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R. Rashed (Ed., Transl.), Sharaf al-Dīn al-Ṭūsī; *Œuvres mathématiques. Algèbre et géométrie au XII^e siècle.* Paris: Société d'édition „Les Belles Lettres", 1986. Vol. 1: 470 pp., ISBN 2-251-35562-6. Vol. 2: 459 pp., ISBN 2-251-35563-4.

These two volumes contain editions and French translations of the three extant mathematical works of Sharaf al-Dīn al-Ṭūsī (who flourished in the end of the 6th century H./12th century A. D.): a brief, hitherto unpublished, treatise about the asymptotes of the hyperbola, a short treatise on elementary geometry that had been studied previously by Suter,[1] and a very long hitherto unpublished text whose title has been lost, but to which we will refer as the *Algebra*. This text is one of the most important mathematical works from the Arabic-Islamic tradition, because it contains the most profound medieval discussion of cubic equations that is known to be extant.

The Arabic-Islamic mathematicians appear not to have known the algebraical solution of the cubic equation. However, in the fourth/tenth century they constructed the roots of various cubic equations geometrically by means of conic sections. In the 11th century, 'Umar al-Khayyām wrote his famous *Algebra*, in which he gave geometrical constructions by means of conic sections of the roots of all types of cubic equations. (Because the medieval Arabic-Islamic mathematics only worked with positive coefficients, they had to distinguish 17 types of cubic equations.) 'Umar al-Khayyām's treatment was incomplete in various respects. Some types of equations, such as $x^3 + c = ax^2 + bx$, do not have roots for all possible choices of the coefficients. In such

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[1] H. Suter, Einige geometrische Aufgaben bei arabischen Mathematikern. *Bibliotheca Mathematica*, 3. Folge, 8 (1907–8), pp. 23–36, reprinted in: H. Suter, *Beiträge zur Geschichte der Mathematik und Astronomie der Araber*, ed. F. Sezgin, Frankfurt (Institut für Geschichte der Arabisch-Islamischen Wissenschaften) 1986, vol. 2, pp. 217–230.

cases, al-Khayyām stated that the root x exist if the conic sections in his construction intersect, but he did not discuss the question for which choices of a, b and c the intersection takes place. Secondly, al-Khayyām's constructions prove the existence of a root x, but they are useless for the computation of x.

Until recently the *Algebra* of al-Khayyām was supposed to be the final word on the subject in the Arabic-Islamic tradition. It now appears that Sharaf al-Dīn went considerably further in his own *Algebra*.

Sharaf al-Dīn divides the cubic equations into two groups. The first group (discussed in Volume 1 of the work under review) consists of the equations which for all (positive) choices of the coefficients have a positive root x, such as $x^3 + ax^2 = c$. For these equations, Sharaf al-Dīn gives the same geometrical construction by means of conic sections as al-Khayyām, but Sharaf al-Dīn adds a careful proof that a point of intersection exists and an algorithm for the numerical approximation of x. This algorithm is a variant of the so-called method of Ruffini and Horner,[2] which seems to be of ancient Chinese origin. Sharaf al-Dīn does not mention that the equation $x^3 + bx = ax^2 + c$ can have two or three positive roots for suitable choices of the coefficients.

The second group (discussed in Volume 2) consists of the equations that do not always have a positive root, such as $x^3 + c = bx$. Sharaf al-Dīn correctly discusses the necessary and sufficient conditions for the existence of a positive root x as well as the number of positive roots. The essence of his results may be stated thus, if we write the equations as $f(x) = c$. Sharaf al-Dīn finds a quantity m such that the equation has zero, one or two solutions if c is greater than, equal to or less than f(m) respectively. In modern terms $f'(m) = 0$, but it should be noted that Sharaf al-Dīn does not use the modern derivative in any sense of the word.[3] For $c < f(m)$, the equation has one root $x_1 > m$ and another root $x_2 < m$. Sharaf al-Dīn constructs these roots by substituting $y = x_1 - m$ and $z = m - x_2$; he shows that y and z satisfy the equations $y^3 + py^2 = d$ (1) and $z^3 + d = pz^2$ (2), with $d = f(m) - c$ and $p > 0$ defined in terms of the coefficients in a way which does not concern us here. The root y of (1) had been constructed and approximated in the first part of the *Algebra*, so that x_1 can also be constructed and approximated. Sharaf al-Dīn proves the existence of z by showing that $z - y$ satisfies a quadratic equation with coefficients dependent on p, d and y. He then gives an algorithm for the approximation of z from (2).

The preceding summary gives an impression of the high level of the discussion of Sharaf al-Dīn, but the modern notation may be mislead-

[2] P. Luckey, Die Ausziehung der n-ten Wurzel und der binomische Lehrsatz in der islamischen Mathematik. *Mathematische Annalen* 120 (1948), pp. 217–274.

[3] J. P. Hogendijk, Sharaf al-Dīn al-Ṭūsī on roots of cubic equations, to appear in *Historia Mathematica*.

ing. Sharaf al-Dīn did not use modern algebraic symbolism and almost all of his proofs are geometrical in the Euclidean style.

The edition of the Arabic text is based on all manuscripts of the *Algebra* that are known to exist. The edition is on the whole very good. In the following list of minor notes to the Arabic text in Volume 1, a notation such as p. 38:15 refers to line 15 of page 38. The notes have been numbered for further reference. Angular brackets ⟨ ⟩ in the Arabic text contain editoreal additions to the manuscripts. **1.** In p. 3:2 Rashed emends the text to *li-ḍilʿin* ⟨*aw*⟩ *li-ākhar*, but the mathematical context requires *li-l-ḍilʿi l-ākhar*, which is also in better agreement with the mss. **2.** On p. 26:16 and p. 26:19 delete ⟨*wa-l-aʿlā*⟩; in the case of the quadratic equation the digits in the upper line do not have to be transposed. **3.** On p. 26:20 delete ⟨*wa-huwa wāḥid*⟩. **4.** On p. 29:10 delete ⟨*min*⟩. **5.** On p. 34:10 delete ⟨*wa-l-aʿlā*⟩; on p. 34:9, a number 3 should be placed in the upper line on the left side of the two zeros. **6.** On p. 35 line 3 of the apparatus read 966655,45576. **7.** On p. 40:13, delete ⟨*murabbaʿ*⟩. **8.** On p. 59:2 and p. 59:8 there should be four rows of digits. Rashed does not mention the fact that the rows are displaced in the manuscript L, for example 1088 on p. 59:8 appears in the manuscript after *al-thānī* on p. 59:10. **9.** On p. 63:18 Rashed emends *al-ḥāl* in the ms. to *al-māl* (the square), but the mathematical context requires the emendation *al-ḥāṣil* ("the result"; the "square" which Rashed has in mind is treated on p. 64:7). Also delete ⟨*wa-fī ʿadad al-amwāl*⟩ on p. 63:17. **10.** On p. 64:9 delete ⟨*musaṭṭaḥan*⟩. Notes to the Arabic text in Volume 2 will appear elsewhere.[3]

The French translation is very literal, and it renders Sharaf al-Dīn's thought on the whole accurately. The notes to the Arabic text entail the following changes to the translation: **1.** p. 3:2–3 for "parallèle à l'un des côtés du triangle ou à l'autre" read "parallèle à l'autre côté du triangle". **2.** On p. 26:23 and p. 26:31 delete "et la ligne supérieure". Note that Rashed does not use angular brackets in connection with the French translations of the words he added to the Arabic text. **3.** On p. 26:32 delete "c'est un". **4.** On p. 29:7 for "provient donc de" read "est donc". **5.** On p. 34:15–16 delete "et de la ligne supérieure". **7.** On p. 40:16 for "le carré de PS" read "PS". **8.** With reference to p. 59:3 and 59:15 compare note **8** to the text. **9.** On p. 63:29–30 delete "par le nombre des carrés, de multiplier le carré ⟨du deuxième nombre cherché⟩" (the pointed brackets indicate that the last four words are not in the edited Arabic text). **10.** On p. 64:14 delete "sur une ligne". Note that *musaṭṭaḥ* (rectangle) is also mistranslated as "line" elsewhere, thus on p. 64:6 change "que nous plaçons dans une ligne" to "que nous posons un rectangle", on p. 64:8 change "la ligne" to "le rectangle", similarly on p. 63:11. On p. 64:19 the correct translation "rectangle" is used. The following notes can be added: On p. 8:4 *abadan* is translated as "indéfiniment", but I would prefer "continuellement", because Sharaf al-Dīn proves that the hyperbola approaches its asymptotes continuously, not that the distance can become less than any given positive quantity. Page 39:2 "le principe de la question": this is in fact the original problem. Page 50:11 the number 321 should be placed under 670. Notes to the translation in Volume 2 will appear elsewhere.[3]

Rashed added to the text and the translation a mathematical transcription (a summary of the argument in modern notations, such as $AB^2 + AC.AD$), and a commentary (essentially another summary in modern algebraical symbolism and a discussion of some tacit assumptions that Sharaf al-Dīn makes). The transcription and the commentary follow the text rather closely. Volume 1 also contains an introduction and a chapter on the numerical approximations in the *Algebra* in a very heavy notation, including reconstructions of the tables that were left out by an anonymous scribe.

In the introduction, Rashed relates the *Algebra* of Sharaf al-Dīn to modern mathematical concepts and methods in a way which seems irrelevant or even misleading from a historical point of view. Thus, calling the substitution $y = x - m$ an "affine transformation" will no doubt impress the reader unfamiliar with modern mathematics, but the term is of no help in understanding the text of Sharaf al-Dīn or its historical setting. Similarly, the words "global and algebraical" (with reference to the work of Al-Khayyām) versus "local and analytical" (with reference to the work of Sharaf al-Dīn) are vague, ill-defined, and extremely confusing. It seems to me that in discussions of Arabic texts one should not put too much emphasis on modern mathematical concepts which are not explicitly attested in the Arabic-Islamic tradition. Possible (mathematical) relationships with modern mathematics do not determine the value of the contributions of the Arabic-Islamic mathematicians, and their contributions are even more impressive if one realizes how limited their mathematical resources were.

Thanks to the publication of the edition and the French translation of the *Algebra* the reader can now obtain a first-hand knowledge of the contents and significance of the work of Sharaf al-Dīn. The two volumes under review are therefore a valuable addition to our knowledge of Arabic-Islamic mathematics.

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