Roshdi Rashed: *Géométrie et dioptrique au Xe siècle: Ibn Sahl, al-Qūhī et Ibn al-Haytham*. Paris: Les Belles Lettres, 1993 (Collection sciences et philosophie arabes). cliii + 630 + vii pp. ISBN 2-251-35563-4.

In April 1995 I was invited by the book review editor of the journal *Physis* (Rome) to review R. Rashed, *Géométrie et dioptrique au Xe siècle: Ibn Sahl, al-Qūhī et Ibn al-Haytham* for *Physis*. I wrote the review and submitted it in September 1995. Later, Rashed's close colleague Dr Morelon (Paris) submitted a review of the same book to *Physis*. The editors of *Physis* published Morelon's review and on April 28, 2000, they informed me that Dr Morelon submitted his review in 1997, and that it was not possible for them to publish the review which I had submitted in 1995 to *Physis*.

The following review is an edited version of the review which I submitted to *Physis*. I have made some additions in order to discuss the recent publications [11], [1] and [2], which are closely related to the book under review.

The focus of this book is Abū Sa'd al-'Alā' ibn Sahl, a late 10th-century Islamic mathematician who wrote a number of works on mathematics and optics in Arabic. The book contains editions, French translations and commentaries of the following of his extant works:

1. The treatise on burning mirrors and burning lenses. This is al-'Alā''s most important work, which Rashed has put together from two incomplete Arabic manuscripts. In this treatise, al-'Alā' discusses the elliptic burning mirror and the parabolic burning mirror, which were known in antiquity, and the hyperbolic burning lense, which seems to have been his own invention. Al-'Alā' rotates a segment of a single-branch hyperbola around its transverse axis and he considers the convex lense bounded by the resulting hyperbolic surface of revolution and a plane perpendicular to the axis. He proves that rays of light parallel to the axis, which enter the lense through the perpendicular plane, will be refracted by the convex surface to one of the foci of the hyperbola. In this proof he assumes a property of refracting rays which is mathematically equivalent to the sine law of refraction.

In the book under review, and in the earlier article [10], Rashed argues that al-'Alā' was in possession of the sine law of refraction. Professor A.I. Sabra points out that there are problems of interpretation because al-'Alā' does not mention the equivalent of the sine law explicitly. In any case, al-'Alā''s discussion of hyperbolic lenses raises interesting questions about the history of Arabic theories of reflection and refraction. For example, one would like to know whether Ibn al-Haytham (ca. 1000) knew the work of al-'Alā'. (To the reviewer this appears to be doubtful.) Another question is: how did al-'Alā' find his mathematical equivalent of the sine law of refraction?

Rashed does not answer this question in his book, nor in his article [10]. The following explanation could be suggested. The phenomenon of refraction had already been investigated by Ptolemy in Book V of his *Optics*. Since al-'Alā' cites Ptolemy's *Optics* elsewhere, we may assume that he was familiar with Ptolemy's treatment of refraction. Ptolemy's tables of refraction suggest that rays can only be refracted from glass or water to air if the rays make an angle with the normal which is less than a certain maximal angle α of incidence, with $\alpha < 90°$ ($\alpha = 45°$ for glass, $\alpha = 54°$ for water, see [9, pp. 229, 234]). If the angle between the ray and normal is greater than α, the ray is not refracted but reflected. Now suppose that al-ʿAlāʾ asked himself the question: what shape must a burning lense have? That is to say, what is the shape of a lense such that all the rays parallel to an axis are to be refracted to a single point *F* on the axis? Clearly, the lense must have rotational symmetry around the axis, so the surface of the lense is obtained by rotating some plane curve around the axis. The only curves which were studied in medieval Arabic mathematics were the circle, the ellipse, the parabola and the hyperbola. Now consider the angle of incidence of a ray parallel to the axis, which ray hits the curve at any point *P*. We want this ray to be refracted to the point *F* on the axis, so the angle between the normal at *P* and the line through *P* parallel to the axis must be less than α, for any point *P* on the curve. Of the four curves mentioned, only the hyperbola can have this property. Thus al-ʿAlāʾ could have found the hypothesis that the surface of the lense must be a hyperbolic surface of revolution. From this hypothesis one can derive, by analysis, the property of incident and refracted rays equivalent to the sine law. What we find in the text is the synthesis which corresponds to this analysis. This hypothetical scenario could also explain why no mention was ever made of a numerical index of refraction, because al-ʿAlāʾ need not have made any experiments.

The practical importance of the subject must not be over-emphasized. Al-ʿAlāʾ describes 'constructions ' of conic sections by means of a complicated system of pulleys and ropes, but it is not clear if such a machine could work in practice, and it does not seem easy to construct a hyperbolic lense in this way.

For the relation between the hyperbolic lense and the sine law of refraction, the reader can also consult the very clear explanation in [7] (without reference to history).

The other works of al-ʿAlāʾ published by Rashed in the book under review are the following:

2. A short work in which it is argued that the celestial sphere is not completely transparent. According to the text, al-ʿAlāʾ wanted to include this little treatise in a commentary on Book V of Ptolemy's *Optics*.

3. A short work on conic sections. Rashed says (p. lxxxii) that the 'properties studied (by al-ʿAlāʾ in his treatise) are comparable to some of those which Apollonius treats, for example in propositions 38 to 40 of Book III of the *Conics*'. Surprisingly, Rashed does not seem to have realized (cf. his index) that the propositions in this treatise are in fact the same as *Conics* III:16. The aim of the treatise was therefore to provide an alternative proof of *Conics* III:16. The treatise is unrelated to projective geometry.

4. A commentary to a treatise by Abū Sahl al-Kūhī on the astrolabe, see text c. below.

Rashed appended to these texts his editions of two fragments of Book VII of Ibn al-Haytham's *Optics*, the treatise by Ibn al-Haytham on the burning sphere, the revision of this treatise by Kamāl al-Dīn al-Fārisī, and three more texts:

a. The anonymous *Synthesis of the Problems Analysed by Abū Sa'd al-'Alā' ibn Sahl.* This text contains ten easy preliminary theorems, followed by syntheses of three problems. Unfortunately, Rashed does not give the interesting historical background of the first two problems. In the first problem, one is required to inscribe a triangle in a given circle in such a way that the rectilinear extensions of the three sides pass through three given collinear points (pp. 167-175). This problem was solved by Apollonius in his lost work *On Tangencies* (which was extant in Arabic) as a preliminary to his famous construction of a circle tangent to three given circles, and al-'Alā''s analysis is related in a non-trivial way to the solution of the same problem given by Pappus of Alexandria in *Collection* VII, prop. 117 [6, pp. 246-249]. The second problem (edited and translated on pp. 175-179) is historically related to the *Neuseis* of Apollonius. See the discussion of this part of the *Synthesis of the Problems* in the paper [4, pp. 197-198], which Rashed does not cite. The third problem is a variation of a problem which was studied in connection with the regular heptagon. The relationship between the third problem, the regular heptagon and the work of the 4th/10th-century geometer al-Shannī are extensively discussed in the paper [3], which Rashed does not mention either. Thus on p. cxxxv of his introduction, Rashed mentions "un autre événement ... curieusement resté inaperçu des historiens," namely a quotation by al-Shannī, which is discussed in [3, p. 257]. In footnote 19 of the same page, he quotes from an unpublished Arabic manuscript, although the same passage in the manuscript was edited and translated in [3] (quotation [III,13] on [3, p. 251], Arabic text on [3, p. 321]), and so on. In 1979, A. Anbouba identified the anonymous *Synthesis of the Problems* as a work by the 10th century geometer Abū l-Jūd. This identification is supported by the fact that the *Synthesis of the Problems* contains serious mathematical mistakes, which Rashed appears not to have noticed. Rashed's suggestion that the anonymous author was al-Shanni (p. cxxxvi-vii) is therefore implausible. See for further details [3, pp. 258-263].

b. A short fragment by al-'Alā' on the area of a segment of the circle. The fragment is found in a text by al-Sijzī (4th/10th century) called *On the Selected Problems* ..., and Rashed added it to his book because he wanted to include all extant works by al-'Alā'. The existence of fragments by al-'Alā' in this text by al-Sijzī was first noticed in [4, p. 193]. When the book under review was published, Rashed apparently had not yet read al-Sijzī's text *On the Selected Problems* .... completely. Rashed has recently published two more fragments by al-'Alā' in al-Sijzī's *On the Selected Problems* ... in [11, pp. 81-95]. For one of these fragments see also [5, pp. 160-162], the other fragment will be discussed below.

c. The treatise on the astrolabe by Abū Sahl al-Kūhī, on which al-'Alā' wrote a commentary. This treatise by al-Kūhī consists of two Books. In Book 1, al-Kūhī discusses the various projections of the celestial sphere which can be used for the construction of the astrolabe, and he explains the stereographic projection of azimuthal and altitude circles (the pole of projection is of course the celestial north or south pole). In Book 2, which is incompletely extant, al-Kūhī solves various geometrical problems connected with his construction of the astrolabe. Al-Kūhī's plane of construction is the plane of the local meridian, and the relevant other circles which occur in the problems (such as circles on the astrolabe and altitude circles on the celestial sphere) are represented by their images after rotation around their intersections with the meridian plane. This procedure is traditional in so-called analemma constructions which were widely used in later Greek and medieval Islamic astronomy. The following example will illustrate the problems. In section 2 of Book 2, al-Kūhī assumes that the latitude of the locality and the (rotated!) stereographic projection of an almucantar is given, and he assumes that we know one more thing: either (a) the pole of projection or (b) the centre of the celestial sphere, or (c) the radius of the sphere, or (d) the line segment between the celestial pole and the zenith, and so on. For all these possibilities, he wants to construct the intersection of the celestial sphere with the meridian plane, so he can construct all other astrolabe curves. The example shows that most or all of these problems are of mathematical (recreational) interest only. The practical situation which comes closest to these problems is an astrolabe on which almost all lines are erased, but in that case we would always know (b) so the other cases are irrelevant. The problems are based on the standard construction of the astrolabe circles and in this sense they are related to stereographic projection. The solutions to these problems, however, contribute little if anything to the "theory of projection", and they are completely unrelated to projective geometry.

The astrolabe text by al-Kūhī has also been published by Professor Len Berggren [2] in an edition which appeared in 1994, but which was circulated among colleagues as early as 1991 in essentially the same form in which it appeared. Rashed's suggestions about a dependency in [11, p. 80 note 3] are therefore untenable.[1] On the other hand, it appears that the three most important

[1] The last part of one of the solutions of the problem 6 of section 2 of Book 2 is missing, and it has been reconstructed by Rashed in p. 220 in the book under review, and by Berggren [2, pp. 221-222]. Morelon [8] and Abgrall [1, p. 63, note 172] have naively concluded from the resemblance between these reconstructions that [2] is based on the book under review. The resemblance between the two reconstructions can be easily explained, however. Problem 6 of section 2 is closely related to problem no. 2 of section 2, and the last parts of the solutions are similar. The missing end of the synthesis of problem no. 6 can therefore be reconstructed by making a few trivial changes in the extant end of the synthesis of problem no. 2. Compare p. 220 and p. 214 line 20 - p. 215 of the book under review, and [2, pp. 221-222] with [2, pp. 228-229]. (In Berggren's subdivision of Book 2, problems 2 and 6 of section 2 in the book under review belong to sections 3 and 5, respectively.)

corrections to the book under review in Rashed's recent article [11] are similar to the corresponding passages in Berggren's edition.[2]

The third fragment by al-'Alā' which Rashed published in [11] gives interesting new information on al-Kūhī's treatise on the astrolabe, although Rashed does not make the relationship clear. In Book 2 of this treatise, al-Kūhī solves two problems by means of preliminary constructions of his treatise *Establishing Points on Lines*, and al-'Alā' shows in the fragment how these problems can be solved without reference to al-Kūhī's *Establishing Points on Lines*. Al-'Alā' also gives a list of problems which give an idea of the missing part of Book 2 of al-Kūhī's treatise.

In spite of the above-mentioned setbacks, the Arabic texts and French translations in this book are a major contribution to the history of Arabic optics and mathematics.

ACKNOWLEDGEMENT

I wish to thank Professor A.I. Sabra (Cambridge, Mass.) for stimulating discussions of the history of the sine law of refraction.

REFERENCES

[1] P. Abgrall, La géométrie de l'astrolabe au Xe siècle, *Arabic Sciences and Philosophy* 10 (2000), 7-77.
[2] J. Len Berggren, Abū Sahl al-Kūhī's Treatise on the Construction of the Astrolabe with Proof: Text, Translation and Commentary, *Physis* Nuova Serie 31 (1994), 141-252.
[3] Jan P. Hogendijk, Greek and Arabic Constructions of the Regular Heptagon,

[2] 1. The correction to p. 208 line 8 of the book under review in [11, p. 96] is similar to [2, p. 235 line 11] (Arabic text), compare the translation "[the centre of the sphere will be known]" in [2, p. 163 line 24].

2. Rashed's correction to his wrong emendation in p. 193 line 8 in the book under review in [11, p. 98] is similar to Berggren's emendation of the same passage, see the Arabic text in [2, p. 249 line 6], the apparatus in [2, p. 211 no. 45], the translation in [2, pp. 150-151] and Figure I,1 in [2, p. 180]. Note that the point labeled "I" in Rashed's correction corresponds to the point labeled "W" in Berggren's edition.

3. In [11, p. 98], Rashed adds the following passage to the edited Arabic text in the book under review, p. 193 lines 14-17, to restore the mathematical sense: <fa-muthallath ABD shabīh bi-muthallath AHE, fa-zāwiyat ADB mithl zāwiyat AEH, lākin zāwiyat ADB mithl zāwiyat AGB, fa-zāwiyat AEH mithl zāwiyat AGB, wa-zāwiyat ZAE mushtarika fī hādhā al-mithāl.>

This passage is almost identical to the following passage, which Berggren added in [2, p. 249 lines 12-14] to restore the mathematical sense of the Arabic text:

<fa-muthallath ABD shabīh bi-muthallath AHE, fa-zāwiyat ADB musāwiya li-zāwiyat AEH, wa-zāwiyat AGB musāwiya li-zāwiyat ADB li-annahumā 'alā qaws AB, fa-zāwiyat AGB musāwiya li-zāwiyat AEH.>

Archive for History of Exact Sciences 30 (1984), 197-330.

[4] Jan P. Hogendijk, Arabic Traces of Lost Works of Apollonius, *Archive for History of Exact Sciences* 35 (1986), 187-253.

[5] Jan P. Hogendijk, Traces of the Lost "Geometrical Elements" of Menelaus in Two Texts of al-Sijzī, *Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften* 13 (1999-2000), 129-164.

[6] A. Jones (ed.), *Pappus of Alexandria. Book VII of the Mathematical Collection.* New York (Springer) 1986. Sources in the History of Mathematics and Physical Sciences 8.

[7] Mohsen Maesumi, Parabolic Mirrors, Elliptic and Hyperbolic Lenses, *American Mathematical Monthly* 99 (1992), 558-560.

[8] R. Morelon, review of: *R. Rashed, Géométrie et dioptrique au Xe siècle. Ibn Sahl, al-Qūhī et Ibn al-Haytham,* in: *Physis* 36 (1999), 281-284.

[9] Claudius Ptolemaeus, *L'optique de Claude Ptolemée*, ed., A. Lejeune, Louvain: Publications Universitaires, 1956.

[10] R. Rashed, A Pioneer in Anaclastics: Ibn Sahl on Burning Mirrors and Lenses, *Isis* 81 (1990), 464-491.

[11] R. Rashed, Ibn Sahl et al-Qūhī: Les Projections, Addenda & Corrigenda, *Arabic Sciences and Philosophy* 10 (2000), 79-100.

JAN P. HOGENDIJK*