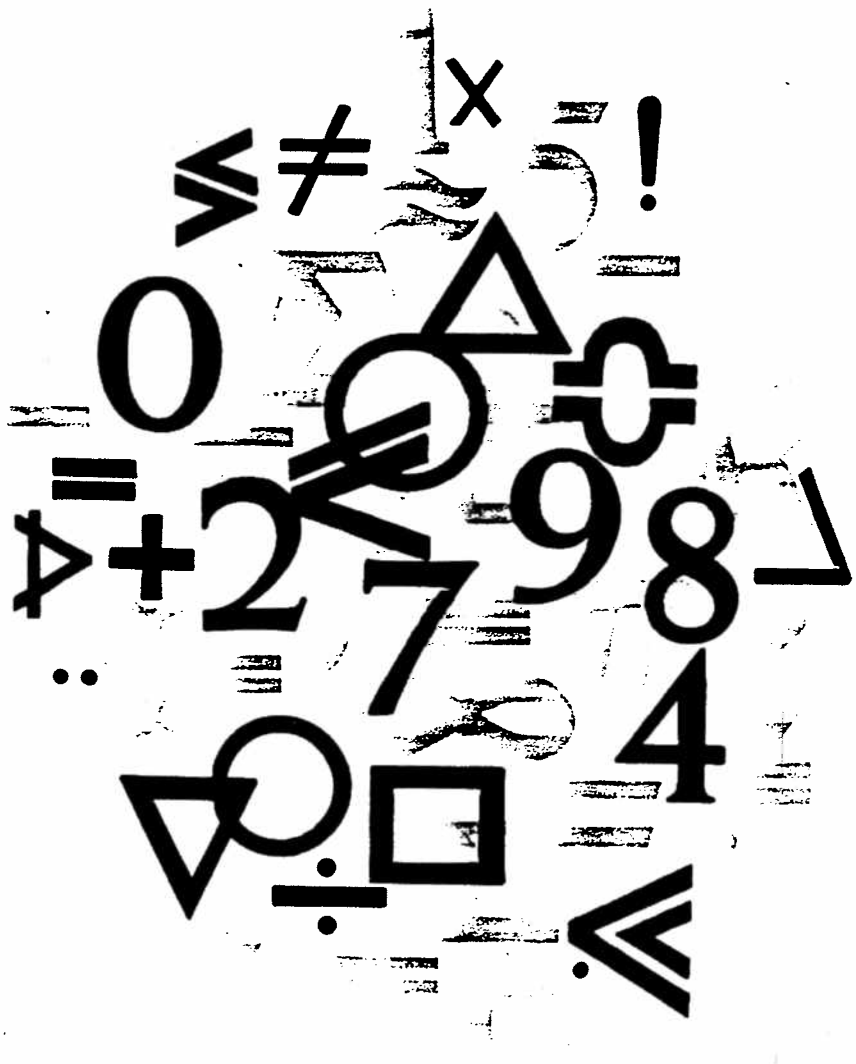

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A SUMMARY OF THE MATHEMATICAL WORKS OF GHIYATH UD-DIN JAMSHID KASHANI

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ABSTRACT

In this article we briefly study Ghiyath ud-din Jamshid Kashani's personal life as well as his mathematical achievement. We discuss his **Meftah al-hesab, Resaleh-e muhitiyyah, Resaleh-e vatar va jaib,** and **Talkhieth al-meftah.** In particular, we look at his following works: extraction of the nth root; expansion of $(a+b)^n$; calculation of the radii of inscribed and circumscribed circles for a regular n-gon as well as the area of the n-gon; fundamental theorem; approximation of π up to sixteen decimal places; creation and solution of the cubic equation $ax = b + x^3$; approximation of sin 1° up to twenty-two decimal places; decimal fractions.

Introduction

Information concerning Kashani and his achievements in mathematics is scattered not only in Iranian libraries and the libraries of other Middle Eastern countries, but also in libraries throughout the world as well. Originally, all of Kashani's work was written in Arabic or Persian, but now most of his well-known work has been summarized and/or translated into English, French, German, Russian, Persian, and other languages.

His name, as he himself has written in the beginning of his works, is Jamshid-ibn Masud-ibn Mahmud-e Tabib-e Kashani mulaqqab be Ghiyath ud-din.

Mulaqqab be Ghiyath ud-din means "known as Ghiyath ud-din." Kashani (or Kashi) means that he is "from the city of Kashan" and ibn means "the son of." Jamshid Kashani (or Kashi) and Jamshid Al-Kashani (or Al-Kashi) are among several variations of his name. Al-Kashani and Al-Kashi are the Arabic versions of his name. Kashani was born in the second half of the fourteenth century in the Tamerlane's (Timur) empire. Unfortunately, Kashani's exact birth date is not known. His birth place is Kashan, a city in the central part of Iran about 160 miles south of the capital Tehran and about 300 miles from the Caspian Sea.

Kashani was one of the most renowned mathematicians in the Islamic world, and he was known for his extraordinary ability to perform very difficult mental computations. Among the many nicknames given to Kashani one could name: "the real pearl of the glory of his age," "the second Ptolemy," "the king of engineers," "the reckoner," and "our master of the world." "A remarkable scientist, one of the most famous in the world, who had a perfect command of the science of the ancients, who contributed to his development, and who could solve the most difficult problems," stated Sultan (King) Ulugh Beg in the introduction to his own zij (zij referring to astronomical tables or astronomical instruments) [2]. Ulugh Beg, himself a great scientist, had much respect for the work of other scientists. Although Kashani is known as a mathematician and astronomer, he was a physician as well. Nothing is documented of Kashani's life until June 2, 1406 (Dhu-al-hijjah 12, 808 A.H.L.), when he observed a lunar eclipse in his native town of Kashan. To support himself financially, like many other scientists of the Middle Ages, he dedicated most of his known scientific treatises to monarchs or other influential rulers of his time in order to get compensation.

Contrary to the ambiguity of Kashani's birth date, the exact date and place of his death is available from several reliable sources. He died on the morning of Wednesday, June 22, 1429 (Ramadan 19, 832 A.H.L.) outside of Samarqand at the observatory he had helped to build in Uzbek (or Uzbekistan). Uzbekistan, part of the old Islamic heartland, is located in the former U.S.S.R., in northeast Iran, in central Asia. While Kashani was one of the closest collaborators and consultants to Ulugh Beg, at times Ulugh Beg was embarrassed by Kashani's poor manners and lack of refinement. It is believed by some that Ulugh Beg might have caused Kashani's death.

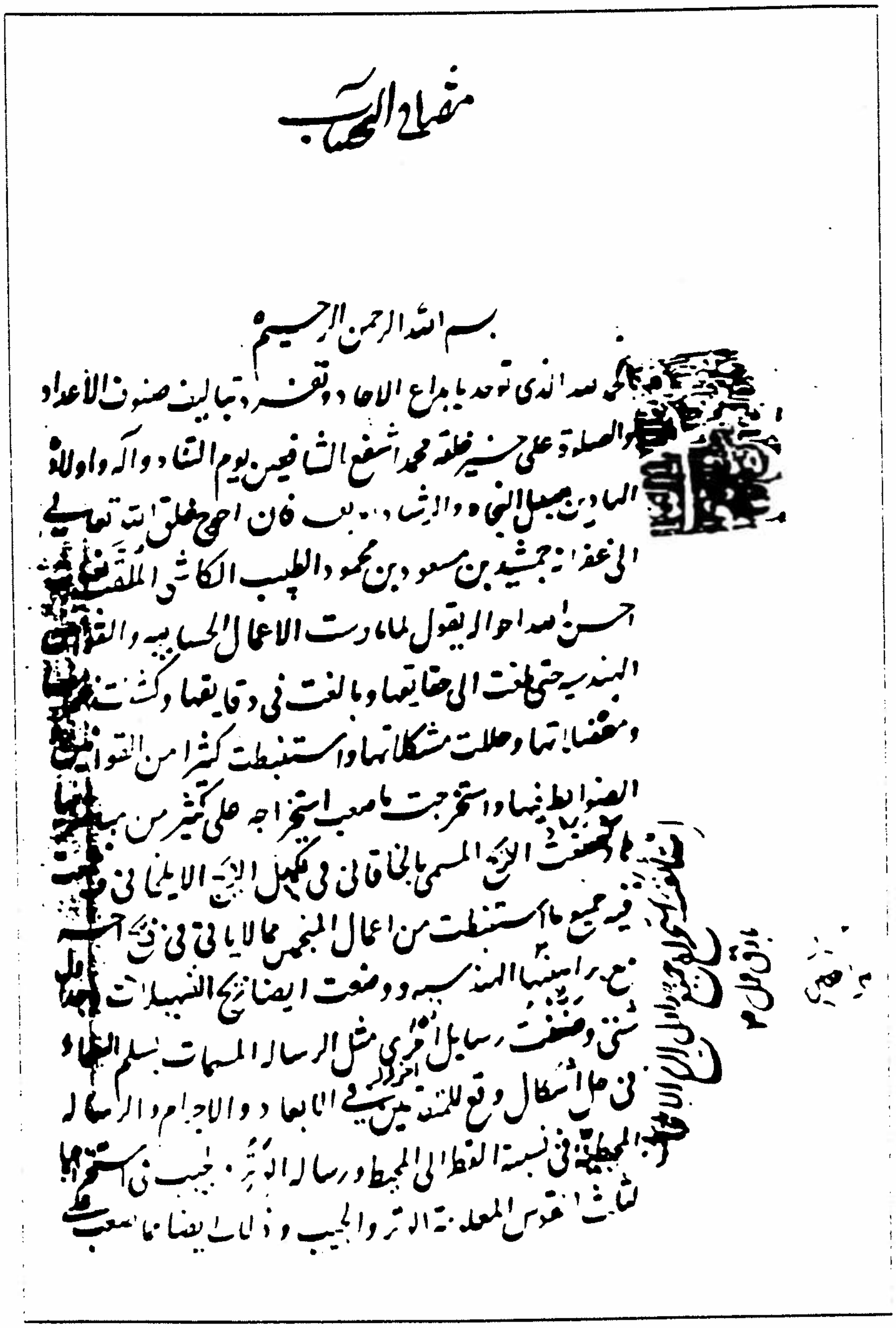
It is beyond the scope of this article to write about Kashani's mathematical achievements in depth. Thus we are going to list his major and most important works in mathematics, and then make some comments concerning the content of each. The four greatest mathematical achievements of Kashani are: **Meftah al-hesab** ("The Key of Arithmetic" or "The Calculators' Key"), **Resaleh-e muhitiyyah** ("The Treatise on the Circumference"), **Resaleh-e vatar va jaib,** ("The Treatise on the Chord and Sine"), and **Talkhieth al-meftah** ("Compendium of the Key"). Other mathematical discoveries of Kashani are in conjunction with his many works on astronomy.

Meftah al-hesab

After more than seven years of hard work, Kashani completed **Meftah al-hesab** in Arabic on March 2, 1427 (Jamadi al-awwal 3,830 A.H.L.), and dedicated this remarkable mathematical achievement to Sultan Ulugh Beg, grandson of Timur and the ruler of Samarqand. Among other places, a handwritten copy of **Meftah al-hesab** exists at the central library of the University of Tehran, Number 6287. Plate A shows a photocopy of the first page of this copy. **Meftah al-hesab,** Kashani's best known work, was written as a textbook and was used not only as a textbook, but also as an encyclopedia of elementary mathematics by a broad range of students, as well as accountants, astronomers, architects, land surveyors, merchants, and other professionals. It is divided into five books preceded by an introduction, each book consisting of several chapters. The titles of these five books, as well as the number of chapters in each book, respectively, are: "On the Arithmetic of Integers" (six chapters), "On the Arithmetic of Fractions" (twelve chapters), "On the Computation of Astronomers" (six chapters), "On the Measurement of Plane Figures and Bodies" (nine chapters preceded by an introduction), and "On the Solutions of Problems by Means of Algebra" (four chapters). The reader is reminded that Kashani discussed the extraction of the *n*th root in chapter five of both book 1 and book 3. For translations of the introduction of **Meftah al-hesab** into French, German, and Persian, see Woepcke [12], Luckey [6], and Qurbani [8], respectively. Luckey, along with the translation of the introduction, also has written an explanation of some parts of **Meftah al-hesab**. Moreover, Luckey's article [7], concerning the extraction of the nth root of an integer and the binomial expansion in Islamic mathematics, is based on **Meftah al-hesab**. Dakhel has written a translation as well as an explanation of the fifth chapter of the third book of **Meftah al-hesab** in English [3].

In the first book of **Meftah al-hesab**, Kashani describes in detail a general method of extracting *n*th root of an integer in the decimal system as well as the sexagesimal system. Later in the nineteenth century this method was rediscovered by European mathematicians, and now it is known as Ruffini-Horner's method. However, many Persian mathematicians, including Abu Rayhan-e Biruni, Omar Khayyam, and Nasir ud-din Tusi, worked on the extraction of roots of integers with some success.

Kashani claimed that the general method of extracting roots of integers was his own discovery. Also, the reader needs to know that in the first book he gave a general rule for the expansion of $(a + b)^n$ (for any positive integer n) which is the same rule that we are using nowadays. Kashani put the coefficients of the expansion $(a + b)^n$ in a triangular table and made use of it in the extraction of the *n*th root of integers. Nowadays, this triangle is called the Pascal triangle. Kashani acknowledged that the expansion of $(a + b)^n$ and the triangular table that he used is due to his predecessors, by which he probably meant Omar Khayyam and Nasir ud-din



مفتاح الحساب

بسم الله الرحمن الرحيم

الحمد لله الذي توحد بإبداع الآحاد وتفرد بتأليف صنوف الأعداد
والصلاة على خير خلقه محمد أشفع الشافعين يوم التناد وآله وأولاده
الهادين سبل النجاة والرشاد وبعد فإن أحوج خلق الله تعالى
إلى غفرانه جمشيد بن مسعود بن محمود الطبيب الكاشي الملقب [illegible]
أحسن الله أحواله يقول لما مارست الأعمال الحسابية والقواعد
الهندسية حتى وقفت على حقائقها وبلغت في دقائقها وكشفت [illegible]
ومعضلاتها وحللت مشكلاتها واستنبطت كثيرا من القواعد
والضوابط فيها واستخرجت ما صعب استخراجه على كثير من [illegible]
[illegible] صنفت الزيج المسمى بالخاقاني في تكميل الزيج الايلخاني في [illegible]
فيه جميع ما استنبطت من أعمال المنجمين مما لا يأتي في زيج [illegible]
مع براهينها الهندسية ووضعت أيضا زيج التسهيلات [illegible]
شتى وصنفت رسائل أخرى مثل الرسالة المسماة بسلم السماء
في حل إشكال وقع للمتقدمين في الأبعاد والأجرام والرسالة
المحيطية في نسبة القطر إلى المحيط ورسالة الوتر والجيب في استخراجهما
لثلث القوس المعلومة الوتر والجيب وذلك أيضا مما صعب

Plate A. First page of **Meftah al-hesab.**

Tusi. Also in the first book, he presented a method for calculation of $a^n - b^n$, where n is any positive integer.

In the fourth book of **Meftah al-hesab**, Kashani presented numerous geometric formulas including the following, where R and r are radii of circumscribed and inscribed circles of a regular n-gon, S is the area of the regular n-gon, and a is the length of each side of the regular n-gon.

$$R = \frac{a}{2\sin\left(\frac{180}{n}\right)^{\circ}},$$

$$r = \frac{a}{2}\cot\left(\frac{180}{n}\right)^{\circ},$$

and

$$S = \frac{a^2 n}{4}\cot\left(\frac{180}{n}\right)^{\circ}.$$

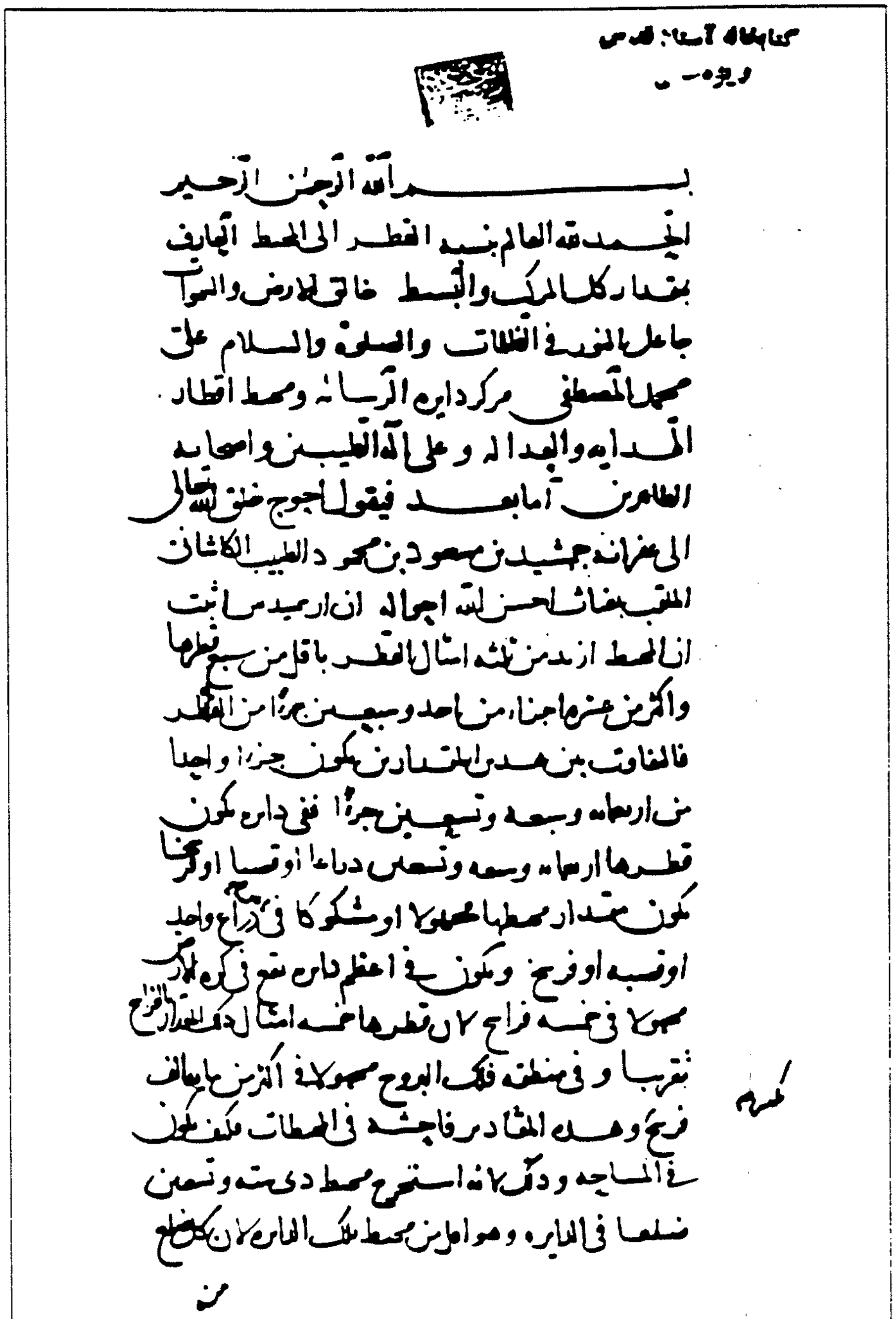
He used the above formula for S to calculate the areas of regular n-gons ($n = 3$, 5-10, 12, 15, 16) in both sexagesimal and decimal systems. Also, in the fourth book he presented several trigonometric identities including those identities which are known as the laws of sines and cosines.

Resaleh-e muhitiyyah

Kashani completed **Resaleh-e muhitiyyah** in July 1424 (Shaban 827 A.H.L.). This masterpiece of computational technique consists of an introduction, ten chapters, and an epilogue. The original copy, which was written long-hand in Arabic by Kashani himself, was donated by Nader Shah (King Nader) in 1145 A.H.S. to the library of the Astan-e Quds-e Razavi, Mashhad, Iran (Number 162 of the mathematics collection and Number 5389 of the general collection of the central library of the Astan-e Quds-e Razavi). Plate B shows a photocopy of the first page of this resaleh. For the first time, Luckey translated with comments the entire book into German in 1949 [5], and the German version was published in Berlin, posthumously. Qurbani translated the introduction of this resaleh into Persian [8], and he promised to publish a Persian as well as a French translation of this resaleh. However, the author has not been able to verify whether or not Qurbani kept his promise.

In the first chapter of the **Resaleh-e muhitiyyah**, Kashani proved his fundamental theorem, which is the basis for the rest of his calculation in this resaleh.

Theorem. If on a semi-circle with the diameter $AB = 2r$ and the center O we consider an arbitrary arc AG, and we call the middle of the arc GB the point D, and we draw the chord AD, then we have


کتابخانه آستان قدس

بسم الله الرحمن الرحيم
الحمد لله العالم بنسبة القطر الى المحيط [illegible]
بمقدار كل المركب والبسيط خالق الارض والسموات
جاعل النور في الظلمات والصلوة والسلام على
محمد المصطفى مركز دائرة الرسالة ومحيط اقطار
الهداية والعدالة وعلى آله الطيبين واصحابه
الطاهرين اما بعد فيقول احوج خلق الله تعالى
الى غفرانه جمشيد بن مسعود بن محمود الطبيب الكاشاني
الملقب بغياث احسن الله احواله ان ارشميدس اثبت
ان المحيط ازيد من ثلثة امثال القطر باقل من سبع قطرها
واكثر من عشرة اجزاء من احد وسبعين جزءا من القطر
فالتفاوت بين هذين المقدارين يكون جزءا واحدا
من اربعمائة وسبعة وتسعين جزءا ففي دائرة يكون
قطرها اربعمائة وسبعة وتسعين ذراعا او قصبا او فرسخا
يكون مقدار محيطها مجهولا او مشكوكا في ذراع واحد
او قصبة او فرسخ ويكون في اعظم دائرة تقع في كرة الارض
مجهولا في خمسة فراسخ لان قطرها خمسة امثال ذلك العدد [من] الفراسخ
تقريبا وفي منطقة فلك البروج مجهولا في اكثر من [illegible]
فرسخ وهذا المقدار فاحش في المحيطات فكيف يكون
في المساحة ودقائق لانه استخرج محيط ذي ستة وتسعين
ضلعا في الدائرة وهو اطول من محيط تلك الدائرة لان كل ضلع
من

Plate B. First page of **Resaleh-e muhitiyyah.**

$$r(2r + \mathrm{lc}(AG)) = (\mathrm{lc}(AD))^2 ,$$

where lc(AG) and lc(AD) represent the lengths of the chords AG and AD, respectively.

He deduced that in a unit circle, if the arc AG is equal to 2φ radian, then $\mathrm{lc}(AD) = 2\sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$. From this he obtained the identity $\sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \sqrt{\frac{1+\sin\varphi}{2}} \left(\varphi < \frac{\pi}{2}\right)$, which is known as the J. H. Lambert identity. It is interesting to note that this identity appears in J. H. Lambert's work in 1770, while Kashani proved this identity in 1424 (827 A.H.L.), 346 years earlier. Kashani also used his fundamental theorem to calculate the value of π, correct to sixteen decimal places. To obtain $2\pi = 6.2831853071795865$ (this is equivalent to 2π = 6;16,59,28,1,34,51,46,14,50 in sexagesimal system), he calculated the perimeters of inscribed and circumscribed polygons, in a given circle, having $3(2^{28}) = 805306368$ sides. This number of sides would have given him the desired accuracy. Kashani stated in advance his requirement for the accuracy of the approximation of π. He wanted the value of 2π to be so accurate that when the same method is applied to calculate the circumference of the earth, the round-off error would not be more than the width of a horse's hair. This approximation of π, in 1424, was a remarkable achievement for that time, and it far surpassed all approximation of π by all previous mathematicians throughout the world, including Archimedes and Ptolemy (Archimedes' approximation in 250 B.C. was π = 3.14 and Ptolemy's result in circa 150 A.D. was $\pi = 3.14166$).

It took European scientists 186 years to improve on Kashani's approximation of π. François Viete, 155 years later in 1579, obtained a value for π, correct only to eleven decimal places. Adrian Romain, in 1593, calculated the value of π correct to fifteen decimal places, and, finally, Ludolf van Ceulen obtained the value of π correct to thirty-five decimal places in 1610. Kashani was well aware of the importance of his discovery and to make sure that the digits in the approximation of π are preserved exactly the same way as he obtained them, he put the digits of his result in a couplet in Arabic as well as Persian. After finding an approximation for the value of π in the **Meftah al-hesab**, Kashani wrote: "Nobody except God knows the true value of π."

Resaleh-e vatar va jaib

The exact completion date of **Resaleh-e vatar va jaib** is not known. But, since Kashani in the introduction of **Meftah al-hesab** proudly talks about this resaleh, one would guess that it was completed before 1427. Unfortunately, the original copy has been destroyed. However, Qazi Zadeh-e Roumi wrote an account of this resaleh in Arabic which was published in 1299 A.H.L., in Tehran, Iran. Two copies of Qazi Zadeh's version of this resaleh exist in the Ketabkhaneh-e Melli-e Malek (National Library of Malek), Numbers 3536/1 and 3180 in Tehran, and this Arabic version has been translated into Russian by Juschkewitsch [4]. Also, brief explanations of this resaleh exist in English (Aaboe [1]), French (Sedillot [10]), German (Schoy [9]), and Persian (Qurbani [8]).

The main part of this resaleh involves the creation and solution of the cubic equation $ax = b + x^3$ (a and b are constants) and approximation of sin 1° from sin 3° using the trigonometric identity $\sin 3x = 3\sin x - 4\sin^3 x$, which he obtained using the above cubic equation. Kashani used a circle of radius r to obtain this cubic equation and an iterative method (fixed-point iteration) to solve this cubic equation to obtain sin 1° = 0.0174524064372835103712, correct to twenty-two decimal places (which is equivalent to sin 1° = 1;2,49,43,11,14,44,16,26,17, correct to ten sexagesimal places). Not only was this the most fascinating and creative method of approximation, but it was the first approximation method in the history of mathematics, and the most significant achievement in medieval algebra. The best previous approximation, correct to four sexagesimal places, were obtained by two other Muslim scientists Abu-al-Wafa and Ibn Yunus, in the tenth century.

Talkhieth al-meftah

Talkhieth al-meftah, written in Arabic, was completed on August 7, 1421 (Shaban 7, 824 A.H.L.), and as its title suggests it is an early and abbreviated version of **Meftah al-hesab**. Among other libraries, a hand-written copy of **Talkhieth al-meftah** exists at the central library of the University of Tehran, Number 868/3. Plate C is a photocopy of the first page of this copy of **Talkhieth al-meftah**. It consists of three chapters and it contains about one-eighth of the material in **Meftah al-hesab**. There is nothing about the third book of **Meftah al-hesab** in **Talkhieth al-meftah**. A Persian translation of the titles of thirty chapters of **Talkhieth al-meftah** is given in Qurbani [8].

Decimal Fractions

Kashani discovered decimal fractions in 1423, and his discovery was inspired by sexagesimal fractions. For the first time, Luckey, in this book about **Meftah al-hesab** [6], announced that Kashani indeed was the true discoverer of decimal fractions. Prior to Luckey's announcement, and before Europeans learned about **Meftah al-hesab** and **Resaleh-e muhitiyyah** , they acknowledged Simon Stevin in 1585, Joost Burgi in 1592, and François Viete circa 1600 and others as discoverers of decimal fractions. Apparently mathematicians before Kashani felt a need for decimal fractions and some of them unknowingly used them, but Kashani introduced decimal fractions methodically. It is apparent that his notion of decimal

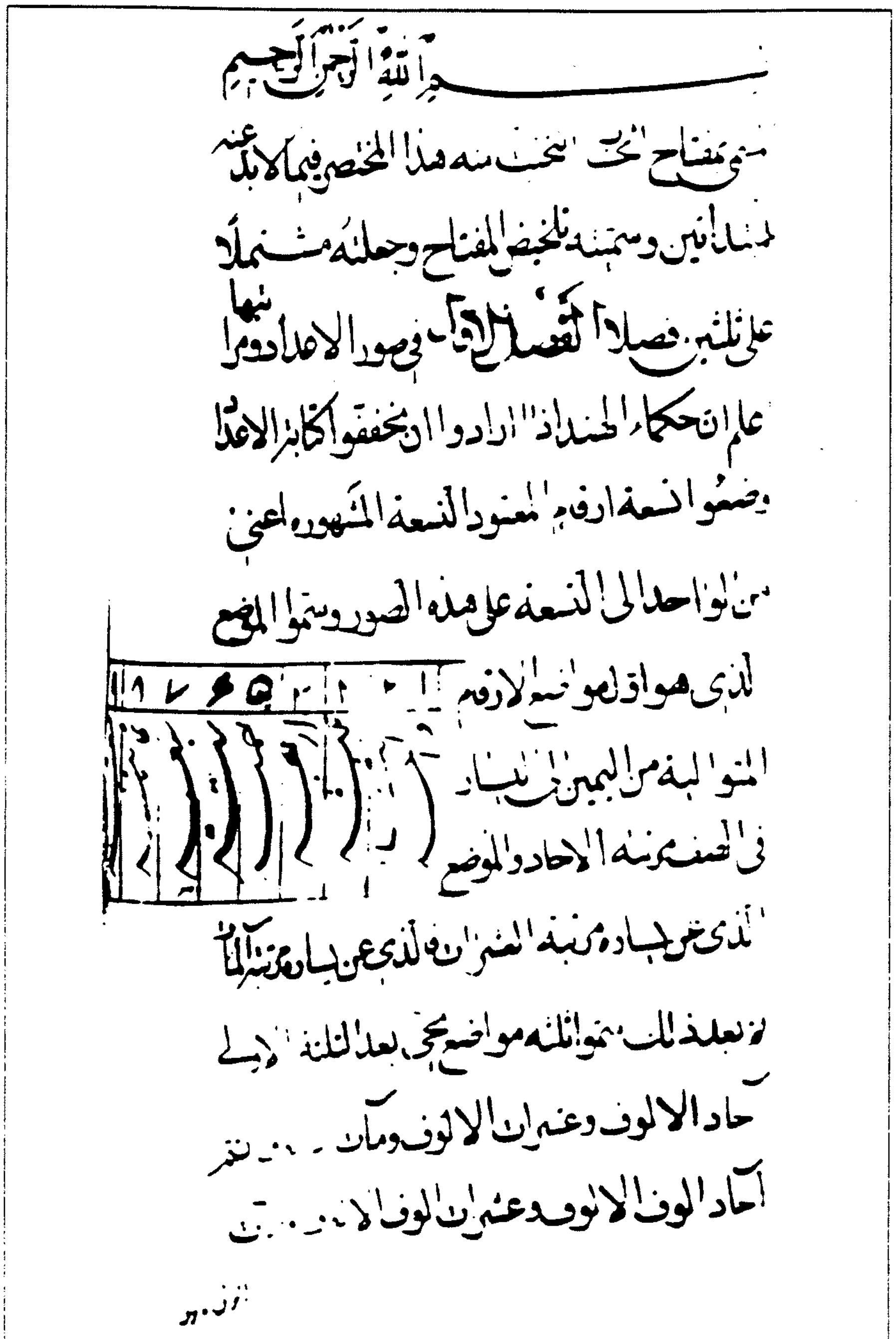


Plate C. First page of **Talkhieth al-meftah.**

fractions was basically the same as ours. Kashani has used decimal fractions in both **Meftah al-hesab** and **Resaleh-e muhitiyyah**, and he explained them wherever he used them.

Remarks

1. The start of the Islamic calendar is the year 622 A.D., when prophet Muhammad migrated from his hometown of Makkeh to the city of Madineh, both cities in the Arabian peninsula. Prophet Muhammad's migration is called hijrat. Nowadays, there are two Islamic calendars in use. One is the lunar calendar and the other is the solar calendar. The lunar calendar is 354 days long, while the solar calendar is 365 days long (the same length as the Gregorian calendar). In this article, "A.H.L." and "A.H.S." are used for these two calendars, respectively. The twelve months of the Arabic lunar calendar are: Muharram, Safar, Rabi al-awwal, Rabi al-thani, Jamadi al-awwal, Jamadi al-thani, Rajab, Shaban, Ramadan, Shawwal, Duh-al-qadah, Dhu-al-hijjah; while the twelve months of the Persian solar calendar are: Farvardin, Ordibehesht, Khordad, Teer, Mordad, Shahrivar, Mehr, Aban, Azar, Day, Bahman, Esfand. The beginning of the solar calendar year is the first day of Farvardin (the first day of spring–March 21st), whereas the beginning of the lunar calendar year is the first day of Muharram. Needless to say, the lunar calendar does not coincide with the seasons, whereas the solar calendar does.

2. We note that while in Persian we write (as we see in this article) Makkeh, Madineh, jihrat, resaleh (treatise), **Meftah al-hesab**, **Resaleh-e muhitiyyah**, **Resaleh-e vatar va jaib**, and **Talkhieth al-meftah**; in Arabic, we write Mecca, Medina, hijrah, risala, **Miftah al-hisab, Risala al-muhitiyya, Risala al-watar wa'l jaib,** and **Talkhieth al-miftah**, respectively.

3. Most of Kashani's work was written in sexagesimal (base 60) system. In sexagesimal system the digits are separated by commas, and the integral and fractional parts by semicolons. For example: 9,25;37,16 in base 60 is $9(60) + 25 + 37(60^{-1}) + 16(60^{-2})$ in base 10.

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